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SOME REMARKS ON THE STRUCTURE OF THE LIE ALGEBRA OF FORMAL VECTOR FIELDS

by Bruce L. Reinhart

A formal vector field in $\mathbb{R}^m$ is an expression

$$\sum_{i=1}^{m} A^i(x^1,..,x^m)\partial/\partial x^i$$

where $A^i(x^1,..,x^m)$ is a formal power series in the variables $(x^1,...,x^m)$. The Lie bracket which is defined for polynomial vector fields extends immediately to the set $\mathfrak{a}_m^\infty$ of formal vector fields, since the terms of degree k in the product depend only on terms of degree $\leq k+1$ in the factors. The cohomology of $\mathfrak{a}_m^\infty$ with real coefficients gives rise to characteristic classes for foliations of codimension m with trivial normal bundle, essentially because $\mathfrak{a}_m^\infty$ is in some sense the Lie algebra for the topological groupoid Γ_m of germs of diffeomorphisms of $\mathbb{R}^m$. Thus, for the study of foliations, it is useful to understand the structure of this algebra. Certain extensions occur in the inductive construction of $\mathfrak{a}_m^\infty$ from Lie algebras of truncated polynomial vector fields. In this paper, the cohomology groups which classify these extensions are studied.

Let $\mathfrak{gp}_m^\infty$ be the subalgebra of $\mathfrak{a}_m^\infty$ consisting of formal vector fields with constant term 0. This algebra in some sense corresponds to a group, the group of ∞-jets of diffeomorphisms of $\mathbb{R}^m$ that leave the origin fixed. $\mathfrak{gp}_m^\infty$ has finite dimensional quotient algebras $\mathfrak{gp}_m^k$, whose coefficients are polynomials of degree $\leq k$. The corresponding Lie group GP_m^k consists of k-jets of origin-fixing diffeomorphisms of $\mathbb{R}^m$. (It is important to note that in all calculations for finite k, terms of degree $>k$ are discarded.) For every $k \geq 1$, there are exact sequences of Lie algebras

$$0 \to \mathfrak{n}_{m,k}^{k+1} \to \mathfrak{gp}_m^{k+1} \to \mathfrak{gp}_m^k \to 0,$$

where $\mathfrak{n}_{m,k}^{k+1}$ is defined by this sequence.

Lemma. $\mathfrak{n}_{m,k}^{k+1}$ <u>is an abelian Lie algebra, and the action of</u> $\mathfrak{gp}_m^k$ <u>in</u> $\mathfrak{n}_{m,k}^{k+1}$ <u>depends only on</u> $\mathfrak{gp}_m^1 = \mathfrak{gl}_m$. <u>The sequence splits for</u> $k = 1$ <u>and also for</u> $m = 1$, $k = 2$.

Proof. Since the coefficients in $\mathfrak{n}^{k+1}_{m,k}$ are homogeneous polynomials of degree $k + 1$, all the brackets are killed by truncation. In fact, all brackets involving terms of degree $k + 1$ and of degree $\ell \geq 2$ are killed by truncation, which proves the second part of the statement. The splittings mentioned arise from injections because in these cases, no truncation actually occurs in calculating the bracket.

For $k = 1$, the corresponding splittings of Lie group sequences also arise from the obvious injection of the general linear group into the higher order jet groups. For the case $m = 1$, $k = 2$ the splitting takes the polynomial mapping $a_1x + a_2x^2$ in the variable x to the mapping $a_1x + a_2x^2 + \frac{a_2^2}{a_1}x^3 + \frac{a_2^3}{a_1^2}x^4 + \cdots$, truncated wherever one pleases or not at all.

Theorem. <u>None of the above exact sequences except those mentioned in the lemma split</u>. <u>Also</u>

$$\dim H^2\left(\mathfrak{gp}^k_1 ; \mathfrak{n}^{k+1}_{1,k}\right) = \begin{cases} 0 & k = 1,2 \\ 2 & k = 5,7 \\ 1 & \text{otherwise} \end{cases}$$

Note that the cohomology group mentioned in the theorem classifies extensions modulo split extensions. This theorem will be proved later in the paper, and explicit cocycles representing generators and representing the canonical extension will be given.

This calculation for the case $m = 1$ may have a more general meaning than appears at first, because of the following proposition.

Proposition. <u>Let</u> $m \geq 1$ <u>and</u>

$$\rho(x) = \sum_{i=1}^{m} x^i\, \partial/\partial x^i .$$

<u>Then</u> $\mathfrak{a}^{\infty}_m$ <u>is the vector space direct sum of the subalgebra consisting of volume preserving vector fields and the subalgebra consisting of multiples of</u> $\rho(x)$. <u>If</u> $m = 1$, <u>the latter subalgebra is</u> $\mathfrak{gp}^{\infty}_1$. <u>The projection of a homogeneous polynomial vector field of degree</u> $k \geq 1$ <u>onto the second subalgebra is given by</u>

$$P\left(\sum_{i=1}^{m} A^i\, \partial/\partial x^i\right) = \frac{1}{m+k-1}\left(\sum_{i=1}^{m} \frac{\partial A^i}{\partial x^i}\right)\rho(x) .$$

Proof. By direct calculation, $P^2 = P$, P is the identity on any multiple of $\rho(x)$, and the kernel of P is the divergence free vector fields. Thus P gives rise to a direct sum splitting of the homogeneous fields of each positive degree. (The operator P has been used by Vagner [7] and the splitting of the homogeneous fields has also been studied by Terng [6].) The divergence free fields, including all the fields of degree 0, form a subalgebra since there is a purely formal calculation which establishes this fact. If p is homogeneous of degree k and q is homogeneous of degree ℓ, then

$$[p(x)\rho(x), q(x)\rho(x)] \quad = \quad (\ell - k)p(x)q(x)\rho(x).$$

Note that the coefficients in the last formula depend only on the degree of the polynomials, so that the structure of the subalgebra generated by $\rho(x)$ reflects very strongly the structure of $\mathfrak{gp}_1^\infty$. Thus, it appears that the relations among $\mathfrak{a}_m^\infty$, the divergence-free fields, and $\mathfrak{gp}_1^\infty$ are worthy of further study. Another open question is the meaning of the nonstandard extensions that occur for $m = 1$ and $k = 5,7$. Since elements of $\mathfrak{a}_1^\infty$ give rise to symplectic vector fields in $\mathbb{R}^2$, one may ask whether these extensions are related to the strange behaviour of the 4-jets of the symplectic algebra in $\mathbb{R}^2$ (Gelfand, Kalinin, and Fuks [4]). Another reason for studying these questions now is that there are beginning to be general theorems about Lie groups, such as GP_m^k, which are not either semi-simple or solvable (such as Dani [2,3], Brezin and Moore [1], Moore [5]).

Proof of the theorem. The proof consists of calculations with cochains, using the basis consisting of the formal vector fields

$$x^{i_1} \ldots x^{i_r}\, \partial/\partial x^i .$$

For $\mathfrak{gp}_m^k$, $1 \le r \le k$ and for $\mathfrak{n}_{m,k}^{k+1}$, $r = k + 1$. If $m > 1$ and $k > 1$, then the cocycle representing the canonical extension is easily seen not to be a coboundary. For $m = 1$, let f be a cochain and let

$$f(a,b) \quad = \quad f(x^a d/dx, x^b d/dx).$$

Then any cochain can be modified by a coboundary to obtain $f(1,b) = 0$ for $b \ge 2$. Furthermore, no nonzero cocycle which satisfies this condition can be a coboundary. Thus, it remains to determine the conditions for a cochain satisfying this condition to be a cocycle. By considering $\delta f(1,b,c)$, one sees that a cocycle must satisfy

$$f(a,b) = 0 \quad \text{for} \quad a + b \neq k + 2.$$

Thus $H^2 = 0$ for $k = 1,2$, while for $k = 3$ (respectively 4) the value of $f(2,3)$ (respectively $f(2,4)$) is arbitrary. For $k = 5$, the values of $f(2,5)$ and $f(3,4)$ are arbitrary. For the canonical extension, the values are $f(2,5) = 3$ and $f(3,4) = 1$. For $k = 6$, the values of $f(2,6)$ and $f(3,5)$ are subject to the condition that $-2f(3,5) + f(2,6) = 0$, while the canonical extension is given by $f(2,6) = 4$ and $f(3,5) = 2$. For $k = 7$, the values of $f(2,7)$, $f(3,6)$, and $f(4,5)$ are subject to the one condition

$$-f(4,5) - 3f(3,6) + 2f(2,7) = 0$$

while the canonical extension is given by $f(2,7) = 5$, $f(3,6) = 3$, and $f(4,5) = 1$. For $k \geq 8$, the number of conditions for a cochain to be a cocycle increases very rapidly, and it is always possible to find enough so that the only cocycles are multiples of the canonical cocycle, which is given by $f(a,b) = b - a$ provided $a + b = k + 2$.

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