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A CHANGE OF RINGS THEOREM FOR LOCAL RINGS

by

Clas Löfwall

Let $f\colon (A,m,k) \longrightarrow (B,n,l)$ be a local homomorphism of local noetherian rings. Suppose also that char(l) = 0. Define the graded vector space V as

$$V = s\ker(QTor^A(k,k) \otimes_k l \xrightarrow{Qf_*} QTor^B(l,l))$$

(Q means the functor defined by forming the set of indecomposable elements, and s is the shift operator acting as $(sW)_i = W_{i-1}$).

Let further $\Gamma(V)$ denote the (graded) commutative algebra on V and let $Pol(V)$ denote the (graded) bicommutative Hopf algebra on V. Define Y as $X \otimes_A B$, where X is a minimal A-resolution of k. In [2] Theorem 3.5 Avramov (implicitly) states that the Hopf algebra $Ext_Y(l,l)$ is the <u>tensor product</u> of $Pol(V^*)$ and the Hopf algebra kernel of

$$f^*\colon Ext_B(l,l) \longrightarrow Ext_A(k,k) \otimes_k l .$$

This is however not true! But it <u>is</u> true that there is an exact sequence of Hopf algebras

$$1 \longrightarrow Pol(V^*) \longrightarrow Ext_Y(l,l) \longrightarrow \ker(f^*) \longrightarrow 1$$

<u>and</u> that $Pol(V^*)$ is contained in the graded centre of $Ext_Y(l,l)$ (a result of Calle Jacobsson [10]) but the sequence does not split in general (Calle Jacobsson was the first to give a counter-example to this).

The following theorem or rather its first corollary may be seen as the corrected version of Theorem 3.5 in [2]. It should be pointed out that, in [2], Avramov proves that the sequence in the theorem below is exact as a sequence of <u>coalgebras</u> and also that η_Y^* and f^* are maps of Hopf algebras. The main contribution of the present paper is the fact that γ_1 is a map of <u>Hopf</u> algebras. The proof of Theorem 1 does not use Avramovs result. For a proof of the fact that $im(\gamma_1)$ is contained in the graded centre of $Ext_Y(l,l)$ the reader is referred to the paper [10] by C. Jacobsson (he proves this when $A \longrightarrow B$ is flat, but the same proof is valid in the general case).

After this was written, Avramov has given an independent proof of Theorem 1 in arbitrary characteristic.

Theorem 1 There is an exact sequence of coalgebras

$$\begin{array}{l} 1 \longrightarrow \mathrm{Pol}(V^*) \xrightarrow{\gamma_1} \mathrm{Ext}_Y(1,1) \xrightarrow{\eta_Y^*} \mathrm{Ext}_B(1,1) \xrightarrow{f^*} \mathrm{Ext}_A(1,1) \otimes_k 1 \xrightarrow{\gamma_2} \mathrm{Pol}(s\,V^*) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ ((s\,V^*)^i = (V^*)^{i+1}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1 \end{array}$$

where γ_1 , η_Y^* and f^* are maps of Hopf algebras, η_Y^* is induced by the unit map η_Y: $B \longrightarrow Y$. Moreover $\mathrm{im}(\gamma_1)$ is contained in the (graded) centre of $\mathrm{Ext}_Y(1,1)$ (in other words, $\gamma_1(V^*)$ is contained in the centre of the Lie algebra of primitive elements in $\mathrm{Ext}_Y(1,1)$).

Combining theorem 1 with the Eilenberg-Moore spectral sequence $\mathrm{Ext}_{H(Y)}(1,1) \Rightarrow \mathrm{Ext}_Y(1,1)$ we get

Corollary 1 With the same notations as in theorem 1 we have a spectral sequence of Hopf algebras

$$\mathrm{Ext}^{p,q}_{\mathrm{Tor}^A(B,k)}(1,1) \Rightarrow E$$

and an exact sequence of Hopf algebras

$$1 \longrightarrow \mathrm{Pol}(V^*) \xrightarrow{\gamma} E \longrightarrow \mathrm{Ext}_B(1,1) \xrightarrow{f^*} \mathrm{Ext}_A(k,k) \otimes_k 1$$

with $\mathrm{im}(\gamma)$ contained in the graded center of E .

If B is A-flat, $\mathrm{Tor}^A(B,k) = B \otimes_A k$ and the spectral sequence degenerates, yielding the following

Corollary 2 Suppose in addition that B is A-flat, then there is an exact sequence of Hopf algebras with $\mathrm{im}(\gamma)$ contained in the graded center of $\mathrm{Ext}_{B \otimes_A k}(1,1)$

$$1 \longrightarrow \mathrm{Pol}(V^*) \longrightarrow \mathrm{Ext}_{B \otimes_A k}(1,1) \xrightarrow{g^*} \mathrm{Ext}_B(1,1) \xrightarrow{f^*} \mathrm{Ext}_A(k,k) \otimes_k 1$$

where g^* is induced by the natural projection $g : B \longrightarrow B \otimes_A k$. Moreover, $|V| < \infty$ and $V_i = 0$ for odd i.

Remark **The fact that $|V| < \infty$ is due to André [5], the fact that** $V_i = 0$ for odd i is due to Avramov [3].

Theorem 1 may be generalized to cover cases where Y is not necessarily of the form $X \otimes_A B$.

Definition We say that "derivations may be extended" in $Y = B\langle\ldots S\ldots\rangle$ if there is for each S a derivation $j_S\colon Y \longrightarrow Y$ (commuting with the differential) such that $j_S(S) = 1$ and $j_S(S') = 0$ if $\deg(S') \leq \deg(S)$ and $S' \neq S$.

Theorem 2 Suppose (B,n,l) is a local ring, $\mathrm{char}(l) = 0$, and Y is a DG B-algebra obtained by adjoining variables such that $Y \otimes_B l$ has zero differential. Then there is an exact sequence of coalgebras

$$1 \longrightarrow \mathrm{Pol}(V^*) \longrightarrow \mathrm{Ext}_Y(1,1) \xrightarrow{\eta_Y^*} \mathrm{Ext}_B(1,1) \xrightarrow{f^*} \mathrm{Hom}_B(Y,1) \longrightarrow \mathrm{Pol}(s\,V^*) \longrightarrow 1$$

where η_Y^* is a map of Hopf algebras and $V = s\ker(Q(Y \otimes_B 1) \xrightarrow{Qf_*} Q\mathrm{Tor}^B(1,1))$. The maps f_* and f^* are induced by a map $f\colon Y \longrightarrow U$ where U is the acyclic closure of B ($\longrightarrow 1$). Suppose also that "derivations may be extended" in Y. Then $\mathrm{Pol}(V^*)$ is the Hopf algebra kernel of η_Y^* and contained in the graded center of $\mathrm{Ext}_Y(1,1)$ and the map $\mathrm{Ext}_B(1,1) \longrightarrow \mathrm{im}(f^*)$ induced by f^* is a Hopf algebra map.

Corollary 3 Suppose (B,n,l) is a local ring and Y a DG B-algebra obtained by adjoining variables and let η_Y and ε_Y denote the unit and augmentation maps (Y is augmented to l in the natural way). Suppose

a) $\mathrm{Hom}_B(Y,l)$ has zero differential

b) $\varepsilon_Y^* : \mathrm{Ext}_B(l,l) \to \mathrm{Ext}_B(Y,l) = \mathrm{Hom}_B(Y,l)$ is surjective

c) derivations may be extended in Y.

Then there is an exact sequence of Hopf algebras

$$1 \to \mathrm{Ext}_Y(l,l) \xrightarrow{\eta_Y^*} \mathrm{Ext}_B(l,l) \xrightarrow{\varepsilon_Y^*} \mathrm{Hom}_B(Y,l) \to 1$$

This is a particular case of Avramovs Theorem 2.2 in [1]. It may be applied e.g. for Y equal to a "step" in the Tate-resolution of l over B, or for $Y = X \otimes_A B$ where $(A,m,k) \to (B,n,l)$ is a small homomorphism and X is a minimal resolution of k over A. Also we have the following

Corollary 4 Suppose $B = l[[x_1,\ldots,x_n]]/(f_1,\ldots,f_r) + \mathfrak{a}$ where f_i are quadratic forms in $x_1,\ldots,x_n$ and $\mathfrak{a}$ is an ideal contained in $(x_1,\ldots,x_n)^3$. Let A be the subalgebra of $\mathrm{Ext}_B(l,l)$ generated by the one-dimensional elements and $U = (B \otimes_l A^*, d)$ (defined in [11]).

Then there is an exact sequence of Hopf algebras

$$1 \to \mathrm{Ext}_U(l,l) \to \mathrm{Ext}_B(l,l) \to A \to 1$$

(which is split by the natural map $A \to \mathrm{Ext}_B(l,l)$).

Moreover if $m^4 = 0$ the Eilenberg - Moore spectral sequence $\mathrm{Ext}_{H(U)}(l,l) \Rightarrow \mathrm{Ext}_U(l,l)$ degenerates yielding a (split) exact sequence of Hopf algebras

$$1 \to T(s\,H(U)) \to \mathrm{Ext}_B(l,l) \to A \to 1$$

which is a result in [11]

Proof It was proved in [11] that U satisfies a), b) and c) of corollary 3, and that $A = \mathrm{Hom}_B(U,l)$.

Proof of theorem 2: Let Z be the acyclic closure of Y ($\rightarrow 1$) and consider the following exact sequence of differential Γ-algebras

$$1 \rightarrow Y \otimes_B 1 \rightarrow Z \otimes_B 1 \rightarrow Z \otimes_Y 1 \rightarrow 1 .$$

This yields an exact sequence of complexes

$$0 \rightarrow Q(Y \otimes_B 1) \rightarrow Q(Z \otimes_B 1) \rightarrow Q(Z \otimes_Y 1) \rightarrow 0 .$$

According to Gulliksen ([9] Th. 1.6.2) $Z \otimes_Y 1$ has zero differential and $Y \otimes_B 1$ has zero differential by assumption.

According to Gulliksen ([9] Th.3.2.3), since char(1) = 0, the natural map $Z \otimes_B 1 \rightarrow \mathrm{Tor}^B(1,1)$ induces an isomorphism $H\,Q(Z \otimes_B 1) \rightarrow Q\,\mathrm{Tor}^B(1,1)$. Hence there is an exact sequence (with naturally defined maps)

$$0 \rightarrow s^{-1}V \rightarrow Q(Y \otimes_B 1) \rightarrow Q\,\mathrm{Tor}^B(1,1) \rightarrow Q(Z \otimes_Y 1) \rightarrow V \rightarrow 0 \qquad (2)$$

This gives an exact sequence of Γ-algebras

$$1 \rightarrow \Gamma(s^{-1}V) \rightarrow Y \otimes_B 1 \rightarrow \mathrm{Tor}^B(1,1) \rightarrow Z \otimes_Y 1 \rightarrow \Gamma(V) \rightarrow 1 .$$

Dualizing this and observing that (by Eilenberg-Moore spectral sequence) $\mathrm{Tor}^Y(1,1) = Z \otimes_Y 1$ we get the first part of the theorem. The fact that $\mathrm{Ext}_Y(1,1)$ is a Hopf algebra and $(\eta_Y)^*$ is a Hopf algebra map has been proved by Avramov (to appear in a forthcoming paper). However, for our purposes we will use another definition of $\mathrm{Ext}_Y(1,1)$ and in the appendix we prove that this definition coincides with the one used by Avramov.

The definition of the Γ-Hopf algebra $\mathrm{Tor}^Y(1,1)$ is parallel to the case where $Y = B$. As was said above Gulliksen proves that the acyclic closure Z of Y ($\longrightarrow 1$) has the property that $Z \otimes_Y 1$ has zero differential but also that the Y-derivations on Z defined up to a certain degree, commuting with the differential, may be extended to the whole of Z. As Γ-algebras we have $\mathrm{Tor}^Y(1,1) = Z \otimes_Y 1$. A Γ-Hopf algebra structure is defined on $\mathrm{Tor}^Y(1,1)$ precisely when $Q^*(Z \otimes_Y 1)$ is given a structure of a graded Lie algebra (cf. [12],[6],[14]) and this is done as follows. Let $\mathrm{Der}_Y(Z)$ denote the set of graded derivations $Z \longrightarrow Z$ which are zero on Y. This is a graded Lie algebra. The map $[d_Z, \cdot] : \mathrm{Der}_Y(Z) \longrightarrow \mathrm{Der}_Y(Z)$ makes $\mathrm{Der}_Y(Z)$ to a differential Lie algebra, and hence $H(\mathrm{Der}_Y(Z))$ is a Lie algebra. There is a natural map $\phi : \mathcal{Z}\mathrm{Der}_Y(Z) \longrightarrow Q^*(Z \otimes_Y 1)$ defined in the following way. If $j \in \mathrm{Der}_Y(Z)$ is a homogeneous element then $\varepsilon_Z \circ j : Z \longrightarrow 1$ induces a derivation $Z \otimes_Y 1 \longrightarrow 1$ and this map is zero on decomposable elements, since it is non-zero in just one degree and satisfies the derivation rule. Hence $j \in \mathrm{Der}_Y(Z)$ defines $\phi(j) \in Q^*(Z \otimes_Y 1)$. The map ϕ is onto, since if $\alpha \in Q^*(Z \otimes_Y 1)^r$ then $\tilde{\alpha}$ may be defined such that

$$\begin{tikzcd}
Z_r \arrow[dr] \arrow[dd, "\tilde{\alpha}"'] & & \\
& (Z\otimes_Y 1)_r \arrow[r] \arrow[d] & Q(Z\otimes_Y 1)_r \arrow[dl, "\alpha"] \\
Z_0 \arrow[r, "\varepsilon"] & 1 &
\end{tikzcd}$$

commutes

and if $\tilde{\alpha}$ is defined as zero in degrees below r, then $\tilde{\alpha}$ is a Y-derivation $Z_{\leq r} \longrightarrow Z$ and this derivation may be extended to a derivation $j \in \mathcal{Z}\mathrm{Der}_Y(Z)$ such that $\phi(j) = \alpha$. The kernel of ϕ is $\{j \in \mathcal{Z}\mathrm{Der}_Y(Z);\ \varepsilon \circ j = 0\} = \mathcal{B}\mathrm{Der}_Y(Z)$. This is true since if $\varepsilon \circ j = 0$, then there is a map of complexes $r: Z \longrightarrow Z$ such that $[d,r] = j$. But since Z is obtained from Y by adjunction of variables, it is possible to define r successively such that $r \in \mathrm{Der}_Y(Z)$. Also, $\varepsilon \circ [d,r] = 0$, since $dZ \subset nZ$ and r is B-linear. Hence ϕ induces an isomorphism $H(\mathrm{Der}_Y(Z)) \longrightarrow Q^*(Z \otimes_Y 1)$, which gives the desired Lie algebra structure on $Q^*(Z \otimes_Y 1)$.

We now turn to the question whether $(\eta_Y)_* : \mathrm{Tor}^B(1,1) \longrightarrow \mathrm{Tor}^Y(1,1)$ is a Hopf algebra map. A map $Y_1 \xrightarrow{f} Y_2$ of DG free Γ-B-algebras induces a Hopf algebra map $\mathrm{Tor}^{Y_1}(1,1) \longrightarrow \mathrm{Tor}^{Y_2}(1,1)$ in the following way. (One could also have a change of rings $B_1 \longrightarrow B_2$, Y_i a B_i-algebra and

$$\begin{tikzcd}
B_1 \arrow[r] \arrow[d] & B_2 \arrow[d] \\
Y_1 \arrow[r] & Y_2
\end{tikzcd}$$

commutative, but we restrict ourselves to the case with B fixed). Let Z_i be the acyclic closure of Y_i ($\longrightarrow 1$). According to ([9] lemma 1.8.6)

f may be extended to a map $\tilde{f} : Z_1 \to Z_2$ and this induces a map of Γ-algebras $Z_1 \otimes_{Y_1} 1 \to Z_2 \otimes_{Y_2} 1$. We will prove that this map also is a map of Hopf algebras. If $j \in Q^*(Z_i \otimes_Y 1)$ let $\tilde{j}$ denote an element in $\mathcal{Z}\mathrm{Der}_{Y_i}(Z_i)$ such that $\varepsilon_i \circ \tilde{j} = j$ where $\varepsilon_i : Z_i \to 1$ is the augmentation. Suppose $j_a , j_b \in Q^*(Z_2 \otimes_{Y_2} 1)$. Then $[j_a,j_b]$ is mapped to

$$[j_a,j_b] \circ \tilde{f} = \varepsilon_2 \circ [\tilde{j}_a,\tilde{j}_b] \circ \tilde{f} = \varepsilon_2 \circ \tilde{f} \circ [(\widetilde{j_a \circ \tilde{f}}),(\widetilde{j_b \circ \tilde{f}})] =$$
$$= \varepsilon_1 \circ [(\widetilde{j_a \circ \tilde{f}}),(\widetilde{j_b \circ \tilde{f}})] = [j_a \circ \tilde{f}, j_b \circ \tilde{f}] .$$

$$\begin{tikzcd}
Z_1 \arrow[r, "\tilde{f}"] \arrow[d, "(\widetilde{j_a \circ \tilde{f}})"'] & Z_2 \arrow[r, "j_a"] \arrow[d, "\tilde{j}_a"] & 1 \\
Z_1 \arrow[r, "\tilde{f}"] \arrow[d, "(\widetilde{j_b \circ \tilde{f}})"'] & Z_2 \arrow[r, "j_b"] \arrow[d, "\tilde{j}_b"] \arrow[ur, "\varepsilon_2"] & 1 \\
Z_1 \arrow[r, "\tilde{f}"] \arrow[urr, bend right, "\varepsilon_1"'] & Z_2 \arrow[ur, "\varepsilon_2"] &
\end{tikzcd}$$

the squares are homotopy commutative.

Let now Y satisfy the additional assumption that derivations may be extended and consider the Hopf algebra map $\mathrm{Tor}^B(1,1) \to \mathrm{Tor}^Y(1,1)$ induced by $\eta_Y : B \to Y$. I claim that the Γ-algebra kernel is a Hopf algebra, i.e. the image of the map $Q^*\mathrm{Tor}^Y(1,1) \xrightarrow{\alpha} Q^*\mathrm{Tor}^B(1,1)$ is a Lie ideal in $Q^*\mathrm{Tor}^B(1,1)$. This gives the assertion of the theorem that $\mathrm{Ext}_B(1,1) \to \mathrm{im}(f^*)$ is a Hopf algebra map. Let Z be the acyclic closure of Y ($\to 1$) and U the acyclic closure of B ($\to 1$) . There are maps of augmented DG B-algebras making the diagram below homotopy commutative

$$\begin{tikzcd}
Z & \\
Y \arrow[u, "\delta"] \arrow[r, "\beta"] & U \arrow[ul, "\gamma"']
\end{tikzcd}$$

The map $U \xrightarrow{\gamma} Z$ induces the canonical map $U \otimes_B 1 = \mathrm{Tor}^B(1,1) \to \mathrm{Tor}^Y(1,1) = Z \otimes_Y 1$. Suppose $j \in Q^*(U \otimes_B 1)$ is in $\mathrm{im}(\alpha)$. Then there is $j'' \in \mathcal{Z}\mathrm{Der}_Y(Z)$ such that $\varepsilon_Z \circ j'' \circ \gamma = j$ and

$j' \in \mathcal{Z}\mathrm{Der}_B(U)$ such that $\varepsilon_U \circ j' = j$. Suppose also $j_1 \in Q^*(U \otimes_B 1)$ and $j_1' \in \mathcal{Z}\mathrm{Der}_B(U)$ such that $\varepsilon_U \circ j_1' = j_1$ and $j_1'' \in \mathcal{Z}\mathrm{Der}_B(Z)$ such that $\varepsilon_Z \circ j_1'' \circ \gamma = j_1$ (j_1'' exists since γ is a homotopy equivalence). Since derivations may be extended in Y, by assumption, there is $j_1''' \in \mathcal{Z}\mathrm{Der}_B(Y)$ such that $\varepsilon_Y \circ j_1''' = j_1 \circ \beta$

$$\begin{tikzcd}
 & Z \arrow[rr, "j_1''"] & & Z \arrow[rr, "j''"] & & Z \\
Y \arrow[ur, "\delta"] \arrow[dr, "\beta"'] \arrow[rr, "j_1'''"] & & Y \arrow[ur, "\delta"] \arrow[dr, "\beta"'] & & & \\
 & U \arrow[uu, "\gamma"] \arrow[rr, "j_1'"] & & U \arrow[uu, "\gamma"'] \arrow[rr, "j'"] & & U \arrow[uu, "\gamma"']
\end{tikzcd} .$$

All diagrams above are homotopy commutative since

$\varepsilon_Z \circ j'' \circ \gamma = j = \varepsilon_U \circ j' = \varepsilon_Z \circ \gamma \circ j'$ and

$\varepsilon_Z \circ j_1'' \circ \gamma = j_1 = \varepsilon_U \circ j_1' = \varepsilon_Z \circ \gamma \circ j_1'$ and

$\varepsilon_U \circ \beta \circ j_1''' = \varepsilon_Y \circ j_1''' = j_1 \circ \beta = \varepsilon_U \circ j_1' \circ \beta$.

It follows from the diagram that also $j_1'' \circ \delta$ and $\delta \circ j_1'''$ are homotopic, since γ is a homotopy equivalence. Now

$$[j,j_1] = \varepsilon_U \circ (j' \circ j_1' \pm j_1' \circ j') = \varepsilon_Z \circ (j'' \circ j_1'' \pm j_1'' \circ j'') \circ \gamma .$$

This is an element in $\mathrm{im}(\alpha)$ if $\varepsilon_Z \circ [j'',j_1''] \circ \delta = 0$. But this is true since $j'' \circ \delta = 0$ and $j_1'' \circ \delta$ and $\delta \circ j_1'''$ are homotopic.

Hence we have proved that $\mathrm{im}(\alpha)$ is a Lie ideal and we have proved all assertions of theorem 2 but the fact that $\mathrm{Pol}(V^*)$ is the Hopf algebra kernel of η_Y^* if derivations may be extended in Y . To do this we will use André's notion of "special variables", see [5] .

From (2) above it follows that V^* may be interpreted as $\ker(Q^*(Z \otimes_Y 1) \longrightarrow (HQ(Z \otimes_B 1))^*)$. If $Z = Y\langle \ldots T, \ldots \rangle$ then a basis for V^* may be chosen to consist of duals of variables T , such that dT has a component in Y which is not in $nY + I^{(2)}Y$. Since derivations may be extended in Y , there is a derivation j defined up to $\deg(dT)$

such that $j(dT) = 1$, i.e. V^* has a basis consisting of duals of special variables. But André has proved that if a special variable is even then it may be adjoined in any order with respect to the other variables. We will prove that André's result holds also for special variables of odd degree (at least when char(l) = 0). From this it easily follows that if T is a special variable then the derivation j_T associated to T may be chosen to satisfy $j_T(T') = \delta_{TT'}$ and then $[j_T, j_{T'}] = 0$ if T and T' are special variables. For the fact that $[j_T, j_{T'}] = 0$ if just one of T and T' is special (which gives the assertion that $V^* \subset \mathrm{center}(Q^*\mathrm{Ext}_Y(l,l))$) we refer to the paper of C. Jacobsson [10].

André's result

André proves the following ((B,n,l) is a local ring)

Theorem Suppose $Y \subset U \subset Z$ are DG B-algebras and U is obtained by adjunction of an even special variable S , $dS = s \in Y$, Z by a variable T , $dT = t \in U$. Then there is $t' \in Y$, with t' and t homologous such that $Y \subset V \subset Z$ where $V = Y\langle T'; dT' = t'\rangle$ and $Z = V\langle S; dS = s\rangle$, also s is a special cycle of V .

Explanation. S is called a special variable and dS a special cycle if there is a derivation j such that $j(dS) = 1$.

The proof of the theorem follows easily from the following two lemmata.

Lemma 1 Suppose X is a DG B-algebra and R is a variable of odd degree. Then the inclusion map

$$X \longrightarrow X\langle R;\, dR = 0\rangle\langle S;\, dS = R\rangle$$

is a quasi-isomorphism.

Lemma 2 Suppose Y is a DG B-algebra and s is a special cycle of odd degree. Then there is a DG-algebra X of Y , such that $Y \simeq X\langle R;\, dR = 0\rangle$.

If char(l) = 0 lemma 1 is true also for R of <u>even</u> degree. Lemma 2 may be generalized to

Lemma 3 Suppose Y is a DG B-algebra and $S \in Y$ an element such that there exists a derivation j with $j(S) = 1$. Then there is a DG sub-algebra X of Y such that $Y \simeq X\langle R;\, dR = s\rangle$ where $s = dS \in X$.

We will prove lemma 3 for the case when S is of even degree. To do this we will first prove

Lemma 4 Suppose S is an element of even degree in a DG algebra Y and j a derivation such that $j(S) = 1$. For each $y \in Y$ let y' denote the element $y - j(y)S + j^2(y)S^{(2)} - \dots$ ($j^n(y) = 0$ for big enough n). Then $y = y' + j(y)'S + j^2(y)'S^{(2)} + \dots$ and $j(y') = 0$.

Proof The second claim follows from the fact $j(j^n(y)S^{(n)}) = j^{n+1}(y)S^{(n)} + j^n(y)S^{(n-1)}$. The first follows from

$$\sum_{n=0} j^n(y)'S^{(n)} = \sum_{n,k=0} (-1)^k j^{n+k}(y)S^{(k)}S^{(n)} = \sum_{p=0} j^p(y) \sum_{n+k=p} (-1)^k S^{(k)}S^{(n)} = y$$

since if $p>0$ $\sum_{n+k=p} (-1)^k S^{(k)}S^{(n)} = \sum_{n+k=p} (-S)^{(k)}S^{(n)} = (S - S)^{(p)} = 0$.

Proof of lemma 3:

Put $X = \{y \in Y;\, j(y) = 0\}$. Define $\tau\colon X\langle R;\, dR = s\rangle \longrightarrow Y$ as the inclusion map on X and $\tau(R) = S$. The map $\sigma\colon Y \longrightarrow X\langle R;\, dR = s\rangle$ is defined by $\sigma(y) = y' + j(y)'R + j^2(y)'R^{(2)} + \dots$. The map τ is a map of DG algebras and τ is an isomorphism if $\tau \circ \sigma = \mathrm{id}_Y$ and $\sigma \circ \tau = \mathrm{id}_{X\langle R\rangle}$.
Lemma 4 gives the first equality and for the second it is enough to prove that if $y = a_0 + a_1S + a_2S^{(2)} + \dots$ with $a_i \in X$, then $j^k(y)' = a_k$. But $j^k(y) = \sum_{n\geq k} a_n S^{(n-k)}$, $j^k(y)' = \sum_{l\geq 0} (-1)^l j^{k+l}(y)S^{(l)} = \sum_{l\geq 0} (-1)^l \sum_{n\geq k+l} a_n S^{(n-k-l)}S^{(l)}$

$$= \sum_{n-k\geq 0} a_n \sum_{l=0}^{n-k} S^{(n-k-l)}(-S)^{(l)} = \sum_{n-k\geq 0} a_n (S-S)^{(n-k)} = a_k .$$

APPENDIX

The definition of $\mathrm{Ext}_F(k,k)$ as a Hopf algebra when F is an augmented differential algebra.

When $F = R$ is a local ring Assmus [7] has defined a diagonal on $\mathrm{Tor}^R(k,k)$ in the following way (modified by Avramov).

Let P be a resolution of k over R and $\varepsilon\colon P \longrightarrow k$ the augmentation. Then $\mathrm{Tor}^R(k,k) = H(k \otimes_R P)$ and the diagonal is defined as the composition

$$\begin{aligned} H(k \otimes_R P) &\xleftarrow[\approx]{H(\varepsilon\otimes 1)} H(P \otimes_R P) \xrightarrow{H(\rho)} H(P \otimes_R P \otimes_P P \otimes_R P) \xleftarrow[\approx]{\kappa} \\ &\longleftarrow H(P \otimes_R P) \otimes_{H(P)} H(P \otimes_R P) \xrightarrow[\approx]{H(\varepsilon\otimes 1)\,\otimes\, H(\varepsilon\otimes 1)} H(k \otimes_R P) \otimes_k H(k \otimes_R P) \end{aligned} \tag{1}$$

where $\rho(x \otimes y) = x \otimes 1 \otimes 1 \otimes y$ and κ is the Künneth isomorphism. The fact that the dual of this map coincides with the (opposite of) the Yoneda product was proved by Levin [9 , Th. 2.3.3].

If F is a DG (differential graded) augmented algebra, $\mathrm{Tor}^F(k,k)$ is defined in [13]. It may be computed as $H(k \otimes_F Y)$ where Y is any DG F-module such that $H(Y) = k$ and $Y^{\#}$ is $F^{\#}$-flat (the sign # means: Forget the differential). Also in this case Assmus' definition of the diagonal makes sense, and $\mathrm{Tor}^F(k,k)$ is, as in the case $F = R$, a Hopf algebra which as an algebra is a free Γ-algebra, i.e. a free graded commutative algebra with divided powers. A proof with all details of this fact has recently been carried out by Avramov [4] . Another approach to prove this goes as follows. Gulliksen [9] has proved that the acyclic closure Y of F has the property that $k \otimes_F Y$ has zero differential. This shows that $\mathrm{Tor}^F(k,k)$ is equal to $\Gamma(V)$, the free Γ-algebra on a graded vector space V. A Hopf-Γ-algebra structure on $\mathrm{Tor}^F(k,k)$ is obtained precisely when V^* is given a graded Lie algebra structure (

The Lie algebra structure on V^* is defined in the text of this paper, using the isomorphism $V^* \cong H(\mathrm{Der}_F(Y))$. This Lie product induces a product on the enveloping algebra of V^*, which is equal to the dual of $\mathrm{Tor}^F(k,k)$ and denoted by $\mathrm{Ext}_F(k,k)$. Also, $\mathrm{Ext}_F(k,k) = (k \otimes_F Y)^* = \mathrm{Hom}_F(Y,k) \cong$ $\cong \mathrm{HHom}_F(Y,Y)$. Hence $\mathrm{Ext}_F(k,k)$ is in a natural way an associative algebra, defined by composition of maps. This <u>is</u> the enveloping algebra of V^*. Indeed, there is a map of Lie algebras $H(\mathrm{Der}_F(Y)) \to \mathrm{HHom}_F(Y,Y)$ and $\mathrm{HHom}_F(Y,Y)$ is generated as an algebra by the image of this map. From this it follows that there is a surjection $U(V^*) \to \mathrm{HHom}_F(Y,Y)$ and this is a bijection because of the standard assumption of locally finite dimensionality.

Now the question arises if this product <u>coincides</u> with the dual of Assmus' diagonal on $\mathrm{Tor}^F(k,k)$. One has to be a bit careful with the choice of Y. The map $\mathrm{HHom}_F(Y,Y) \to \mathrm{HHom}_F(Y,k)$ is not always an isomorphism (but if $F = R$, R a local ring, then Y may be any R-projective resolution of k). It is however an isomorphism if Y is the acyclic closure of F, or if Y is an F-resolution of k in the sense of Moore [13]. Anyway, whatever Y is, one may consider the product of those elements of $\mathrm{HHom}_F(Y,k)$ coming from $\mathrm{HHom}_F(Y,Y)$. The following theorem is due to Sjödin [15, Theorem 5].

<u>Theorem</u> **Let $F \xrightarrow{\varepsilon_F} k$, $Y \xrightarrow{\varepsilon_Y} k$ be augmented DG algebras and $F \to Y$ a morphism of augmented DG algebras such that $H(\varepsilon_Y)$ is an isomorphism and $Y^\#$ is $F^\#$-flat. Let Δ be defined by (1) (with $P=Y$, $R=F$, $\varepsilon=\varepsilon_Y$) and let $f,g \in Z\mathrm{Hom}_F(Y,Y)$. Consider $H(1 \otimes \varepsilon_Y of)$ and $H(1 \otimes \varepsilon_Y og)$ as elements of $\mathrm{Hom}_k(H(k \otimes_F Y), k)$ (identifying $k \otimes_F k$ and k). Then the product of these elements induced by the dual of Δ is $H(1 \otimes \varepsilon_Y ofog)$.**

<u>Proof</u> **We will repeat Sjödins proof, since the reader might have some difficulties to find the reference and also, if he finds it, he might have some difficulties with the notations used there. The proof depends on the**

following diagram, where $\mu_k: k \otimes_k k \to k$ is the canonical isomorphism and likewise $\mu: H(k \otimes_F Y) \otimes_k k \to H(k \otimes_F Y)$. Also we write ε as short for ε_Y.

$$\begin{tikzcd} H(k\otimes_F Y) \arrow[r, "H(1\otimes g)"] \arrow[d, "\Delta"'] & H(k\otimes_F Y) \arrow[d, "\Delta"'] & = & H(k\otimes_F Y) \arrow[r, "H(1\otimes\varepsilon o f)"] & k \\ H(k\otimes_F Y)\otimes H(k\otimes_F Y) \arrow[r, "1\otimes H(1\otimes g)"] & H(k\otimes_F Y)\otimes H(k\otimes_F Y) \arrow[rr, "1\otimes H(1\otimes\varepsilon)"] & & H(k\otimes_F Y)\otimes k \arrow[u, "\mu"] \arrow[r, "H(1\otimes\varepsilon o f)\otimes 1"] & k\otimes k \arrow[u, "\mu_k"'] \end{tikzcd}$$

By examining the definition of Δ it is easy to see that the left square is commutative, the middle square is commutative since $H(1\otimes\varepsilon)$ is a counit for $H(k\otimes_F Y)$ (for a proof of this, see [4]). The right square is commutative since $H(1\otimes\varepsilon o f)$ is k-linear. Hence,

$$\Delta^*(\mu_k o[H(1\otimes\varepsilon o f) \otimes H(1\otimes\varepsilon o g)]) = \mu_k o(H(1\otimes\varepsilon o f) \otimes 1)o(1 \otimes H(1\otimes\varepsilon))o(1 \otimes H(1\otimes g)o\Delta =$$

$$= H(1 \otimes \varepsilon o f)oH(1\otimes g) = H(1 \otimes \varepsilon o f o g).$$

Remark 1. The proof depends only on the fact that $H(1\otimes\varepsilon)$ is a counit on $H(k\otimes_F Y) = \mathrm{Tor}^F(k,k)$. When this is proved (which is an easy check of diagrams, see 15,p.32] or 4 and when it is proved that the composition of maps on $\mathrm{HHom}_F(Y,Y)$ induces a Lie algebra structure on V^* where $H(k\otimes_F Y) = \Gamma(V)$ (which is most easily seen by identifying V^* and $H(\mathrm{Der}_F(Y))$), then we get a proof of the coassociativity of Δ and of the compatibility of Δ with the divided powers.

Remark 2. In some cases there are other possible definitions of the product on $\mathrm{Ext}_F(k,k)$. Suppose $\mathrm{Ext}_F(k,k) \cong \mathrm{Ext}_{(\ X,d)}(k,k)$ as Hopf algebras and $(\Lambda X,d)$ is a minimal model (i.e. X is a graded vector space, $dX \subset \Lambda^{\geq 2}X$, d is of degree -1 , Λ has the same meaning as Γ which we have used above). This occurs if e.g. F and $(\Lambda X,d)$ are equivalent under the equivalence relation generated by quasi-isomorphisms (maps inducing isomorphism in homology). Let L be the graded Lie algebra such that $\mathrm{Ext}_F(k,k) = U(L)$.

Then L may be interpreted as sX with Lie structure induced by the dual of the map $d_2 : X \to X \wedge X$ (with an unpleasant sign involved). The proof of this may be found in [8].

If the augmented DG algebra F is an augmented k-algebra, it is possible to generalize the use of the Bar-resolution which is available in the non-differential case. If I is the augmentation ideal of F, the tensor algebra on I^* now has two differentials, one arising from the coproduct on I^* and one arising from the differential on I. The total complex is easily seen to be a differential algebra which induces an algebra structure on the cohomology, which by definition is $\mathrm{Ext}_F(k,k)$ as a vector space. That this product is the right one may be seen by an explicit "lifting" formula (the same as in the classical case). Details may be found in [4]. Or at least the formulas, the rest is left to the reader ...

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