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The chains on the loops and 4-dimensional homotopy types

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In this lecture I want to describe some results on a problem of J.H.C. Whitehead:

> Classify algebraically the homotopy types of 4 - dimensional polyhedra!

In 1948 J.H.C. Whitehead solved this problem for simply connected 4 - dimensional polyhedra; he first used the 'enriched cohomology ring' [7] and later the 'certain exact sequence' [9] as the classifying invariants which determine the homotopy type. On the other hand he showed in [8] that two 3 - dimensional polyhedra are homotopy equivalent iff the cellular chain complexes of their universal coverings are homotopy equivalent. These results rely on the following facts (a) and (b) respectively:

(a) Each simply connected 4 - dimensional CW - complex X is homotopy equivalent to the mapping cone C_f of a map f between 1 - point unions of 2 - dimensional and 3 - dimensional spheres,

$$f: \bigvee_A S^3 \vee \bigvee_B S^2 \longrightarrow \bigvee_C S^3 \vee \bigvee_D S^2$$

Here A,B,C and D denote index sets.

(b) Each chain map between chain complexes of universal coverings is geometrically realizable up to dimension 3.

For 4 - dimensional polyhedra which are not simply connected there is no analogue of (a) and we cannot use (b). Therefore the solution of the general problem has to be different from the solutions in the special cases described by J.H.C. Whitehead.

We are not yet able to solve the problem completely but we describe a result which solves the problem 'up to the prime 2'. Moreover, we solve the problem if the second homotopy group π_2 satisfies the condition that multiplication by 2 is an isomorphism or that $\pi_2 = \mathbb{Z}$ is the group of integers.

Our classifying invariant is the chain algebra given by the chains on the loop space. In fact, for all connected polyhedra this chain algebra determines the chains of the universal covering, see (24). Therefore this invariant is more powerful than the chains of the universal covering used by Whitehead.

Also we describe a small model of this chain algebra, which extends the result of Adams and Hilton [1] to the non simply connected case.

The results described in this lecture are just a few items in my forhtcoming book 'On the homotopy classification problem'. This book will contain all proofs and the explicit constructions.

We say a CW - complex X is reduced if the 0 - skeleton of X consists of a single point, which is the base point of X. All maps and homotopies which we consider are basepoint preserving. Let $\mathcal{CW}$ be the category of reduced CW - complexes and of cellular maps. Its homotopy category $\mathcal{CW}/\simeq$ is equivalent to the homotopy category of all path connected CW - spaces. Let

(1) $$\hat{C}_*: CW \longrightarrow \mathit{Chain}^{\wedge}$$

be the functor of <u>cellular chains on the universal covering</u>. Here $\mathit{Chain}^{\wedge} = \mathit{Chain}^{\wedge}_{\mathbb{Z}}$ is the following category. Objects are pairs (π, C) where π is a group and where C is a positive chain complex of free (right) $\mathbb{Z}[\pi]$-modules with $C_0 = \mathbb{Z}[\pi]$. A morphism in $\mathit{Chain}^{\wedge}$ is a pair (φ, f): $(\pi, C) \longrightarrow (\pi', C')$ where $\varphi: \pi \longrightarrow \pi'$ is a homomorphism between groups and where f is a φ-equivariant chain map, that is $df = fd$ and $f(x \cdot \alpha) = (fx) \cdot (\varphi\alpha)$ for $x \in C$, $\alpha \in \pi$; $f_0 = \mathbb{Z}[\varphi]$. The functor in (1) is given by

$$\hat{C}_*(X) = (\pi_1 X, C_* \hat{X})$$

where $\hat{X}$ is the universal covering of X in which we fixed a basepoint $*$. Thus for each map $f: X \longrightarrow Y$ in CW there is a unique basepoint preserving covering map $\hat{f}: \hat{X} \longrightarrow \hat{Y}$. This map is cellular and induces a $\pi_1(f)$-equivariant chain map, $\hat{f}_*: C_*\hat{X} \longrightarrow C_*\hat{Y}$, between the cellular chain complexes. The functor $\hat{C}_*$ in (1) carries the map f to the pair $(\pi_1(f), \hat{f}_*)$. Two morphisms (φ, f), (φ', f') in $\mathit{Chain}^{\wedge}$ are homotopic if $\varphi = \varphi'$ and if there is a φ-equivariant map $\alpha: C \longrightarrow C'$ of degree $+1$ with $\alpha_0 = 0$ and $d\alpha + \alpha d = f' - f$. The functor (1) induces the functor

(2) $$\hat{C}_*: CW/\simeq \longrightarrow \mathit{Chain}^{\wedge}/\simeq$$

between homotopy categories. As a variant of the Whitehead theorem we have

(3) A map f in CW is a homotopy equivalence if and only if $\hat{C}_* f$ is a homotopy equivalence in $\mathit{Chain}^{\wedge}$.

Moreover, for the full subcategory CW^n of n-dimensional complexes in CW Whitehead proves in [8]:

(4) On $CW^3/\simeq$ the functor $\hat{C}_*$ is full and on $CW^2/\simeq$ the functor $\hat{C}_*$ is full and faithful.

We derive from (3) and (4)

(5) Two 3-dimensional complexes X, Y in CW are homotopy equivalent if and only if their chain complexes $\hat{C}_*X$, $\hat{C}_*Y$ are homotopy equivalent in $Chain^\wedge$.

Such a result is not true for 4-dimensional complexes, but we get

(6) <u>Theorem</u>: *Let X, Y be 4-dimensional complexes in CW and let*

$$F = (\varphi, f) : \hat{C}_*X \longrightarrow \hat{C}_*Y$$

be a map in $Chain^\wedge$. Then we associate with the triple (X, Y, F) an obstruction element

$$O_{X,Y}(F) \in \hat{H}^4(X, \varphi^*\Gamma(\pi_2 Y))$$

with the following property:
The homotopy class of F in $Chain^\wedge/\simeq$ is realizable by a map in CW if and only if $O_{X,Y}(F) = 0$.

Here $\hat{H}^4$ denotes the cohomology with local coefficients and Γ is the quadratic construction of Whitehead [9]; Γ is a functor so that $\Gamma(\pi_2 Y)$ is a $\pi_1(Y)$-module which via φ^* is considered as a $\pi_1(X)$-module.

If X and Y are simply connected we can derive from the theorem the result of Whitehead in [7]. In the general case we get:

(7) <u>Corollary</u>: *Two 4-dimensional complexes X, Y in $\mathbf{CW}$ are homotopy equivalent if and only if there is a homotopy equivalence*

$$F = (\varphi, f) : \hat{C}_* X \simeq \hat{C}_* Y$$

in $\mathrm{Chain}^{\wedge}$ with $O_{X,Y}(F) = 0$.

Thus we have to compute the obstruction $O_{X,Y}(F)$ in terms of invariants of X and Y respectively. This is not yet completely done. We know however, how to compute the image of this obstruction under the following homomorphism:

(8) $$\tau_* : \hat{H}^4(X, \varphi^* \Gamma(\pi_2 Y)) \longrightarrow \hat{H}^4(X, \varphi^*(\pi_2 Y \otimes \pi_2 Y)) .$$

Here $\tau : \Gamma(A) \longrightarrow A \otimes A$ is the canonical homomorphism associated to the quadratic map $q : A \longrightarrow A \otimes A$, $q(a) = a \otimes a$. Clearly, if $A = \pi_2 Y$ is a $\pi_1 Y$-module then τ is a homomorphism of $\pi_1(Y)$-modules. It is well known that τ is an isomorphism if $A = \mathbb{Z}$ and that τ admits a natural retraction if multiplication by 2 is an isomorphism on A. Thus in these cases τ_* in (8) is injective and therefore the image $\tau_* O_{X,Y}(F)$ vanishes iff $O_{X,Y}(F)$ vanishes. We can describe the element $\tau_* O_{X,Y}(F)$ in terms of F and the chain algebras $C_* \Omega X$ and $C_* \Omega Y$ which are given by the cubical chains on the loop spaces ΩX and ΩY respectively. For the computation of $\tau_* O_{X,Y}(F)$ we only need to know small models of these chain algebras.

To this end we describe the connection between the chain algebra $C_* \Omega X$

and the chain complex $\hat{C}_* X$.

Let R be a subring of the rationals $\mathbb{Q}$. We introduce the following diagram of categories and functors:

(9)
$$\begin{array}{ccc} \mathcal{CW} & \xrightarrow{C_*\Omega\otimes R} & \mathcal{DA}_R \\ \Big\downarrow{\scriptstyle \hat{C}_*\otimes R} & & \Big\downarrow{\scriptstyle \hat{B}} \\ \mathit{Chain}^{\wedge}_R & \xrightarrow{\hat{\alpha}} & \mathcal{DF}\hat{\mathcal{M}}_R \end{array}$$

The functor $\hat{C}_*$ was defined in (1); if in the definition of $\mathit{Chain}^{\wedge}$ we replace the ring Z by the subring R of $\mathbb{Q}$ we obtain the category Chain_R, and the functor $\hat{C}_* \otimes R$.

The subcategory $\mathcal{DA}_R$ of chain algebras is defined as follows : A chain algebra A, over R, is a graded, associative algebra A, over R, with unit together with a differential $d : A \to A$ of degree -1 and an augmentation $\varepsilon : A \to R$ such that

(a) A , as a module, is a free R-module,

(b) (A,d) is a chain complex with $A_i = 0$ for $i < 0$,

(c) ε is an algebra homomorphism and a chain map,

(d) the multiplication $\mu: A\otimes A \longrightarrow A$ is a chain map.

We say A is <u>good</u> if

(e) the homology in degree 0 , $H_0 A$, is free as an R-module.

Let $\mathcal{DA}_R$ be the <u>category of good chain algebras</u>. The maps are of degree zero and preserve μ, d and ε.

Clearly, $A = C_*\Omega X \otimes R$ is a chain algebra in $\mathcal{DA}_R$. Here ΩX is the Moore loop space which has an associative multiplication. The homology in degree 0 is

(10) $$H_0(C_*\Omega X \otimes R) = R[\pi_1 X] .$$

For a chain algebra we have the canonical projection

(11) $$\lambda\colon A \longrightarrow H_0 A = H$$

which is trivial in degree ≥ 1. Via λ the algebra $H = H_0 A$ is an A-module. Let BA be the reduced bar construction and let $\tau\colon BA \longrightarrow A$ be the canonical twisting cochain. Then via (11) the <u>two-sided bar construction</u>

(12) $$\hat{B}A = H \otimes_\tau BA \otimes_\tau H$$

is defined, see [3], [4]. This is an object in the following category $\mathcal{DF}\hat{\mathcal{M}}_R$. Objects of $\mathcal{DF}\hat{\mathcal{M}}_R$ are pairs $(H, \hat{C})$ where H is an augmented (non graded) algebra with unit which is free as an R-module and where $\hat{C}$ is a chain complex of H-bimodules with the following properties:

(a) $\hat{C}_n$ is a free H-bimodule and the differential d is a map of H-bimodules, $n \in \mathbb{Z}$,

(b) $\hat{C}_0 = H \otimes H$, $\hat{C}_n = 0$ for $n < 0$, A map $\sigma = (\varphi, \sigma): (H, \hat{C}) \longrightarrow (H', \hat{C}')$ is a homomorphism $\varphi: H \longrightarrow H'$ between augmented (non graded) algebras together with a φ-biequivariant chain map $\sigma: \hat{C} \longrightarrow \hat{C}'$ with $\sigma_o = \varphi \otimes \varphi$ in degree 0.

The functor $\hat{\alpha}$ in (9) carries the object (π, C) to $(H, \hat{C})$ with

(13) $$H = R[\pi] \quad \text{and} \quad \hat{C} = C \otimes_H \bar{\xi}^*(H \otimes H^{op}) \quad .$$

We identify an H-bimodule with a right $H \otimes H^{op}$-module. Here H^{op} is the opposite algebra and $H \otimes H^{op}$ is the enveloping algebra, see [2]. If $H = R[\pi]$ we have the canonical homomorphism

$$\xi: H \longrightarrow H^{op} \; , \; \xi[\alpha] = [\alpha^{-1}]$$

which yields

(14) $$\bar{\xi}: H \longrightarrow H \otimes H^{op} \; , \; \bar{\xi}(x) = x \otimes \xi x \; .$$

Via $\bar{\xi}$ the algebra $H \otimes H^{op}$ is a left H-module which is denoted by $\bar{\xi}^*(H \otimes H^{op})$. The functor $\hat{\alpha}$ is defined on maps in the obvious way.

Now all categories and functors in diagram (9) are defined. We introduce homotopy categories by localizing with respect to weak equivalences:

(15) Definition: We call $f: X \longrightarrow Y$ in $\mathcal{CW}$ a twisted R-equivalence if f induces isomorphisms

$$f_*: \pi_1 X \cong \pi_1 Y \qquad \text{and}$$
$$f_*: \pi_n X \otimes R \cong \pi_n Y \otimes R \quad , \quad n \geq 2 \quad .$$

Let $Ho_R \mathcal{CW}$ be the category obtained by localizing $\mathcal{CW}$ with respect to twisted R-equivalences. //

In $\mathcal{DA}_R$ a weak equivalence is a map which induces isomorphisms in homology. In $\widehat{Chain}_R$ and $\mathcal{DF}\hat{M}_R$ weak equivalences are pairs (φ, f) where φ is an isomorphism and where f induces isomorphisms in homology. Moreover, twisted R-equivalences are the weak equivalences in $\mathcal{CW}$. Then we get:

(16) <u>Proposition</u>: *All functors in (9) carry weak equivalences to weak equivalences.*

For $\hat{C}_* \otimes R$ this is part of the R-local Whitehead theorem, for $\hat{B}$ this is proven in [5].

Moreover, we have the following important result:

(17) <u>Theorem</u>: *(A) A map f in $\mathcal{CW}$ is a twisted R-equivalence if and only if the induced map $\hat{C}_* f \otimes R$ is a weak equivalence in $\widehat{Chain}_R$.*

(B) A map g in $\mathcal{DA}_R$ is a weak equivalence if and only if the induced map $\hat{B}g$ is a weak equivalence in $\mathcal{DF}\hat{M}_R$.

Part (A) is a variant of the Whitehead theorem, see (3), part (B) seems to be new. For A in $\mathcal{DA}_R$ the homology of $\hat{B}A$ is denoted by

(18) $$\mathrm{Tor}_A(H_0A, H_0A) = H_*(\hat{B}A) \quad .$$

Therefore, (B) is equivalent to

(19) <u>Theorem</u>: *A map $f: A \longrightarrow B$ in $\mathcal{DA}_R$ induces isomorphisms in homology if and only if*

$$Tor_f(H_0f, H_0f): Tor_A(H_0A, H_0A) \longrightarrow Tor_B(H_0B, H_0B)$$

is an isomorphism.

This result is well known in the connected case $H_oA = H_oB = R$. We now localize all categories in diagram (9) with respect to weak equivalences. By (16) we obtain the diagram of functors:

(20)
$$\begin{tikzcd}
\mathrm{Ho}_R\mathcal{CW} \arrow[rr, "C_*\Omega\otimes R"] \arrow[dd, "\hat{C}_*\otimes R"'] \arrow[dr, dashed, "A_R"] & & \mathrm{Ho}\mathcal{DA}_R \arrow[dd, "\hat{B}"] \\
& \hat{\alpha}^*\mathrm{Ho}\mathcal{DA}_R \arrow[ur, dashed] \arrow[dl, dashed] \arrow[dr, phantom, "\text{pull}"] & \\
\mathrm{Ho}\mathcal{C}hain_R^{\hat{}} \arrow[rr, "\hat{\alpha}"'] & & \mathrm{Ho}\mathcal{DF}\hat{\mathcal{M}}_R
\end{tikzcd}$$

In extension of equation (10) we get:

(21) <u>Theorem</u>: *Diagram (20) commutes, that is, there is a natural equivalence*

$$t:\ \hat{B}(C_*\Omega X\otimes R) \xrightarrow{\sim} \hat{\alpha}(\hat{C}_*X\otimes R)$$

in $\mathrm{Ho}\mathcal{DF}\hat{\mathcal{M}}_R$.

Let $\hat{\alpha}^*\mathrm{Ho}\mathcal{DA}_R$ be the pull back category. Then commutativity of the diagram yields the functor A_R in (20). The functor $\hat{\alpha}$ is faithful so that the pull back category can be considered as a subcategory of $\mathrm{Ho}\mathcal{DA}_R$.

(22) <u>Remark</u>: This result seems only to be known in the case $X = K(\pi,1)$. In this case we have the equivalence

$$C_*\Omega X \otimes R \sim R[\pi]$$

in $\mathcal{D}A_R$ and

$$\hat{B}(R[\pi]) = \hat{B}(R[\pi], R[\pi])$$

is the <u>normalized bar construction</u> of the non graded algebra $R[\pi]$.
From (21) we deduce the classical equations

$$\hat{H}_*(R[\pi], \Gamma) = \hat{H}_*(K(\pi,1), {}_\xi\Gamma) \quad ,$$

$$\hat{H}^*(R[\pi], \Gamma) = \hat{H}^*(K(\pi,1), \Gamma_\xi) \quad .$$

Here Γ is a $R[\pi]$-bimodule and the left side is the <u>Hochschild (co-) homology</u> of the algebra $R[\pi]$. The right side is the (co-) homology of the group π , compare chapter X, theorem 5.5 in [5]. //

For an algebra A in $\mathcal{D}A_R$ we have the <u>one-sided bar construction</u> (see [3], [4]):

(23) $$BA \otimes_\tau H \quad , \quad H = H_0 A \quad ,$$

where H is an A-module by $\lambda: A \longrightarrow H_0 A$. From (21) we deduce

(24) <u>Corollary</u>: *There is a natural equivalence*

$$B(C_*\Omega X \otimes R) \otimes_\tau H \xrightarrow{\sim} \hat{C}_* X \otimes R$$

in $HoChain^{\wedge}$ *where* $H = R[\pi] = H_0 \Omega X$.

<u>Proof</u>: For $A = C_*\Omega X \otimes R$ and $C = \hat{C}_* X \otimes R$ let

$$1 \otimes \varepsilon: \Lambda = H \otimes H^{op} \longrightarrow H \otimes R = H \quad .$$

Then we get

$$BA \otimes_\tau H = \hat{B}(A) \otimes_\Lambda (1 \otimes \varepsilon)^* H$$
$$\sim (\hat{\alpha} C) \otimes_\Lambda (1 \otimes \varepsilon)^* H = C \quad . \qquad //$$

The corollary shows that the chain algebra $C_*\Omega X \otimes R$ determines up to weak equivalence the chain complex of the universal covering as mentioned in the introduction.

We are now ready to state our results on the classification problem for 4-dimensional polyhedra:

(25) <u>Theorem</u>: *Let X and Y be CW-complexes of dimension ≤ 4 and let $1/2 \in R \subset \mathbb{Q}$. Then X and Y are equivalent in $Ho_R CW$ if and only if $A_R(X)$ and $A_R(Y)$ are equivalent in the pull back category $\hat{\alpha}^* Ho \mathcal{D} A_R$.*

(26) <u>Theorem</u>: *Let X and Y be CW-complexes of dimension ≤ 4 and assume that $\pi_2 Y = \mathbb{Z}$ or that multiplication by 2 is an isomorphism on $\pi_2 Y$. Then the complexes X and Y are homotopy equivalent in CW if and only if $A_{\mathbb{Z}}(X)$ and $A_{\mathbb{Z}}(Y)$ are equivalent in $\hat{\alpha}^* Ho \mathcal{D} A_{\mathbb{Z}}$.*

(27) <u>Remark on the proof of (25) and (26)</u>: Let $A = C_*\Omega X \otimes R$ and let $B = C_*\Omega Y \otimes R$. We show that for a map

$$F = (\varphi, f) : \hat{C}_* X \otimes R \longrightarrow \hat{C}_* Y \otimes R$$

there is an obstruction

$$O_{A,B}(\hat{\alpha} F) \in \hat{H}^4(X, \varphi^*(\pi_2 Y \otimes \pi_2 Y) \otimes R)$$

with the following properties. We have:

$$O_{A,B}(\hat{\alpha}F) = 0$$

iff there exists a map $\bar{F}: A \longrightarrow B$ in $\mathrm{Ho}\mathcal{DA}_R$ such that for the equivalence t in (21)

$$t(\hat{B}\bar{F}) = (\hat{\alpha}F)t$$

in $\mathrm{Ho}\mathcal{DF}\hat{\mathcal{M}}_R$. Moreover, for τ_* in (8) we have

$$\tau_* O_{X,Y}(F) = O_{A,B}(\hat{\alpha}F) \quad .$$

This shows that (26) is a consequence of (7), since the assumptions in (26) imply that τ_* is injective. Similarly we prove (25). //

Next we show that the localized categories in diagram (20) can be replaced by homotopy categories. We have already introduced the notion of homotopy in $\mathcal{C}hain\hat{}$ in (2). We have an isomorphism of categories

(28) $$\mathrm{Ho}\mathcal{C}hain\hat{}_R = \mathcal{C}hain\hat{}_R/\simeq \quad .$$

Similarly, we have

(29) $$\mathrm{Ho}\mathcal{DF}\hat{\mathcal{M}}_R = \mathcal{DF}\hat{\mathcal{M}}_R/\simeq$$

where two maps $(\varphi,\sigma),(\varphi',\sigma')$ in $\mathcal{DF}\hat{\mathcal{M}}_R$ are homotopic if $\varphi=\varphi'$ and if there is a φ-biequivariant map α of degree $+1$ with $\alpha_o = 0$ and $d\alpha + \alpha d = \sigma' - \sigma$.

We now consider $\mathrm{Ho}_R\mathcal{CW}$. We say that a space X is a <u>twisted</u> R-<u>space</u> if X is a complex in $\mathcal{CW}$ for which the universal covering is an R-space, that is $\pi_n X = \pi_n \hat{X} = \pi_n \hat{X}\otimes R$, $n \geq 2$. Let $\mathcal{CW}_R$ be the full subcategory of $\mathcal{CW}$ consisting of twisted R-spaces. Then we have the canonical equivalence of categories :

(30) $$\mathrm{Ho}_R\mathcal{CW} \sim \mathrm{Ho}_R\mathcal{CW}_R = \mathcal{CW}_R/\simeq \quad .$$

The equivalence is induced by the inclusion $\mathcal{CW}_R \subset \mathcal{CW}$.

Moreover, let $\mathcal{DFA}_R$ be the full subcategory of $\mathcal{DA}_R$ consisting of chain algebras A for which the underlying algebra is a free associative algebra the generators of which have augmentation 0 , we write $A = (T(V), d)$ where V is the set of generators of A . We introduce the notion of homotopy in $\mathcal{DFA}_R$ as follows: Two maps $f, g: A \longrightarrow B$ are homotopic if there is a map $\alpha: A \longrightarrow B$ of degree 1 of the underlying graded modules with

$$\alpha d + d\alpha = g - f$$
$$\alpha(xy) = (\alpha x)(gy) + (-1)^{|x|}(fx)(\alpha y) \quad .$$

Homotopy '$\simeq$' is a natural equivalence relation on $\mathcal{DFA}_R$.

(31) Theorem: *We have the canonical equivalence of categories*

$$Ho\mathcal{DA}_R \sim Ho\mathcal{DFA}_R = \mathcal{DFA}_R/\simeq \quad .$$

The equivalence is induced by the inclusion $\mathcal{DFA}_R \subset \mathcal{DA}_R$. This result seems to be known only in case R is a field, see [6]. It is in fact available if R is a principle ideal domain.

By use of the equivalences in (28),(29),(30) and (31) we can replace all categories in diagram (20) by homotopy categories. This is important for computations, in particular the chain algebra $C_*\Omega X \otimes R$ can be replaced by a free chain algebra $A = (T(V), d)$ with a small number of generators:

(32) Theorem: *Let X be a CW-complex in* CW *with cellular chain complexes C_*X. (Here $C_nX = H_n(X^n, X^{n-1})$ is the free abelian group generated by the n-cells of X.) For*

$$V_n = \begin{cases} s^{-1}C_1X \oplus s^{-1}C_1X & , \ n=0 \ , \\ C_1X \oplus s^{-1}C_2X & , \ n=1 \ , \\ s^{-1}C_{n+1}X & , \ n \geqq 2 \end{cases}$$

there is a differential d on the tensor algebra $T(V)$ such that the chain algebra $A = (T(V), d)$ is equivalent to $C_*\Omega X$ in $Ho\mathcal{D}A_{\mathbb{Z}}$.

This result shows that Theorems (25) and (26) can be used for explicit computations.

Theorem (32) is known for the special case that X has trivial 1-skeleton $X^1 = *$. Then we have $C_1X = 0$ and $V = s^{-1}\tilde{C}_*(X)$, and the differential of the theorem is the one constructed by Adams and Hilton in [1]. The method of Adams and Hilton relies on the Moore comparison theorem for spectral sequences which is not available in the non simply connected case. Our proof of theorem (32) is totally different and uses a new technique. All details and many more facts related to the results above will be contained in my forthcoming book 'On the homotopy classification problem'.

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