Astérisque

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Astérisque, tome 113-114 (1984), p. 198-205
<http://www.numdam.org/item?id=AST_1984__113-114__198_0>

SPACES WHOSE RATIONAL HOMOLOGY AND DE RHAM HOMOTOPY ARE BOTH FINITE DIMENSIONAL

by

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1. INTRODUCTION

Let S be a path connected space with rational minimal model $(\Lambda X,d)$ - cf [4]. We say S is of type F ([1]) if dim X and dim $H(\Lambda X)$ are both finite.

Now $H(\Lambda X)\cong H^*(S;Q)$ (singular cohomology) while $X\cong\pi^*_{DR}(S)$ by the definition of ΛX and of $\pi^*_{DR}(S)$. Moreover, if S is 1-connected and dim $H^p(S;Q)<\infty$ for all p then $X\cong\mathrm{Hom}_{\mathbf{Z}}(\pi_*(S);Q)$. In this case dim X^p=rank $\pi_p(S)$, and the condition dim $X<\infty$ can be restated as: $\pi_p(S)$ is finite for sufficiently large p.

Henceforth we consider a fixed S of type F and denote by n the degree of its fundamental class: $H^p(\Lambda X)=0$, $p>n$. We also adopt the convention that $|x|$ denotes the degree of a homogeneous element of a graded vector space, and we work over Q as ground field.

2. THE SPACE $\pi^*_{DR}(S)$.

In [1] is shown that $X^p=0$, $p>2n-1$. Here we will show that at most one element in a homogeneous basis of X has degree $\geq n$. More precisely, let q be the largest integer with $X^q \neq 0$. According to [3], q is odd.

Theorem 1. Suppose $q>n$. The algebra $(\Lambda X,d)$ is then of the form $\Lambda X \cong \Lambda(y,x)\otimes \Lambda Z$, where:

(i) $Z^p = 0$, $p \geq n$ (ii) $|x| = q$

(iii) $dx = y^k$ (some $k \geq 2$).

Moreover, if $\bar{d}$ is the differential in ΛZ obtained from d by putting $y=x=0$ in ΛX, then $H^p(\Lambda Z,\bar{d})=0$, $p \geq |y|$.

Corollary: $H(\Lambda X,d) \cong (\Lambda y/y^k)\otimes H(\Lambda Z,\bar{d})$, as $\Lambda y/y^k$-modules.

Proof: Write $X^{odd}=P$ and $X^{even}=Q$. Define a second differential d_σ in ΛX by the conditions $d_\sigma : X \to \Lambda Q$ and $d-d_\sigma$: $X \to P.\Lambda X$. By [2], $H^p(\Lambda X,d_\sigma) = 0$, $p>n$.

Fix a homogeneous basis y_i of Q.

Since d_σ maps the indecomposable elements of $(P\otimes\Lambda Q)^q$ injectively into $(\Lambda Q)^{q+1}$ and since it maps $(P\otimes\Lambda Q)^q$ onto $(\Lambda Q)^{q+1}$ there is an indecomposable $x_1 \varepsilon (P\otimes\Lambda Q)^q$ such that $d_\sigma x_1$ has the form

(1) $$d_\sigma x_1 = y_{i_1}^{k_1}\cdot \ldots \cdot y_{i_r}^{k_r}, \quad k_\nu > 0.$$

Choose x_1 so that $|y_{i_1}|$ is minimized and so that (once y_{i_1} is fixed) k_1 is maximized.

Denote by $(\Lambda W,d')$ the differential algebra obtained from $(\Lambda X,d_\sigma)$ by dividing by y_{i_1}. We observe first that $y_{i_2}^{k_2}\cdot \ldots \cdot y_{i_r}^{k_r}=\Phi$ is not a coboundary in ΛW. Indeed, if we could write $\Phi=d'\Psi$ we would have $d_\sigma\Psi=\Phi+y_{i_1}\Phi_1$, whence $d_\sigma(x_1-y_1^{k_1}\Psi)=y_{i_1}^{k_1+1}\Phi_1$. It would follow that one of the constituent monomials of $y_{i_1}^{k_1+1}\Phi_1$ was of the form $d_\sigma v$, v an indecomposable element of $(P\otimes\Lambda Q)^q$ and this would contradict our hypothesis on x_1 above.

Now $(\Lambda W,d')$ is if the form $(\Lambda x_1,0)\otimes(\Lambda Y,d'')$. Hence $y_{i_2}^{k_2}\cdot\ldots\cdot y_{i_r}^{k_r}$ is not a coboundary in $(\Lambda Y,d'')$. In particular, if n' is the maximum degree in which $H(\Lambda W)\neq 0$,

$$n' \geq q + \sum_{\nu=2}^{r} k_\nu |y_{i_\nu}|. \tag{2}$$

On the other hand by [2; Theorem 3]

$$n' = n + |y_{i_1}| - 1. \tag{3}$$

It follows that $|y_{i_1}|>(q-n) + \sum_{2}^{r} k_\nu |y_{i_\nu}|$ and hence $k_\nu=0$, $\nu\geq 2$. We thus obtain (calling y_{i_1} simply y_1) that $d_\sigma x_1=y_1^k$ for some k.

Write $\Lambda X=\Lambda(y_1,x_1)\otimes\Lambda Z_1$. The induced projection $\rho:\Lambda X\to\Lambda Z_1$ determines a differential $\overline{d}_\sigma$ in ΛZ_1. We show now that

$$H^p(\Lambda Z_1,\overline{d}_\sigma) = 0 \qquad \text{if } p\geq|y_1| \text{ or } p\geq\tfrac{n}{2}. \tag{4}$$

Indeed if m is the maximum degree in which $H(\Lambda Z_1,\overline{d})\neq 0$ then by [2; Theorem 3]

$$n = m + (k-1)|y_1|.$$

Since $d_\sigma x_1 = y_1^k$ we have

$$q + 1 = k|y_1|$$

and these two equations imply (1).

In view of [1] we have

$$(5) \qquad Q = (y_1)\oplus Q^{<|y_1|} \quad \text{and} \quad X = (x_1)\oplus X^{<|n|} .$$

Now we show that

$$(6) \quad H^{|y_1|}(\Lambda X^{<|y_1|},d) = H^{|y_1|+1}(\Lambda X^{<|y_1|},d) = 0.$$

Because ([2]) there is a spectral sequence converging from $H(\ ,d_\sigma)$ to $H(\ ,d)$ it is sufficient to prove (6) with d_σ replacing d. Now (5) shows that the projection ρ restricts to a map $\rho_1:(\Lambda X^{<|y_1|},d_\sigma)\to(\Lambda Z_1,\overline{d}_\sigma)$ which is injective in degrees $\le|y_1|+1$ and surjective in degrees $\le|y_1|$. Thus (6) follows from (4). From (6) we may deduce an element $\Omega\varepsilon(\Lambda X^{<|y_1|}\cap P\cdot\Lambda X)^{|y_1|}$ such that $d(y_1+\Omega)=0$.

Since $X^{<n}=(y_1)\oplus Z_1$ we conclude that $H(\Lambda X^{<n},d_\sigma)\cong \Lambda y_1\otimes H(\Lambda Z_1,\overline{d}_\sigma)$, using (4). Since $q+1=k|y_1|$ it follows further from (4) that $\dim H^{q+1}(\Lambda X^{<n},d)\le 1$. Moreover, if $(y_1+\Omega)^k$ were a d-coboundary in $\Lambda X^{<n}$ then y_1^k would be a d_σ-coboundary in $\Lambda X^{<n}$, which is impossible. Hence $(y_1+\Omega)^k=dx$ for some indecomposable element x. Put $y=y_1+\Omega$ and choose an automorphism of ΛX which fixes y and

carries x to an element of X.

q.e.d.

Remark. Call $(\Lambda X, d)$ exceptional if one is in the case of Theorem 1, and ordinary otherwise. One sees easily that if $(\Lambda Z, \bar{d})$ is ordinary, then $(\Lambda X, d) \cong (\Lambda(x,y), d) \otimes (\Lambda Z, \bar{d})$. There are, however, simple examples in which $(\Lambda Z, \bar{d})$ is also exceptional and the isomorphism of the corollary cannot even be made multiplicative.

3. DIMENSION OF $H^*(S)$.

Theorem 2. $\dim H^*(S) = \dim H(\Lambda X) \le 2^n$. This inequality is sharp when S is an n-torus.

Proof: In [1] is shown that

$$\dim H^*(S) \le \prod_1^q 2b_i$$

where $2b_1 - 1, \ldots, 2b_q - 1$ are the degrees of a basis of P. Moreover it is shown there that $\sum b_i \le n$. If $b_i > 1$ then $2b_i \le 2(b_i - 1)2$ and so $\prod_1^q 2b_i \le 2^{\Sigma b_i} \le 2^n$.

q.e.d.

4. LEFSCHETZ NUMBER.

Suppose $f: S \to S$ is a continuous map. It induces $\Phi: (\Lambda X, d) \to (\Lambda X, d)$, and $H(\phi) : H(\Lambda X) \to H(\Lambda X)$ is identified with f^*, so that in particular the Lefschetz number of f is given by

$$L(f) = \sum_p (-1)^p \text{ trace } H^p(\phi).$$

To calculate $L(f)$ we extend the coefficients (by tensoring) to $\mathbb{C}$. Let ψ be the semisimple part of ϕ. It is a semisimple automorphism of ΛX and hence we can suppose it preserves X. Because ψ is the semisimple part of ϕ it is a polynomial in ϕ in each $(\Lambda X)^p$, and so commutes with d. Since ψ also preserves X it commutes with d_σ. Hence

$$L(f) = \sum(-1)^p H^p(\psi,d) = \sum(-1)^p \text{ trace } H^p(\psi,d_\sigma).$$

Choose a homogeneous bases $y_1,\ldots,y_r$ and $x_1,\ldots,x_q$ of Q and P such that $\psi y_i=\alpha_i y_i$ and $\psi x_j=\beta_j x_j$, and such that α_i $(i\le s)$ and $\beta_j (j\le t)$ are the eigenvalues distinct from 1. Putting $y_1=\ldots=y_s=0$ we arrive at a factor model $(\Lambda\overline{X},\overline{d}_\sigma)$ of the form $(\Lambda(y_{s+1},\ldots,y_r)\otimes\Lambda(x_{t+1},\ldots,x_q),\overline{d}_\sigma)\otimes(\Lambda(x_1,\ldots,x_t),0)$. The Lefschetz number of the induced endomorphism $\overline{\psi}$ is the product of the Euler characteristic χ of the first factor with $\prod_{i=1}^{t}(1-\beta_i)$.

Define a model $(\Lambda X\otimes\Lambda U,D)$ extending $(\Lambda X,d_\sigma)$ by putting $U=(u_1,\ldots,u_s)$ and $Du_i=y_i$. A spectral sequence converges from $H(\Lambda X\otimes\Lambda U)$ to $H(\Lambda\overline{X},\overline{d}_\sigma)$ and so we conclude that

$$L(f)\cdot\prod_{i=1}^{s}(1-\alpha_i) = \chi\prod_{i=1}^{t}(1-\beta_i).$$

Finally let $|y_i|=2a_i$ and $|x_i|=2b_i-1$. We can apply [2] to obtain

Theorem 3: With the notation above $q-t \geq r-s$. Moreover, $L(f)=0$ if $q-t>r-s$, and

$$L(f) = \frac{\prod_1^t (1-\beta_i) \prod_{t+1}^q b_i}{\prod_1^s (1-\alpha_i) \prod_{s+1}^r a_i}, \quad \text{if } q-t=r-s.$$

Remark: Let $\overline{\phi}$ denote the linear part of ϕ. Then $\overline{\phi}$ is the action of f on $\pi_\psi^*(S)$. If S is 1-connected this is dual to the action of $f_\#$ in $\pi_*(S)$. In this case $\alpha_i (i \leq r)$ and $\beta_j (j \leq q)$ are the eigenvalues of $f_\#$ corresponding to a basis of $\pi_*(S)\otimes\mathbb{C}$. Thus $L(f)$ can be computed from $f_\#$.

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