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Transverse Curvature of Foliated Manifolds

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Let M be a smooth manifold and let ∇ be a linear connection on M. A fundamental problem in differential geometry is to find relations between the curvature of ∇ and the topology of M. We consider the analogue of this fundamental problem for foliated manifolds and basic connections.

Let $(M,\mathfrak{F})$ be a foliated manifold. Let Q be the normal bundle of $\mathfrak{F}$ and let ∇ be a basic connection on Q. Our fundamental problem is then to study the relationship between the curvature of ∇ and the structure of the foliated manifold $(M,\mathfrak{F})$.

We recall a few basic concepts. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the subbundle tangent to the leaves of $\mathfrak{F}$. Let $Q = T(M)/E$ be the normal bundle. Let $\pi:T(M) \to Q$ be the natural projection and let $\chi(M)$, $\Gamma(E)$, $\Gamma(Q)$ denote the sections of $T(M)$, E, Q respectively. A connection $\nabla:\chi(M) \times \Gamma(Q) \to \Gamma(Q)$ is basic [3], transverse [9], adapted [7] if $\nabla_X Y = \pi([X,\tilde{Y}])$ for all $X \in \Gamma(E)$, $Y \in \Gamma(Q)$ where $\tilde{Y} \in \chi(M)$ satisfies $\pi(\tilde{Y}) = Y$. The parallel transport which ∇ induces along a curve lying in a leaf of $\mathfrak{F}$ coincides with the natural parallel transport along the leaves. Let $R:\chi(M) \times \chi(M) \times \Gamma(Q) \to \Gamma(Q)$, $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ be the curvature of ∇.

Question. <u>What influence does</u> R <u>exert on the structure of</u> $(M,\mathfrak{F})$?

We consider this question in the particular case where $\mathfrak{F}$ is Riemannian and ∇ is the unique torsion-free metric-preserving basic connection on Q.

Let M be a compact manifold and let $\mathfrak{F}$ be a codimension-q Riemannian foliation of M. There is a metric g on Q such that the natural parallel transport along a curve lying in a leaf of $\mathfrak{F}$ is an isometry. This is equivalent to the existence of a bundle-like metric in the sense of Reinhart [11].

Lemma [9]. <u>There is a unique metric-preserving basic connection</u> ∇ <u>on</u> Q <u>with zero torsion</u> $(T(X,Y) = \nabla_X \pi Y - \nabla_Y \pi X - \pi[X,Y] = 0$ <u>for all</u> $X,Y \in \chi(M))$.

<u>Remark.</u> ∇ is transversely projectable [9], basic [7] $(R(X,Y) = 0$ for all $X \in \Gamma(E)$, $Y \in \chi(M))$.

Let $p \in M$. Let π_p be a two-dimensional subspace of Q_p and let $\{X,Y\}$ be an orthonormal basis of π_p . The transverse sectional curvature of π_p is defined by $K(\pi_p) = -g_p(R(\tilde{X},\tilde{Y})X,Y)$ where $\tilde{X},\tilde{Y} \in T_p(M)$ satisfy $\pi(\tilde{X}) = X$, $\pi(\tilde{Y}) = Y$. Let $\tilde{M}$ be the universal cover of M and let $\tilde{\mathfrak{F}}$ be the lift of $\mathfrak{F}$ to $\tilde{M}$.

Theorem A [2]. <u>If</u> $\nabla R = 0$ <u>and</u> $K \leq 0$, <u>then</u> $\tilde{M}$ <u>is diffeomorphic to a product</u> $\tilde{L} \times R^q$ <u>where</u> $\tilde{L}$ <u>is the common universal cover of the leaves of</u> $\mathfrak{F}$ <u>and</u> $\tilde{\mathfrak{F}}$ <u>is the product foliation.</u>

<u>Application to Reeb's structure theorem</u> [10] <u>for codimension-one foliations defined by a closed one-form</u>: Let M be a compact manifold and let $\mathfrak{F}$ be a codimension-one foliation of M defined by a nonsingular closed one-form ω . Then $E = \text{kernel}\ (\omega)$. Let $\tilde{Y} \in \chi(M)$ be such that $\omega(\tilde{Y}) \equiv 1$. Then $Y = \pi(\tilde{Y}) \in \Gamma(Q)$. Define a metric g on Q by requiring $g(Y,Y) \equiv 1$. Define a connection ∇ on Q by requiring $\nabla_X Y = 0$ for all $X \in \chi(M)$.

Lemma. g <u>is parallel along the leaves of</u> $\mathfrak{F}$ <u>and</u> ∇ <u>is the unique torsion-free metric-preserving basic connection on</u> Q .

Proof: Let $X \in \Gamma(E)$. Then $0 = d\omega(X,\tilde{Y}) = X\omega(\tilde{Y}) - \tilde{Y}\omega(X) - \omega[X,\tilde{Y}]$ and so $[X,\tilde{Y}] \in \Gamma(E)$. Let $f \in C^\infty(M)$. Then $\nabla_X fY = f\nabla_X Y + (Xf)Y = (Xf)Y = \pi((Xf)\tilde{Y}) = \pi([X,f\tilde{Y}] - f[X,\tilde{Y}]) = \pi([X,f\tilde{Y}]) - f\pi([X,\tilde{Y}]) = \pi([X,f\tilde{Y}])$. Thus ∇ is basic. Clearly ∇ preserves g and so g is parallel along the leaves. Let $Z_1,Z_2 \in \chi(M)$. Then $T(Z_1,Z_2) = T(h\tilde{Y},k\tilde{Y})$ where $h,k \in C^\infty(M)$ and so $T(Z_1,Z_2) = hkT(\tilde{Y},\tilde{Y}) = 0$ proving the lemma.

Since Y is a nowhere zero parallel section, it follows that $R = 0$. Hence $\nabla R = 0$

and $K = 0$. Thus Theorem A implies that $\tilde{M} \cong \tilde{L} \times R$ and $\tilde{\mathfrak{F}}$ is the product foliation which is Reeb's result.

Remark. We may rephrase Theorem A in terms of foliages [8]: If the foliage $W = M/\mathfrak{F}$ admits a Riemannian structure with parallel curvature and non-positive sectional curvature, then W will have (in terms of foliages) a "covering" which will be a smooth manifold diffeomorphic to R^q.

We now consider the relationship between curvature and cohomology. The relevant cohomology theory here is base-like cohomology [11], [12]. A differential r-form ω on M is called base-like if on each coordinate neighborhood U with coordinates $(x^1,\ldots,x^k,y^1,\ldots,y^q)$ respecting the foliation $\mathfrak{F}$, the local expression of ω is of the form

$$\sum_{1\leq i_1<\ldots<i_r\leq q} a_{i_1\ldots i_r}(y^1,\ldots,y^q)dy^{i_1}\wedge\ldots\wedge dy^{i_r} .$$

Equivalently, $i_X\omega = i_X d\omega = 0$ for all $X \in \Gamma(E)$ [13]. Since d preserves such forms, we obtain the base-like cohomology algebra $H^*_{bas}(M) = \bigoplus_{r=0}^{q} H^r_{bas}(M)$.

Theorem B. <u>If</u> $\nabla R = 0$ <u>and</u> $K > 0$, <u>then</u> $H^*_{bas}(M)$ <u>is finite dimensional and</u> $H^1_{bas}(M) = 0$.

Remark. We may rephrase Theorem B in terms of foliages [8]. Let $W = M/\mathfrak{F}$ be the space of leaves (a foliage). We can think of $H^*_{bas}(M)$ as the "De Rham cohomology" of W, $H^*_{De\,R}(W)$. Of course, if W is a smooth manifold, this agrees with the De Rham cohomology algebra of W. In this terminology, Theorem B states: If W admits a Riemannian structure with parallel curvature and positive sectional curvature, then $H^*_{De\,R}(W)$ is finite dimensional and $H^1_{De\,R}(W) = 0$.

Example. Let G be a compact connected Lie group of dimension q and let $\mathfrak{g}$

be the Lie algebra of G. Let M be a compact manifold and suppose ω is a smooth $\mathfrak{g}$-valued one-form of rank q on M satisfying $d\omega + \frac{1}{2}[\omega,\omega] = 0$. Then ω defines a smooth codimension-q foliation $\mathfrak{F}$ on M which is a Lie foliation modeled on G [5]. Let $< \, , \, >$ be a bi-invariant Riemannian metric on G. Then $< \, , \, >$ induces a holonomy-invariant metric on Q with parallel curvature and $K \geq 0$. For example, if $G = S^1$ then $\mathfrak{F}$ is a codimension-one foliation defined by a nonsingular closed one-form. If $\pi_1(G)$ is finite (e.g., if G is semi-simple), then $H^*_{bas}(M) \cong H^*(G)$ [1].

Example. This example uses the suspension construction of Haefliger [6]. Define a left action of $\pi_1(S^1) = \mathbf{Z}$ on S^2 by

$$1 \longmapsto \begin{pmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha & 0 \\ -\sin 2\pi\alpha & \cos 2\pi\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

where $0 < \alpha < 1$ is irrational. Let $M = R \times_Z S^2$ be the associated bundle over S^1 with fiber S^2. The foliation of $R \times S^2$ whose leaves are the sets $R \times \{x\}$, $x \in S^2$ passes to a foliation $\mathfrak{F}$ of M. Since Z acts on S^2 by isometries, the normal bundle of $(M,\mathfrak{F})$ admits a transverse metric with $K \equiv 1$. There are exactly two compact leaves. If L is a non-compact leaf, then $\bar{L}$ is diffeomorphic to the two-dimensional torus and the foliation of $\bar{L}$ by the leaves of $\mathfrak{F}$ is Riemannian with $K \equiv 0$.

We now prove Theorem B. Since $\nabla R = 0$, we have that $N = \tilde{M}/\tilde{\mathfrak{F}}$ is a complete, Riemannian, Hausdorff manifold and the natural map $f:\tilde{M} \to N$ is a fiber bundle [2]. Each covering transformation σ of $\tilde{M}$ induces an isometry $\Psi(\sigma)$. We thus obtain a homomorphism $\Psi:\pi_1(M) \to I(N)$ such that $f \circ \sigma = \Psi(\sigma) \circ f$ for all $\sigma \in \pi_1(M)$ where $I(N)$ denotes the isometry group of N. Let $\Sigma = \text{image}\ (\Psi)$ and let $K = \bar{\Sigma} \subset I(N)$. Let $A^r_K(N)$ be the space of K-invariant r-forms on N and let $A^r_{bas}(M)$ be the space of base-like r-forms on M.

Lemma. <u>There is an isomorphism of cochain complexes</u>

$$\begin{array}{ccccc} \cdots & \xrightarrow{d} & A^r_{bas}(M) & \xrightarrow{d} & \cdots \\ & & \downarrow & & \\ \cdots & \xrightarrow{d} & A^r_K(N) & \xrightarrow{d} & \cdots \end{array} .$$

<u>Thus</u> $H^*_{bas}(M) \cong H^*_K(N)$.

Proof: Let $p:\tilde{M} \to M$ be the covering projection. Let $\omega \in A^r_{bas}(M)$. Then $p^*\omega = f^*\eta$ for a unique r-form η on N . Since $p^*\omega$ is $\pi_1(M)$-invariant, it follows that η is Σ-invariant and hence K-invariant. Conversely, let $\eta \in A^r_K(N)$. Then $f^*\eta \in A^r_{bas}(\tilde{M})$. Since η is Σ-invariant, it follows that $f^*\eta$ is $\pi_1(M)$-invariant and hence $f^*\eta = p^*\omega$ for a unique $\omega \in A^r_{bas}(M)$ proving the lemma.

Lemma. N <u>and</u> K <u>are compact</u>.

Proof: Let $\tilde{Q}$ be the normal bundle of $\tilde{\mathfrak{F}}$ and let $\tilde{g}$ be the lift of g to $\tilde{Q}$. The Riemannian metric on N is the one induced by $\tilde{g}$. Since $\nabla R = 0$, it follows that N has parallel curvature. Thus N is a complete, simply connected, Riemannian locally symmetric space and hence N is Riemannian symmetric. Since $K > 0$, it follows that N has positive sectional curvature. Thus N is compact [14] and K is compact proving the lemma.

Since K is compact, the inclusion $A^*_K(N) \to A^*(N)$ induces an injection $H^*_K(N) \to H^*(N)$ [4]. Since N is compact, $H^*(N)$ is finite dimensional and hence $H^*_{bas}(M)$ is finite dimensional. Since $\pi_1(N) = 0$, we have that $H^1_{bas}(M) = 0$.

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