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FOLIATIONS AND SEPARATION OF VARIABLES

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1. Introduction.

The method of separation of variables has been applied to most of the partial differential equations of mathematical physics to represent solutions as sums of products of eigenfunctions of ordinary differential operators. The classical technique developed gradually as the need to solve particular equations arose (see [4]). Recently some general theories have been proposed; see [2] and [3] for some of the most interesting results and for references to other work. However, as Koornwinder [1] has pointed out, even the recent work has sometimes suffered from a lack of rigor and frequently the basic concepts have not been clearly and explicitly defined.

The theory presented here has a somewhat different emphasis from any known to the writer. All of our considerations here are purely local, that is, we are concerned with linear partial differential operators defined on arbitrarily small neighborhood of a point in Euclidean space. Our basic concept is that of the splitting of such an operator by complementary foliations of dimensions n and m, where the dimension of the Euclidean space is $n+m$. Such a splitting leads to representations of solutions to the corresponding equation by sums of products of eigenfunctions of partial differential operators in n and m variables. Repeated splittings then lead to separations in the usual sense. Usually two separations have been regarded as equivalent if they correspond by the operation of an element of some "obvious" group that leaves the operator or the corresponding equation invariant; however, equivalence of this type will not be emphasized here. Instead, our concept of equivalence will amount to regarding separations as equivalent if their separating variables have the same level surfaces (cf. [4, p.504]).

The main theorem, which is stated and proved in section 2, gives necessary and sufficient conditions for a pair of complementary foliations to split an operator. In section 3, some examples are worked out to show how the theory can be used in specific cases to determine all splittings of an operator. In section 4 some further implications of the technique will be discussed.

2. Splitting an operator.

Let $A_1,\dots,A_n,B_1,\dots,B_m$ be a set of (pointwise) linearly independent 1-forms defined in a neighborhood of 0 in E^{n+m}. Let U denote the ideal in the exterior algebra generated by the A_i's and V the ideal generated by the B_j's. Suppose that $dU \subset U$ and $dV \subset V$ so that U and V define complementary foliations of the neighborhood. The notation $F_U(F_V)$ will be used to denote the smooth functions that are constant on the leaves of the U-foliation (V-foliation), and elements of F_U, for example, will be denoted by u or u_i. Thus if u is in F_U, du is in U. The complementary foliations determine maps P_U and P_V from the elements of degree one in the exterior algebra to U and V in an obvious way.

Let $z = (z_1,\dots,z_{n+m})$ be local coordinates in the neighborhood and let partial derivatives of functions be denoted in the usual way by multi-indices. Let $Lf = \sum_{0=|a|\leq k} c(z)D^a f$ be a homogeneous linear partial differential operator. The pair U,V will be said to split L if there is a positive function $R(z)$ and there are partial differential operators $M:F_U \to F_U$ and $N:F_V \to F_V$ such that for every u in F_U and v in F_V

$$L(uv) = R(vM(u)+uN(v)) \,. \tag{2.1}$$

It will be assumed from now on that L satisfies the following non-degeneracy conditions : (i) either $L(1) \equiv 0$ or $L(1)$ never vanishes, (ii) for some u in F_U, $L_0(u) = L(u)-uL(1)$ does not vanish, and (iii) for some v in F_V, $L_0(v)$ does not vanish. The notation $M_0(u) = M(u)-uM(1)$ and $N_0(v) = N(v)-vN(1)$ will be used below.

Now some necessary conditions for U,V to split L will be derived. First note that $F_U \cap F_V$ consists of the constant functions, hence (2.1) implies

$$L(u) = R(M(u)+uN(1)) \tag{2.2}$$

and

$$L(v) = R(vM(1)+N(v)) \,. \tag{2.3}$$

In particular,

$$L(1) = R(M(1)+N(1)) \,. \tag{2.4}$$

Subtracting the sum of (2.2) multiplied by v and (2.3) multiplied by u from the sum of (2.1) and (2.4) multiplied by uv gives

$$(2.5)\qquad L(uv)-vL(u)-uL(v)+uvL(1) = 0$$

as a necessary condition for splitting. Eliminating $N(1)$ from (2.2) and (2.4) gives

$$(2.6)\qquad L_0(u) = RM_0(u)$$

and similarly

$$(2.7)\qquad L_0(v) = RN_0(v) .$$

Applying the exterior derivative to (2.6) and reducing modulo the ideal U gives

$$(2.8)\qquad d(L_0(u)) = M_0(u)dR \pmod U$$

and similarly

$$(2.9)\qquad d(L_0(v)) = N_0(v)dR \pmod V .$$

If u_1 and u_2 are any elements of F_U, eliminating R from (2.6) and (2.8) gives

$$(2.10)\qquad L_0(u_1)d(L_0(u_2)) = L_0(u_2)d(L_0(u_1)) \pmod U$$

and similarly

$$(2.11)\qquad L_0(v_1)d(L_0(v_2)) = L_0(v_2)d(L_0(v_1)) \pmod V .$$

In case $L_0(u)$ does not vanish for some u in F_U, (2.10) can be written as

$$(2.12)\qquad P_V d(\mathrm{Log}|L_0(u)|) \text{ is independent of } u \text{ for } u \text{ in } F_U .$$

Similarly

$$(2.13)\qquad P_U d(\mathrm{Log}|L_0(v)|) \text{ is independent of } v \text{ for } v \text{ in } F_V .$$

Then (2.6), (2.7), (2.8), and (2.9) give

$$(2.14)\qquad d(\mathrm{Log}\, R) = P_V d(\mathrm{Log}|L_0(u)|)+P_U d(\mathrm{Log}|L_0(v)|) .$$

The right side of (2.14) must be an exact 1-form, that is

(2.15) $$dP_V d(\mathrm{Log}|L_0(u)|)+dP_U d(\mathrm{Log}|L_0(v)|) = 0 .$$

If $L(1) \neq 0$, (2.4) implies that $M(1)+N(1) = R^{-1}L(1)$. Since $dM(1)$ must be in U and $dN(1)$ in V , $d^2M(1) = dP_U d(R^{-1}L(1)) = 0$. Using (2.14) one sees that $P_U d(R^{-1}L(1)) = R^{-1}L(1)(P_U d(\mathrm{Log}|L(1)|)-P_U d(\mathrm{Log}|L_0(v)|)$ and $P_V d(R^{-1}L(1)) = R^{-1}L(1)(P_V d(\mathrm{Log}|L(1)|)-P_V d(\mathrm{Log}|L_0(u)|)$, hence $d^2(M(1)) = 0$ is equivalent to

(2.16) $$P_V d(\mathrm{Log}|L(1)|-\mathrm{Log}|L_0(u)|)\wedge P_U d(\mathrm{Log}|L(1)|-\mathrm{Log}|L_0(v)|)+ dP_U d(\mathrm{Log}|L(1)|-\mathrm{Log}|L_0(v)|) = 0 .$$

The conditions (2.5), (2.10), (2.11), (2.15), and (2.16), which do not involve the unknown function R or the unknown operators M and N , are necessary conditions for U,V to split. But these conditions are also sufficient :

THEOREM. Let L be a linear partial differential operator defined in a simply connected neighborhood of a point of E^{m+n} and suppose that complementary foliations are determined by ideals U and V in the exterior algebra. Assume the nondegeneracy conditions (i), (ii), (iii). Then U,V splits L if and only if (2.5), (2.10), (2.11), (2.15), and (2.16) hold. The function R of (2.1) is determined up to a positive constant multiple. Then when R has been given, M is determined up to an additive constant and R and M together completely determine N.

Proof. The only if part has already been proved. If (2.15), (2.10), and (2.11) (hence (2.12) and (2.12)) hold, R is obtained up to a positive multiple by integrating (2.14) because (2.15) is the integrability condition for (2.14). Assuming (2.14) and (2.16), M(1) can be obtained up to an additive constant by integrating $dM(1) = P_U d(R^{-1}L(1))$ and then N(1) is defined by $N(1) = R^{-1}L(1)-M(1)$. For any u in F_U, M(u) can now be obtained by solving (2.6). Now to see that $M_0(u)$ is in F_U , hence M(u) is in F_U , let u' be any element of F_U such that $L_0(u') \neq 0$.

Taking the exterior derivative of (2.6), dividing by R, and using (2.10) with $u = u_1$ and $u' = u_2$ gives

$$d(M_0(u))+M_0(u)d(\text{Log } R) = R^{-1}d(L_0(u)) = R^{-1}L_0(u)d(\text{Log}|L_0(u')|) \pmod{U}.$$

By (2.14) and (2.6) this implies

$$d(M_0(u))+M_0(u)(P_V d(\text{Log}|L_0(u')|)) = M_0(u)d(\text{Log}|L_0(u')|) \pmod{U}.$$

The desired result, $P_V M_0(u) = 0$, follows by applying P_V to both sides and noting that $P_V = P_V^2$. Similarly one defines $N_0(v)$ by (2.7) and shows that $N_0(v)$ and $N(v)$ are in F_V.

Finally to see that (2.1) is satisfied note that by the definition of L_0, (2.5) can be written $L(uv) = vL_0(u)+uL_0(v)+uvL(1)$. Substituting (2.6), (2.7), and the relation $L(1) = R(M(1)+N(1))$, which was used above to define $N(1)$, gives (2.1). This completes the proof.

3. Examples.

These examples will show how the theorem can be used to determine all splittings of an operator. Let $z = (x_1,\ldots,x_m, y_1,\ldots,y_n)$ and write $A_i = dy_i - \sum_{j=1}^{m} p_i^j dx_j$, $B_j = dx_j - \sum_{i=1}^{n} q_j^i dy_i$. Thus A_i, B_j are given a standard presentation that depends only on the choice of coordinates at $z = 0$. Of course if n or m is greater than one, the p_i^j's or q_j^i's must satisfy certain integrability conditions to insure that $dU \subset U$ or $dV \subset V$.

Example 1. $n = m = 1$, $A_1 = dy-pdx$, $B_1 = dx-qdy$, and $Lf = f_{yyx}$. If u is in F_U , $du \wedge A_1 = 0$. Hence

$$(3.1) \qquad u_x = -pu_y .$$

Similarly if v is in F_V ,

$$(3.2) \qquad v_y = -qv_x .$$

Relations among the higher derivatives of u , for example, can be obtained by differentiating (3.1). In fact, any derivative involving x can be expressed in terms of y derivatives alone. Thus $u_{xx} = p^2u_{yy} + (pp_y-p_x)u_y$. The relation (2.5) reduces to

$$u_{yy}v_x + u_xv_{yy} + 2u_{yx}v_y + 2u_yv_{xy} = 0 ,$$

which upon substituting the results of differentiating (3.1) and (3.2) becomes

$$(3.3) \qquad u_{yy}v_x(1+2pq) + u_yv_{xx}(-pq^2-2q) + u_yv_x(-pqq_x+pq_y+2p_yq-2q_x) = 0 .$$

But the part of the jet of u at $z = 0$ involving y-derivatives can be chosen arbitrarily as can the part of the v jet involving the x derivatives . It follows that the coefficients of $u_{yy}v_x$, u_yv_{xx} , and u_yv_x must vanish. In particular, $1+2pq = pq^2+2q = 0$. But since these equations are inconsistent, L admits no splitting and separation of variables cannot be applied to $f_{yyx} = 0$.

Example 2. Let n,m,A_1,B_1 be as in example 1 and take $Lf = f_{yy}-f_x$, the heat operator. In this case (2.5) is $2u_yv_y = -2u_yv_xq = 0$ and therefore $q = 0$. It is then easily verified that $L_o(v) = -v_x$, and $P_UdL_o(v) = 0$, so (2.11) is satisfied. Also $L_o(u) = u_{yy}-u_x = u_{yy}+pu_y$ and a short calculation shows that (2.10) can only hold for all u_1,u_2 in P_U if

$$(3.4) \qquad 2pp_y = p_{yy}-p_x .$$

If (3.4) does hold, then $P_Vd \operatorname{Log} L_o(u) = -2p_ydx$ and $dP_Vd \operatorname{Log} L_o(u) = 2p_{yy}dx \wedge dy$. Then (2.14) and (3.4) give

(3.5) $$p_{yy} = 2pp_y+p_x = 0 \quad .$$

The general solution of (3.5) is seen by elementary integrations to be $p = (y+b)/(2x+a)$, or $p = \text{const.}$ The level curves of the U foliation are then obtained by integrating $dy = pdx$.

4. Remarks. The main interest in separation of variables stems from the the observation that if $u \in F_U$ and $v \in F_V$ are solutions to the eigenvalue problems $M(u) = -\lambda u$ and $N(v) = \lambda v$, then $f = uv$ satisfies $L(f) = 0$. Thus the solution of $L(f) = 0$ is reduced a pair of eigenvalue problems, each of which involves fewer independent variables. If the splitting process can be continued far enough one eventually arrives at eigenvalue problems for ordinary differential operators.

Sometimes it is desirable to find a splitting of a family of operators $L^\lambda(f) = L(f)+\lambda f$ that works for all values of the real parameter λ . Note, for example, that if $L(f) = f_{xx}+f_{yy}$, the level curves determined by the real and imaginary parts of an holomorphic function h with $h'(z) \neq 0$, determine a splitting of $L(f)$ but the splittings of $L^\lambda(f)$ for $\lambda \neq 0$ are much more restricted. To find conditions for simultaneous splitting note that the addition of the term λf does not affect (2.5), (2.10), (2.11), or (2.15); however (2.16) could hold for some value of λ but not for others. But if (2.16) holds for at least two values of λ , it holds for all values. In fact the derivation of (2.16) shows if it holds for two different values of λ , $d\,P_U d\,R^{-1} \equiv 0$ and conversely this condition implies (2.16) for all values of λ . Using (2.14) it is clear that $d\,P_U\,dR^{-1} = 0$ is equivalent to

(4.1) $$dP_V \operatorname{Log}|L_o(u)| \wedge dP_U \operatorname{Log}|L_o(v)| = dP_U d \operatorname{Log}|L_o(v)| \quad .$$

These observations are summarized in following supplement to the theorem :

Supplement. *Assume the conditions of the theorem and set* $L^\lambda(f) = L(f)+\lambda f$. *Then* (U,V) *splits* L^λ *for all* λ *if and only if* (2.5), (2.10), (2.11), (2.15), (2.16), *and* (4.1) *hold.*

Another easy remark that the reader can verify is that if U,V splits L , then L is formally symmetric if and only if M and N are formally symmetric.

Finally we note that some of the ideas employed here can be applied to nonlinear operators. This line of investigation will be continued in a subsequent paper.

References.

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