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EQUILIBRIUM CONFIGURATIONS OF ROTATING LIQUID MASSES

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0. INTRODUCTION.

Many problems related to surface tension phenomena have been studied in the last years from the general point of view of the Calculus of Variations. By using a well-known argument, based on the principle of virtual works, one is led to a variational formulation of the physical problem in which a certain functional, representing the global energy of the system under consideration, has to be minimized, subject to some "natural" constraints, such as prescribed boundary condition or fixed volume constraints. In general, the energy functional will consist of a "surface integral" plus a "volume integral": the latter corresponds to body forces, as gravity forces, while the former results, for example, from the consideration of the forces acting on the surface of separation between the liquid and the gas surrounding it.

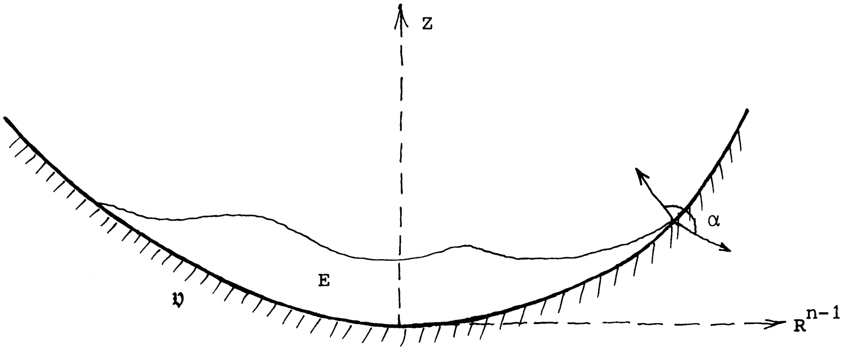
In particular, results on existence and regularity have been recently obtained for the capillary tube and for the sessile and pendent drop (see for example [11], [12]).

In the following we will consider two problems related with rotating liquid masses: the first arises, for instance, in the construction of spincasting contact lenses [7] [17] while the second is related with astrophysics and nuclear physics.

1. ROTATING DROPS IN A VESSEL.

The global energy of an incompressible fluid in an infinite vessel $\mathfrak{V}$ which rotates around the z-axis with constant angular velocity $\sqrt{2\Omega}$ under the combined action of surface and volume forces may be written as

$$(1.1)\qquad \mathcal{F}_\Omega(E) = \int_{\mathfrak{V}} |D\phi_E| + \nu\int_{\partial\mathfrak{V}} \phi_E dH_{n-1} + \int_E H(y,z)\,dy\,dz$$



We denote by $x = (y,z)$, with $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$, an arbitrary point in $\mathbb{R}^n$, ϕ_E is the characteristic function of the set E, $|D\phi_E|$ is the total variation of the Radon measure

$$D\phi_E = \left(\frac{\partial}{\partial x_1}\phi_E, \ldots, \frac{\partial}{\partial x_n}\phi_E\right)$$

(see [2] in these proceedings). Moreover, ν is a coefficient which depends on the liquid and on the material which makes up the walls of $\mathfrak{V}$ (physically, ν represents the cosine of the angle α between the exterior normals to E and to $\partial\mathfrak{V}$, when considered in the contact points between the free surface E and the vessel $\mathfrak{V}$).

Here

$$H(y,z) = gz - \Omega|y|^2 \quad ,$$

where $g > 0$, $\Omega \geq 0$ are given constants. The third integral in (1.1) represents the contribution of energy given by gravitational and kinetic forces, while the first and the second integrals represent the contribution of surface forces. In the following, for brevity, we will write

$$|E| = H_n(E) \quad .$$

We consider the problem:

<u>Problem 1</u>. Minimize the energy functional $\mathcal{F}_\Omega(E)$ in the class

(1.2) $\quad \mathcal{E} = \{E \subset \mathfrak{V} : |E| = 1\} \quad .$

For simplicity we suppose that

$$\mathfrak{V} = \{(y,z) \in \mathbb{R}^n;\ z > |y|^\alpha\} \ , \quad \alpha > 0 \quad .$$

<u>Theorem</u> (Existence results). *If* $-1 < \nu \leq 1$, *we have*

a) $\alpha < 2 \Rightarrow \inf\limits_{\mathcal{E}} \mathcal{F}_\Omega = -\infty \quad \forall\, \Omega \geq 0$

b) $\alpha = 2$, $g < \Omega \Rightarrow \inf\limits_{\mathcal{E}} \mathcal{F}_\Omega = -\infty$

c) $\alpha = 2$, $\Omega < g \Rightarrow$ *there exists* $E_\Omega \in \mathcal{E}$ *minimizing* $\mathcal{F}_\Omega$ *in* $\mathcal{E}$

d) $\alpha = 2$, $\Omega = g$: *the functional* (1.1) *is bounded from below but it is not clear if the minimum is obtained or not*

e) $\alpha > 2 \Rightarrow$ *there exists* $E_\Omega \in \mathcal{E}$ *minimizing* $\mathcal{F}_\Omega$ *in* $\mathcal{E}$ $\forall\, \Omega \geq 0$.

<u>Proof</u>.

a), b) Let $E_j \subset \mathfrak{V}$ be the ball of measure 1 tangent to $\partial\mathfrak{V}$ at the point $(j,0,\dots,0,j^\alpha) \in \partial\mathfrak{V}$ and let r be its radious. We have:

$$\mathcal{F}_\Omega(E_j) \leq n\omega_n r^{n-1} + g(j^\alpha + r) - \Omega(j-2r)^2 = g \cdot j^\alpha - \Omega j^2 + 4\Omega r j + \text{constant}$$

and this last quantity goes to $-\infty$ as j goes to $+\infty$.

c) It is easy to see that there exists a constant $\mu > 0$ such that

(1.3) $\quad \displaystyle\int_{\partial\mathfrak{V}} \phi_E \, dH_{n-1} \leq \int_{\mathfrak{V}} |D\phi_E| + \mu|E| \quad$ for every $E \subset \mathfrak{V}$

(see [8], [21]). Take $E \in \mathcal{E}$. We obtain

(1.4) $\quad \mathcal{F}_\Omega(E) \geq -\mu + (g-\Omega)\displaystyle\int_E z\,dy\,dz \quad \forall\, E \in \mathcal{E}$

that is, $\mathcal{F}_\Omega$ is bounded from below in the class $\mathcal{E}$. Now, let $\{E_j\}$ be a minimizing sequence. In particular, there exists a constant c such that

$\mathcal{F}_\Omega(E_j) \le c \ \forall\, j$. From inequality (1.3) we then obtain

$$c \ge \mathcal{F}_\Omega(E_j) \ge \int_{\mathfrak{D}} |D\phi_{E_j}| + (\nu \wedge 0)\int_{\partial\mathfrak{D}} \phi_{E_j} dH_{n-1} + (g-\Omega)\int_{E_j} zdydz \ge$$

$$\ge \Big[1 + (\nu \wedge 0)\Big]\int_{\mathfrak{D}} |D\phi_{E_j}| + \mu(\nu \wedge 0) + (g-\Omega)\int_{E_j} zdydz \quad .$$

Therefore

$$(1.5) \qquad \int_{\mathfrak{D}} |D\phi_{E_j}| \le \frac{c - \mu(\nu \wedge 0)}{1 + (\nu \wedge 0)} = c_1$$

$$(1.6) \qquad \int_{E_j} zdydz \le \frac{c - \mu(\nu \wedge 0)}{g - \Omega} = c_2$$

From (1.5) and a well-known compactness theorem (see [6], [18]), it follows the existence of a set E_Ω and an increasing sequence $j(k)$ such that

$$(1.7) \qquad E_{j(k)} \xrightarrow{k \to +\infty} E$$

in the $L^1_{loc}(\mathfrak{D})$ topology. Now, from (1.6) we see that actually such a convergence takes place in the $L^1(\mathfrak{D})$ sense, proving that $E_\Omega \in \mathfrak{E}$. The theorem follows from the lower semicontinuity of $\mathcal{F}_\Omega$ with respect to the $L^1(\mathfrak{D})$ topology.

d) In this case (1.5) continues to hold, but (1.6) fails to hold and we are not able to improve the convergence (1.7).

e) Define

$$t_o = \left(\frac{g}{2\Omega}\right)^{\frac{\alpha}{2-\alpha}}$$

in the case $\Omega > 0$, otherwise $t_o = 0$.

From (1.3) we have, for every $E \in \mathfrak{E}$

$$(1.8) \quad \mathcal{F}_\Omega(E) \ge \int_{\mathfrak{D}} |D\phi_E| + (\nu \wedge 0)\left(\int_{\mathfrak{D}} |D\phi_E| + \mu\right) + \int_{E \cap \{z \le t_o\}} H(y,z)dydz + \int_{E \cap \{z > t_o\}} (gz - \Omega z^{\frac{2}{\alpha}-1} \cdot z)dydz \ge$$

$$\ge \int_{\mathfrak{D}} |D\phi_E| + (\nu \wedge 0)\left(\int_{\mathfrak{D}} |D\phi_E| + \mu\right) + \int_{E \cap \{z \le t_o\}} H(y,z)dydz + \frac{g}{2}\int_{E \cap \{z > t_o\}} zdydz \ge c_3 \ \forall\, E \in \mathfrak{E}$$

where $c_3 = c_3(\nu,\mu,\alpha,\Omega,g)$. Therefore, $\inf_{\mathcal{E}} \mathcal{F}_\Omega \geq c_3 > -\infty$.

Let $\{E_j\}$ be a minimizing sequence, in particular there exists a constant c such that $c \geq \mathcal{F}_\Omega(E_j)\ \forall\, j$. From (1.8) we obtain

$$(1.9)\qquad \int_{\mathfrak{D}} |D\phi_{E_j}| \leq \frac{c-\mu(\nu\wedge 0) - \int_{E_j\cap\{z\leq t_o\}} H(y,z)\,dydz}{1+(\nu\wedge 0)} \leq c_4$$

$$(1.10)\qquad \int_{E_j\cap\{z>t_o\}} z\,dydz \leq \frac{c-\mu(\nu\wedge 0) - \int_{E_j\cap\{z\leq t_o\}} H(y,z)\,dydz}{g} \leq c_5$$

and the theorem follows.

2. ROTATING DROPS IN SPACE.

Consider now the case of a liquid drop rotating around its own barycenter with constant angular velocity $\sqrt{2\Omega}$ in the absence of gravity forces.

The global energy of any allowable configuration E is given by

$$(2.1)\qquad \mathcal{F}_\Omega(E) = \int_{R^n} |D\phi_E| - \Omega\int_E |y|^2 dy\,dz \quad .$$

<u>Problem 2.</u> We study the energy functional (2.1) in the class $\mathcal{E}$ defined by

$$(2.2)\qquad \mathcal{E} = \left\{ E\subset \mathbb{R}^n :\ |E| = 1,\ \int_E x_i\,dx = 0,\ (i=1,\dots,n)\right\}$$

that is, among the sets E with prescribed volume and barycenter.

The energy functional being unbounded from below in such a class we must look for a *local minimum* for $\mathcal{F}_\Omega$.

<u>Definition 1.</u> We call $E\in\mathcal{E}$ a local minimum for $\mathcal{F}_\Omega$ if there exists $R>0$ such that

(i) $\quad E \subset\subset B_R = B_R(0)$

(ii) $\mathcal{F}_\Omega(E) \leq \mathcal{F}_\Omega(F)$ for all $F \in \mathcal{E}$, $F \subset B_R$.

Here

$$B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$$

Existence results. To prove the existence of local minimum (at least for small angular velocity) we first state a very simple result (see [1], theorems 2.1 and 2.2).

Theorem 2.1. *Let* R *be such that* $|B_R| > 1$. *Then*

a) *for each* $\Omega \geq 0$ *there exists* $E_\Omega \in \mathcal{E}$, $E_\Omega \subset B_R$ *such that*

$$\mathcal{F}_\Omega(E_\Omega) = \inf\left\{\mathcal{F}_\Omega(F),\ F \in \mathcal{E},\ F \subset B_R\right\} .$$

b) *As* Ω *goes to zero the sets* E_Ω *converge, in the* L^1 *topology, to the ball of measure* 1 *centered at* $(0,\dots,0)$.

The question is to prove that, for small Ω, we have $E_\Omega \subset\subset B_R$. To this aim we must improve the convergence in b).

Let R such that $|B_{R/2}| > 1$ be fixed. For brevity in the following we write $E = E_\Omega$. The first step will be, using theorem 2.1 b), to prove that, for small Ω, there exists t, $R/2 \leq t \leq 3R/4$, such that

$$(2.3)\qquad \int_{\partial B_t} \phi_E \, dH_{n-1} = 0 .$$

Then we improve this result by proving that actually holds

$$(2.4)\qquad \int_{B_R - B_t} \phi_E(x)\, dx = 0 ,$$

thus proving the existence of local minima for small Ω.

To prove (2.3) we use an iterative process analogous to the one used to prove 1 in [2] (in these proceedings). The basic tool is an isoperimetric-type inequality to replace (17) in [16].

For $R/2 \leq t_1 < t_2 < t_3 \leq 3R/4$, let

$$v_1 = |E \cap (B_{t_2} - B_{t_1})|$$

$$v_2 = |E \cap (B_{t_3} - B_{t_2})|$$

$$v = v_1 + v_2 \quad .$$

Moreover, assume the trace of E on ∂B_{t_i} is continuous for $i = 1,2,3$, and define

$$m = \max\left\{\int_{\partial B_{t_i}} \phi_E \, dH_{n-1} \ , \ i = 1,2,3\right\} \quad .$$

Lemma 2.1 (Isoperimetric-type inequality). *There are two constants* c_1 , c_2 *such that, if*

$$(2.5) \qquad v_1 + v_2 \leq \omega_n \left(\frac{(t_3 - t_1)}{4}\right)^n$$

$$(2.6) \qquad 4\, v\, R \leq \frac{t_3 - t_1}{2}$$

$$(2.7) \qquad |E \cap B_{R/2}| \geq \frac{1}{2}$$

then

$$(2.8) \qquad v_1 \wedge v_2 \leq c_1 (m + c_2 v)^N$$

where $N = \frac{n}{n-1}$, $c_1 = c_1(n)$, $c_2 = c_2(n,\Omega,R)$, while $\omega_n = |\{x \in \mathbb{R}^n : |x| \leq 1\}|$.

Sketch of the proof (for details, see [1]). The assumptions (2.5), (2.6) and (2.7) allows precisely to replace the set E by a set $F \in \mathcal{E}$, $F \subset B_R$, such that

$$F = \begin{cases} E \text{ in } B_R - B_{t_3} \\ \text{a ball of measure } v \text{ in } B_{t_3} - B_{\frac{t_1 + t_3}{2}} \\ \left(E \cap B_{t_1}\right)_T \text{ in } B_{\frac{t_1 + t_3}{2}} \end{cases}$$

where $T = (T_1, \ldots, T_n)$ is a suitable translation such that the barycenter of F lies in $(0,\ldots,0)$.

Then, by owing the minimality of E, we have

$$\mathcal{F}_\Omega(E) \leq \mathcal{F}_\Omega(F)$$

which implies, after some calculations (see [1]) the inequality (2.8).

Remark 4. Of course, the term c_2V in the isoperimetric-type inequality together with the necessity to prove (2.5), (2.6), (2.7) at each step of the iterative process make the proof more complicated relatively to the simpler situation considered in [2]. The new difficulties arise because the curvature term

$$-\Omega\int_E |y|^2 dx$$

and particularly because the condition on the barycenter.

Moreover, the method still works (see for details [1]), and we can prove (2.3).

We are now in position to prove the following:

Theorem 2.2 (Existence). *There exists* $\Omega_1 > 0$ *such that, for* $0 < \Omega < \Omega_1$, *there exists* t , $R/2 \leq t \leq 3R/4$ *with*

$$(2.9) \qquad \int_{B_R - B_t} \phi_E(x)\,dx = 0 \quad .$$

Proof. Choosing Ω_1 small enough, we can pick out $t \in (R/2, 3R/4)$ such that (2.3) holds. Put

$$G = E \cap B_t$$

$$v = |E - B_t|$$

and define

$$F = (\mu G)_T$$

where $\mu = \left(\frac{1}{1-v}\right)^{1/n}$ and therefore $|F| = 1$, and $T = (T_1,\ldots,T_n)$ is a suitable translation in order to preserve the barycenter constraint. After some calculations (for details, see [1]), we obtain

$$\mathcal{F}_\Omega(E) - \mathcal{F}_\Omega(F) \geq \left[1 - \left(\frac{1}{1-v}\right)^{\frac{n-1}{n}}\right]\int |D\phi_G| + \text{const.}v + n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}} =$$

$$= \left[- \frac{n-1}{n} v + 0(v^2)\right]\int |D\phi_G| + cv + n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}}$$

and this last quantity is strictly positive for $v > 0$ small. Since v goes to 0 as Ω goes to 0, it follows that it must $v = 0$ for small Ω, thus proving the theorem.

Remark 5. The method used for studying problem 1 and 2 was also used to consider the behaviour of the surface of equilibrium in the capillary tubes when gravity goes to zero (see [9], [10]).

Problem 3. For each $\varepsilon > 0$ let f_ε minimize the functional

$$\mathcal{F}_\varepsilon(f) = \int_A \sqrt{1 + |Df|^2} - \nu\int_{\partial A} f dH_{n-1} + \varepsilon\int_A f^2 dx \tag{2.10}$$

then

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = +\infty, \ \forall\, x \in \bar{A}, \tag{2.11}$$

that is the liquid rises at every point of Ω ; in other words when gravity goes to zero we do not obtain a limit surface. See also [19], [20].
In problem 3 we do not consider a volume constraint but a mean curvature term so the method is simplified.

Remark 6. As we already pointed out (see remark 1) no minimum does exist for problem 1 in the case of $0 < \alpha < 2$. Nevertheless it is possible to prove (see [4]) that, for small Ω, there exist "local minima" (see definition 1) for the functional (1.1).

Remark 7 (Regularity results). The iterative process used for the existence results of problems 1 and 2 can also be used to prove the regularity of the solutions. The ideas are the same as in [16] (see also [2] in these proceedings) but with some modifications due to the condition on the barycenter (problem 2), the presence of curvature terms (problems 1 and 2) and other geometrical conditions (problem 1).

For both problem 1 and problem 2 the regularity result is the following:

Theorem (Regularity). (See [1], [5]). *Let* E *be a set minimizing* $\mathcal{F}_\Omega$ *in* $\mathcal{E}$ *for problem 1 or let* E *be a local minimum for problem 2, then its boundary is an analytic* (n - 1)*-manifold, except possibly for a closed singular set* Σ *whose Hausdorff dimensions does not exceed* n - 8.

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