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RATIONAL HOMOTOPY TYPES OF COFIBRATIONS

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1. Main results.

In this note we study the rational homotopy types of cofibrations. Spaces are always assumed to be connected and simply connected CW-complex and cohomologies are those with rational coefficient. The rational homotopy type of a space is described by Sullivan's theory of minimal model. A space is called a formal space if its minimal model can be constructed by its cohomology ring ([1]). Let X be a formal space and $m(X)$ be its minimal model. Then there is a D.G.A. homomorphism $\rho_X : m(X) \to H(X)$ which induces an isomorphism on the cohomology. There is also a decomposition in each degree i, $m^i(X) = \bigoplus_{j\geq 0} W_j^i$ such that $W_j^i . W_k^m \subset W_{j+k}^{i+m}$, $dW_j^i \subset W_{j-1}^{i+1}$ ([2], [3]).

Let Y a finite complex and formal and $f : Y \to X$ be a map. In general the cofiber of f is not a formal space, so we introduce the following condition :

(P) There is a decomposition $m^i(X) = \oplus W_j^i$ and ρ_Y as above such that $\rho_Y f : m(X) \to H(Y)$ satisfies $\rho_Y f|W_j^i = 0$ for $j \geq 2$. Here we use the same notation f for the induced map between minimal models. Then we have

THEOREM : The cofiber $X \cup_f CY$ is a formal space if the condition (P) is satisfied.

If there are ρ_X and ρ_Y such that $\rho_Y \circ f = f^* \circ \rho_X$ then f is called a formal map. A formal map satisfies condition (P). Hence

COROLLARY 1 : The cofiber of f is a formal space if f is a formal map.

This answers to a problem listed in [4] under the assumption that Y is a finite complex.

Theorem is proved by reducing to the case where f^* is surjective.

2. <u>Proof of Theorem</u>.

The proof of Theorem consists of two steps :

A. Reduction to the case where $\rho_Y \circ f : m(X) \to H(Y)$ is surjective.

B. Reduction to the case where f^* is surjective.

First we prove step A. Let I_1 be an ideal of $H(Y)$ generated by the image of $\rho_X \circ f$ and J_1 be a quotientring of $H(Y)$ by I_1.

Let Y_1 and Z_1 be formal spaces. whose cohomology rings are isomorphic to I_1 and J_1 respectively. Then we have a cofibration

$$Z_1 \xrightarrow{q_1} Y \xrightarrow{p_1} Y_1$$

where q_1 and p_1 are formal maps which induce the quotient map and the inclusion map on cohomologies. Since $q_1^* \circ \rho_Y \circ f : m(X) \to H(Z_1)$ is null homomorphism q_1 f is null homotopic by Theorem (1.2) in [1]. Hence f factors through Y_1 as follows :

$$\begin{array}{ccc} Y_1 & & \\ \uparrow p_1 & \searrow^{f_1} & \\ & & X \\ & \nearrow_{f_2} & \\ Y & & \end{array}$$

Then we have a homotopy commutative diagram of cofibrations.

$$\begin{array}{ccc} Y_1 & \longrightarrow & X \vee SZ_1 \\ \uparrow & & \uparrow \\ Y & \longrightarrow & X \\ \uparrow & & \uparrow \text{0-map} \\ Z_1 & = & Z_1 \end{array}$$

By considering the horizontal cofibers we have a rational homotopy equivalence :

$$X \underset{f}{\cup} CY \qquad\qquad (X \underset{f_1}{\cup} CY_1) \vee SZ_1$$

where SZ_1 is a suspension of Z_1 and it has a same rational homotopy type of wedge of spheres.

Next we take an ideal I_2 of I_1 and do the same process. Proceeding this process we arrived at the situation where image of $\rho_{Y_{n-1}} \circ f_{n-1}$ is itself an ideal in I_{n-1} since $\dim H(Y) < \infty$. Then we have a space Y_n and a map $f_n : Y_n \to X$ such that $\rho_{Y_n} \circ f_n : m(X) \to H(Y_n)$ is surjective and rational homotopy equivalence : $X \underset{f}{\cup} CY \to (X \underset{f_n}{\cup} CY_n) \vee S^{k_i}{}_i$.

Now we prove step B. We take an ideal I_{n+1} of $H(Y_n)$ generated by the image of f_n^* and J_{n+1} be a quotient ring of $H(Y_n)$ by I_{n+1}. Let Y_{n+L} and Z_{n+1} be a corresponding formal spaces. Then we have a diagram :

$$\begin{array}{ccccccc} & & Y_{n+1} & & & \xrightarrow{\quad f_{n+1} \quad} & \\ & & \uparrow p_{n+1} & & & & \\ Z_{n+1} & \xrightarrow{q_{n+1}} & Y_n & \xrightarrow{f_n} & X & \xrightarrow{c_{n+1}} & X \underset{f_n q_{n+1}}{\cup} CZ_{n+1} \end{array}$$

where c_{n+1} is a natural map and we have a map f_{n+1} such that $p_{n+1} \circ f_{n+1}$ is homotopic to $c_{n+1} \circ f_n$. By similar way as in step A we have a rational homotopy equivalence

$$X \underset{f_n}{\cup} CY_n \longrightarrow (X \cup CZ_{n+1}) \underset{f_{n+1}}{\cup} CY_{n+1} \quad .$$

Proceeding this process we have a cofibration

$$Y_{n+k} \to (..(X \cup CZ_{n+1})..) \cup CZ_{n+k} \to (..(X \cup CZ_{n+1})..) \cup CY_{n+k}$$

such that f_{n+k} and rational homotopy equivalence.

$$X \cup CY_n \longrightarrow (X \bigcup_{n+1}^{n+k} CZ) \cup CY_{n+k}$$

where $X \bigcup_{n+1}^{n+k} CZ$ denotes $(..(X \cup CZ_{n+1})..) \cup CZ_{n+k}$.

Now we need to show

LEMMA : Each Z_i , $n+1 \leqslant i \leqslant n+k$, is rational homotopy equivalent to a wedge of spheres and $X \bigcup_{n+1}^{n+k} CZ$ is a formal space.

PROOF : By condition (P) any element in $H(Z_i)$ is in the image of $q_i^* \circ \rho_{Y_{i-1}} \circ f_{i-1}(W_1^*)$. Hence any product of elements in $H(Z_i)$ is zero. Therefore by formality Z_i is a wedge of spheres. Next we show the formality of $X \bigcup_{n+1}^{n+k} CZ$. There is an a selfmap $P_t : Z_{n+1} \to Z_{n+1}$ such that $P_t^* = t^{i+1}$: $H^i(Z_{n+1}) \to H^i(Z_{n+1})$ and grading automorphism $G_t : m(X) \to m(X)$ such that $G_t|W_j^i = t^{i+j}\mathrm{Id}$. Then by condition (P) we have a homotopy commutative diagram :

$$\begin{array}{ccccc} Z_{n+1} & \longrightarrow & X & \longrightarrow & X \cup CZ_{n+1} \\ \downarrow P_t & & \downarrow G_t & & \downarrow \\ Z_{n+1} & \longrightarrow & X & \longrightarrow & X \cup CZ_{n+1} \end{array}$$

and a map G_t^{n+1} such that $(G_t^{n+1})^* = t^i\mathrm{Id}$ on $H^i(X \cup CZ_{n+1})$.

Hence $X \cup CZ_{n+1}$ is a formal space. By induction we can show that $X \bigcup_{n+1}^{n+k} CZ$ is a formal space.

This ends the step B.

Now we consider the case where f^* is surjective.

Let $Y \xrightarrow[f]{} X \xrightarrow[g]{} Z$ be a cofibration such that f^* is surjective. Let $A(Z)$ be a kernel of $f : m(X) \to m(Y)$. Then $A(Z)$ is a free D.G.A. quasi-isomorphic to $m(Z)$. Since c^* is injective the map $\rho_X|A(Z) : A(Z) \to H(X)$ induce isomorphism on $H(Z)$. Hence Z is a formal space and this completes the proof of Theorem.

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