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STRUCTURALLY STABLE CONFIGURATIONS OF LINES OF PRINCIPAL CURVATURE

by

J.SOTOMAYOR and C.GUTIERREZ

ABSTRACT : Sufficient conditions are established for the stability of the configuration defined by the umbilical points and the families of lines of principal curvature of a compact orientable surface immersed in R^3, under small perturbations of the immersion.

1. INTRODUCTION

Directions tangent to a surface M immersed in $\mathbb{R}^3$ along which it bends extremally are called principal directions. The quantities $K \geq k$ measuring the bending are known as principal curvatures. The points on M at which K=k are called umbilical points. Outside the set $\mathcal{U}_M$ of umbilical points, the principal directions define a pair, $\mathcal{L}_M$ and ℓ_M, of line fields, mutually orthogonal called principal line fields on M; $\mathcal{L}_M$ corresponds to the directions of maximal principal curvature K and ℓ_M to those of minimal principal curvature k. The family of integral curves $\mathcal{F}_M$(resp. f_M) of $\mathcal{L}_M$ (resp. ℓ_M) is called principal maximal (resp. minimal) foliation of M, or rather of the immersion of M in R^3.

To every surface M immersed in R^3 it is associated the triple $\mathcal{P}_M = (\mathcal{U}_M, \mathcal{F}_M, f_M)$ which will be called in this paper the principal configuration of M, or rather of the immersion of M in $\mathbb{R}^3$.

The local study of principal configurations received considerable attention in the classical works of Cayley [2], Darboux [3], Picard [10], and Gullstrand [4], among others. These authors attempted to describe the principal configuration of a surface around an umbilical point. However, the arguments used in their conclusions do not always fit the standards of present day rigour, as was pointed out by Hartman and Wintner[7].

The global structure of principal configurations is known only for very rare classical surfaces: surfaces of revolution and surfaces which belong to a triply orthogonal system of surfaces [12]. In the first case the principal foliations are contained in the parallel and meridian curves and the umbilical points form meridian curves. This follows from the remark, probably due to Dupin, according to which two surfaces M_1, M_2 intersecting with constant angle along a curve which is a principal line of M_1 is also a principal line of M_2. In the case of triply orthogonal systems of surfaces, the principal foliations of a surface of one of the systems are obtained intersecting the surface with the elements of the other two systems. This result can be used to visualize the principal configuration of the ellipsoid

$$E(0) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad 0<a<b<c.$$

This is done by considering E(0) as an element of the triply orthogonal family of "confocal quadrics" $E(\lambda)$, $H_1(\lambda)$, $H_2(\lambda)$, defined by

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1,$$

with $\lambda < a^2$ for $E(\lambda)$ (ellipsoids), $a^2<\lambda<b^2$ for $H_1(\lambda)$ (hyperboloids of one sheet) and $b^2<\lambda<c^2$ for $H_2(\lambda)$ (hyperboloids of two sheets) See Fig. 1.1.

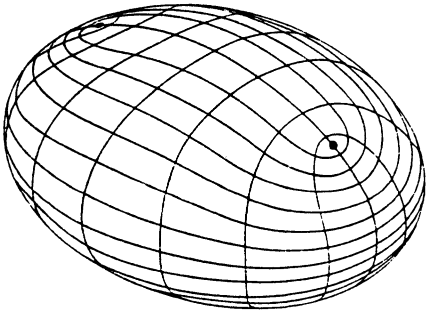


Fig.1.1.;from [12]. Principal configuration of E(0).

In connection with the global features of principal configurations one must also mention Caratheodory's Conjecture, relative to the number of umbilical points on a convex surface. It has been asserted that this number is greater than or equal to 2, for analytic surfaces [6,8]. For smooth, i.e. C^∞, surfaces the problem is open.

The study of the relation between the principal configurations and the focal set of a surface, which is the caustic set of Geometrical optics, was initiated by Gullstrand [4]. The study of the structure of the caustic set fits the context of Thom's Theory of Catastrophes. The structural stability of the focal set at umbilical points under small perturbations of the surface leads to the so called hyperbolic and elliptic umbilical points [11,13].

Recently, physicists have investigated the statistical proportion in which the umbilical points considered by Darboux in [3] as well as the hyperbolic and elliptic umbilical points appear in optical experiments [1].

This paper is devoted to the study of the global features of principal configurations on surfaces in $\mathbb{R}^3$ which remain qualitatively undisturbed under small perturbations of their immersions.

The main result of this work establishes sufficient conditions likely to be also necessary for a compact, oriented, surface M of class C^r, $r \geq 4$, in R^3, or better said for its immersion $\alpha: M \to R^3$, to have a C^s structurally stable principal configuration. This means that for any other surface $\tilde{M}$, sufficiently C^s close to M (i.e. $\tilde{M}=\tilde{\alpha}(M)$, where $\tilde{\alpha}$ is an immersion sufficiently C^s close to α), there is a homeomorphism $h: M \to \tilde{M}$ which maps $\mathcal{U}_M$ onto $\mathcal{U}_{\tilde{M}}$, maps lines of $\mathcal{F}_M$ onto lines of $\mathcal{F}_{\tilde{M}}$ and lines of f_M onto lines of $f_{\tilde{M}}$.

The sufficient conditions for principal structural stability established in this paper are expressed in terms of the umbilical points which must be of D (for Darboux)-type (See section 3), the principal cycles i.e. the compact principal lines, which must have hyperbolic Poincaré first return map (See section 4) and the assymptotic behaviour of non-compact principal lines, specially of umbilical separatrices i.e. principal lines which approach the umbilical points and which separate regions of different patterns of approach to these points.

Some aspects related to principal configurations and which bear some connection to the local part of this paper can be found in the classical works of Darboux and Gullstrand, mentioned above. However, surprisingly enough, neither in the classical nor in the current literature seem to have been focalized the study of surfaces with isolated - hyperbolic or not - principal cycles, not to mention the possibility of more intricate recurrent principal lines.

In [5] it is proved that hyperbolicity constitute the generic case for principal cycles. On the other hand, the presence of non-trivial recurrence turns to be exceptional, at least in the C^2-sense. There have also been constructed examples of non-trivial recurrent lines of principal curvature on tori and spheres immersed in R^3. The first example is related to irrational rotations, while the second happens to be of oscillatory type.

The main result of [5] is that every immersion of a compact oriented surface in R^3, can be aproximated in the C^2-topology by a C^∞ immersion which satisfies the conditions for C^s-principal structural stability, $s \geq 3$, established in this paper in theorem 2.2.4.

In section 2 below are formulated in precise terms the main results of this paper. Their proofs are given in sections 3 to 5.

2. FORMULATION OF THE MAIN RESULTS:

1. Preliminaries

Let M be a compact connected, oriented, two dimensional smooth (i.e. C^∞) manifold. An immersion α of M into R^3 is a map such that $D\alpha_p: TM_p \to R^3$ is one to one, for every $p \in M$. Denote by $\mathcal{I}^r = \mathcal{I}^r(M,R^3)$ the set of C^r-immersions of M into R^3. When endowed with the C^s-topology, $s \leq r$, this set is denoted by $\mathcal{I}^{r,s} = \mathcal{I}^{r,s}(M,R^3)$.

Associated to every $\alpha \in \mathcal{I}^r$ is defined the normal map $N_\alpha: M \to S^2$:

$$N_\alpha(p) = \alpha_u(p) \wedge \alpha_v(p) / | \alpha_u(p) \wedge \alpha_v(p) | \quad ,$$

where $(u,v):(M,p) \to (R^2,0)$ is a positive chart of M around p, $\alpha_u(p) = \frac{\partial \alpha(p)}{\partial u} = D\alpha_p(\frac{\partial}{\partial u}(p))$, $\wedge$ denotes the exterior product of vectors in R^3, determined by a once for all fixed orientation of R^3, and $| \ | = < , >^{1/2}$ is the euclidean norm in R^3.

Since $DN_\alpha(p)$ has its image contained in the image of $D\alpha(p)$ the endomorphism $\omega_\alpha: TM \to TM$ is well defined by

$$D\alpha \,.\, \omega_\alpha = DN_\alpha \quad .$$

It is well known that ω_α is a self adjoint endomorphism, when TM is endowed with metric $< , >_\alpha = \alpha^*< , >$, induced by α from the euclidean metric $<,>$ in R^3. Clearly N_α is well defined and of class C^{r-1} in M.

Let $\mathcal{K}_\alpha = \det \omega_\alpha$ and $\mathcal{H}_\alpha = -\frac{1}{2}$ trace ω_α be the <u>Gaussian</u> and <u>Mean curvatures</u> of the immersion α.

A point $p \in M$ is called an umbilical point of α if $\mathcal{H}_\alpha^2(p) - \mathcal{K}_\alpha(p) = 0$. This means that the eigenvalues of ω_α are equal at p. The set of umbilical points of α will be denoted by $\mathcal{U}_\alpha$.

Outside $\mathcal{U}_\alpha$ the eigenvalues of ω_α are distinct and given by $-\mathcal{H}_\alpha \pm \sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}$. Their opposite values $K_\alpha = \mathcal{H}_\alpha + \sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha})$ and $k_\alpha = \mathcal{H}_\alpha - \sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}$ are called respectivelly <u>maximal</u> and <u>minimal</u> <u>principal curvatures</u> of α. The eigenspaces associated to the principal curvatures define two C^{r-2} line fields $\mathcal{L}_\alpha$, mutually orthogonal in TM (with the metric $< >_\alpha$), called <u>the principal line fields of</u> . They are characterized by Rodrigues equations [12]

$$\mathcal{L}_\alpha = \{ v \in TM; \omega_\alpha v + K_\alpha v = 0\}$$

$$\ell_\alpha = \{ v \in TM; \omega_\alpha v + k_\alpha v = 0\}$$

The integral curves of $\mathcal{L}_\alpha$ (resp. ℓ_α) are called lines of maximal (resp. minimal) principal curvature. The family of such curves i.e. the integral foliation of $\mathcal{L}_\alpha$ (resp. ℓ_α) in $M - \mathcal{U}_\alpha$ will be denoted by $\mathcal{F}_\alpha$ (resp. f_α) and called the maximal (resp. minimal) principal foliation of α.

The triple $\mathcal{P}_\alpha = (\mathcal{U}_\alpha, \mathcal{F}_\alpha, f_\alpha)$ will be called the principal configuration of α.

An immersion $\alpha \in \mathcal{J}^r$ is said to be C^s-principally structurally stable, $s \leq r$, if there is a neighborhood $\mathcal{V}(\alpha)$ of α in $\mathcal{J}^{r,s}$ such that for every $\beta \in \mathcal{V}(\alpha)$ it is possible to find a homemorphism $h = h_\beta : M \to M$ such that $h(\mathcal{U}_\alpha) = \mathcal{U}_\beta$ and $h|M - \mathcal{U}_\alpha$ is a topological equivalence simultaneously between $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ and between f_α and f_β. Shortly it is said that h maps $\mathcal{P}_\alpha$ to $\mathcal{P}_\beta$ or that h is a topological equivalence between $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$.

Bellow are provided sufficient conditions, likely to be also necessary, for an immersion $\alpha \in \mathcal{J}^r$, $r \geq 4$, to be C^s-principally structurally stable, $s \geq 3$.

2. Sufficient Conditions

These conditions are expressed in terms of the umbilical points $\mathcal{U}_\alpha$, the principal cycles i.e. the compact lines of $\mathcal{F}_\alpha$ and f_α and the assymptotic behaviour of non compact principal lines, specially of umbilical separatrices.

The concept of C^s-principal structural stability of $\alpha \in \mathcal{J}^r$ at a point $p \in M$ can be formulated as follows: For every neighborhood $V(p)$ of p in M there must be a neighborhood $v(\alpha)$ of α in $\mathcal{J}^{r,s}$ such that for $\beta \in v(\alpha)$ there must be a point $q = q(\beta)$ in $V(p)$ and a homemorphism $h: W(p) \to W(q)$ between neighborhoods of p and q, which maps p to q and maps $\mathcal{F}_\alpha | W(p)$ and $f_\alpha | W(p)$ respectively onto $\mathcal{F}_\beta | W(q)$ and $f_\beta | W(q)$.

2.1. Propostion. An immersion $\alpha \in \mathcal{J}^r$, $r \geq 4$, is C^3-principally structurally stable at a point $p \in \mathcal{U}_\alpha$ provide the following condition is satisfied:

Condition D: There is a chart $(u,v):(M,p) \to (R^2,0)$ and an isometry Γ of R^3 with $\Gamma(\alpha(p)) = 0$ such that

$$(\Gamma \circ \alpha)(u,v) = (u,v; \frac{k}{2}(u^2+v^2) + \frac{a}{6} u^3 + \frac{b}{2} uv^2 + \frac{c}{6} v^3 + O((u^2+v^2)^2),$$

where

τ)$b(b-a) \neq 0$, and

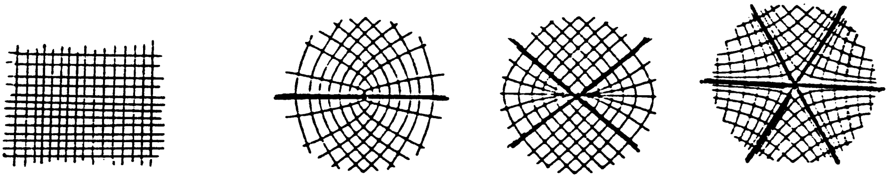
δ)either one of the following inequalities hold

$D_1 : a/b > (c/2b)^2 + 2$

$D_2 : (c/2b)^2 + 2 > a/b > 1, \quad a \neq 2b$

$D_3 : 1 > a/b$.

The local principal configurations are illustrated in Fig. 2.1. Condition D amounts to a condition given by Darboux in [3].



$p \notin \mathcal{U}_\alpha$ $\qquad D_1 \qquad D_2 \qquad D_3$

Fig. 2.1

The index i refers to the number of <u>umbilical separatrices</u> [9] of D_i, $i = 1,2,3$ in each of the foliations $\mathcal{F}_\alpha, \mathcal{G}_\alpha$. Separatrices are drawn in heavy lines in the picture.

The concept of principal structural stability can be localized at principal cycles in the following way:

An immersion $\alpha \in \mathcal{J}^r$ is C^s-principally stable at a principal cycle c if for every neighborhood V(c) of c in M there is a neighborhood $\mathcal{V}(\alpha)$ of α in $\mathcal{J}^r$ such that for $\beta \in \mathcal{V}(\alpha)$ there is a principal cycle $d = d(\beta)$, of β, contained in V(c) and a homeomorphism $h:W(c) \to W(d)$ between neighborhoods of c and d, which maps c to d and maps $\mathcal{F}_\alpha|W(c)$ and $\mathcal{G}_\alpha|W(c)$ respectively onto $\mathcal{F}_\alpha|W(d)$ and ${}_\beta|W(d)$.

<u>2.2. Proposition</u>: An immersion $\alpha \in \mathcal{J}^r$, $r \geq 3$, is C^3-principally structurally stable at a principal cycle c provided one of the following conditions, which are equivalent, is satisfied:

<u>Condition H</u>:

H_1) $$\int_c \frac{d\mathcal{H}_\alpha}{\sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}} \neq 0,$$

H_2) The cycle c is a hyperbolic cycle of the foliation to which it belongs. That is, the Poincaré first return map h associated to a transversal to c at a point q is such that $h'(q) \neq 1$.

<u>2.3. Definition</u>. Let $S^r = S^r(M)$ denote the set of $\alpha \in \mathcal{J}^r$, $r \geq 4$, such that

a) α satisfies condition D of 2.1 at every point $p \in \mathcal{U}_\alpha$

b) α satisfies condition H of 2.2 at each of its principal cycles c.
c) The limit set of every principal line of α is the union of umbilical points and principal cycles.
d) There is no umbilical separatrix of α which is a separatrix of two different umbilical points or twice a separatrix of the same umbilical point.

2.4 Main Theorem. Let $r \geq 4$ and M be a compact oriented two dimensional monifold. The set $S^r = S^r(M)$, defined in 2.3, is open in $\mathcal{J}^{r,3} = \mathcal{J}^{r,3}(M,R^3)$ and every $\alpha \in S^r$ is C^3-principally structurally stable.

In [5] it is proved that S^r is dense in $\mathcal{J}^{r,2}$. A proof of the density of S^r in $\mathcal{J}^{r,s}$, $s \geq 3$, which seems far fetched at this moment due to the difficulty in obtaining C^s approximations by immersions which verify condition c of 2.3, would lead to the actual characterization of the set of C^s-principally structurally stable immersions.

3. UMBILICAL POINTS

Here it will be studied the umbilical points of type D, defined in (2.2.1). In 1. will be shown that condition D does not depend on (u,v) or Γ. The local principal configurations will be established in 2. The proof of (2.2.1) will follow from this analysis.

1. Invariance of Condition D

Let $\alpha : (R^2,0) \to (R^3,0)$ be an immersion of class C^4, which has o as umbilical point and is of the form $\alpha(u,v) = (u,v;h(u,v))$, where h is a C^4 function of the form

(1) $h(u,v) = \frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{d}{2}u^2 v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + r(u,v)$.

where $r = O((u^2+v^2)^2)$. By means of a rotation in the (u,v)-plane, it will be assumed that d=0.

According to [3,12], the differential equation for the lines of principal curvature of α is given by

(2) $[bv + M_1(u,v)]dv^2 - [(b-a)u+cv + M_2(u,v)]dudv - [bv+M_3(u,v)]du^2 = 0.$

Each M_i, i=1,2,3, is of class C^2 and $M_i(u,v) = O((u^2+v^2))$.

Consider the case $M_i \equiv 0$ in (2).
Equation (2) can be written

(3) $bv(\frac{dv}{du})^2 - [(b-a)u+cv](\frac{dv}{du}) - bv = 0.$

Let $b(b-a) \neq 0$ i.e. τ) of Prop. 2.2.1, be satisfied. The following can be remarked:

a) From the homogeneity of (3), $\frac{dv}{du}$ is constant along any straight line passing through the origin.

b) $\frac{dv}{du} \neq 0$ along the line $\{u=0\}$.

c) The line $\{v=tu\}$ is a solution of (3) if and only if $t=0$ or $t=\tau_i$, $i=1,2$, where

$$\tau_1 = \frac{c}{2b} - \sqrt{(c/2b)^2 - a/b + 2}\ ,$$

$$\tau_2 = \frac{c}{2b} + \sqrt{(c/2b)^2 - a/b + 2}\ .$$

d) The condition $\tau_1 \cdot \tau_2 < -1$ (resp. $\tau_1 \cdot \tau_2 > -1$) means that the rays

$$\ell_1 = \{v = \tau_1 u\ ,\ u > 0\}$$

$$\ell_2 = \{v = \tau_2 u\ ,\ u > 0\}$$

determine in the semi plane $\{u > 0\}$ an angle which is greater (resp. smaller) than $\pi/2$.

In synthesis it can be said, in geometric language, that when τ) of (2.2.1.) is satisfied, then

1.1) D_1 of 2.2.1. is satisfied if and only if $\{v=o\}$ is the unique real solution of (3).

1.2) D_2 of 2.2.1. is satisfied if and only if the rays $\ell_o = \{v=0, u > 0\}$, ℓ_1 and ℓ_2 are contained in an open right angular sector, and are different from each other.

1.3) D_3 of 2.2.1. is satisfied if and only if there is no closed right angular sector which contains the three rays ℓ_o, ℓ_1, ℓ_2.

It follows that once τ) of (2.2.1.) is satisfied. The condition δ) as well as the type D_1, D_2, D_3 do not depend on the chart (u,v) or on the isometry Γ used in 2.2.1. In fact, another chart (u',v') and isometry Γ' would give another function

$$h' = \frac{k}{2}((u')^2 + (v')^2) + \frac{a'}{6}(u')^3 + \frac{b'}{2}u'(v')^2 + \frac{c'}{6}(v')^3 + \\ + 0(((u')^2 + (v')^2)^2)\ .$$

to which would be associated another equation (3') which, nevertheless, is related to (3) by a change of variables $(u,v) \to (u',v')$ which is an isometry. This isometry preserves the relations expressed in 1.1), 1.2) and 1.3), since it maps the rays ℓ_o, ℓ_1, ℓ_2 into the correspondant rays ℓ'_o, ℓ'_1, ℓ'_2 of equation (3').

On the other hand, condition τ) of (2.2.1.) also does not depend either on the chart (u,v) or the isometry Γ, since it can be proved to be the coordinate expression of the transversality condition of $j^2\alpha$ at the umbilical point to the submanifold of umbilical second order jets of immersions.

2. Local Principal Configurations

Assume the notation of 1) above. Consider the vector field

$Y = P\frac{\partial}{\partial u} + Q\frac{\partial}{\partial v}$, where

$$P = 2(bv + M_1)$$

$$Q = (b-a)u + cv + M_2 - \rho\sqrt{[(b-a)u + cv + M_2]^2 + 4(bv+M_1)(bv + M_3)}$$

where ρ is 1 or-1 according, to $b-a > 0$ or $b-a < 0$.

When $P \neq 0$, $\frac{dv}{du} = \frac{Q}{P}$ solves equation (2). Therefore Y is tangent to one of the principal foliations of α except possibly when $P = 0$.

When $M_i \equiv 0$, $i = 1,2,3$, and p is an umbilical point of type D_3, (i.e. $b-a \geqslant 0$), the phase portrait of Y is illustrated in Fig. 3.1. The points in the ray $\ell_o = \{v = 0, \ u \geq 0\}$ are singularities of Y.

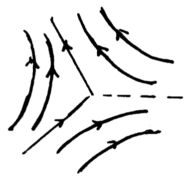


Fig. 3.1

To analyze the phase portrait of Y in general, in an angular sector which contains ℓ_o, it is convenient to perform the blowing up

$$H : \begin{cases} u = s \\ v = ts + v(s) \end{cases}$$

Here $v = v(s)$ is the unique solution of $P(s,v(s)) = 0$, with $v(0)=0$.

Since $2b = \frac{\partial P}{\partial Y}(0,0) \neq 0$, v is well defined and of class C^2, by the Implicit Function Theorem; also $v'(0) = 0$.

The map H is a diffeomorphism of the plane $s \geqslant 0$ onto the plane $u \geqslant 0$; it maps rectangles onto distorted angular sectors.

Call $Z = H_*Y = S\frac{\partial}{\partial s} + T\frac{\partial}{\partial t}$ the induced vector field

$$H_* Y\ (s,t) = DH^{-1}_{H(s,t)}\ Y(H(s,t))$$

Therefore, the functions S and T are given in the plane $s \geqslant 0$ by

$$(1) \quad \begin{aligned} S &= 2[bts + bv(s) + M_1(s,st + v(s))] \\ T &= \frac{1}{s}[- tS - v'(s)S + S_1 - \rho\sqrt{S_1^2 + SS_2}] \ , \end{aligned}$$

where

$$S_1 = (b-a)s + cst + cv(s) + M_2(s, st + v(s))$$

$$S_2 = 2[bst + bv(s) + M_3(s, st + v(s))] .$$

The following expressions hold:

$$S(s,t) = 2(b\ st + st\ U(s,t)), \text{ with } U(0,0) = 0.$$

$$(2) \quad S_1(s,t) = (b-a)s + c\ st + s\ U_1(s,t), \text{ with } U_1(0,0) = 0$$

$$S_2(s,t) = 2b\ st + s\ U_2(s,t), \quad \text{with } U_2(0,0) = \frac{\partial U_2}{\partial t}(0,0) = 0.$$

where U, U_1, U_2 are functions of class C^1.

In fact, defining

$$U(s,t) = \int_0^1 \frac{\partial M_1}{\partial y}(s, r\ st + v(s))\ dr,$$

it holds, by Hadamard's Lemma, that

$$S(s,t) = 2(b\ st + b\ v(s) + M_1(s, v(s)) + st\ U(s,t)) = \\ = 2(b\ st + st\ U(s,t)),$$

since $b\ v(s) + M_1(s, v(s)) = 0$, by definition. Write $M_2(s,st + v(s)) = M_2(s,v(s)) + st \int_0^1 \frac{\partial M_2}{\partial y}(s, r\ st + v(s))dr$. Write $V^1(s) = \int_0^1 v'(rs)\ dr$, and $M_2(s,v(s)) = s \int_0^1 \left[\frac{\partial M_2}{\partial x}(rs, rv(s)) + \frac{\partial M_2}{\partial y}(rs, rv(s))V^1(s) \right] dr$.

Therefore,

$$c\ v(s) + M_2(s,st + v(s)) = s\ [cV^1(s) + V_1^1(s) + V^1(s)V_2^1(s) + tV_3^1(s,t)],$$

where

$$V_1^1(s) = \int_0^1 \frac{\partial M_2}{\partial x}(rs, rv(s))dr ,$$

$$V_2^1(s) = \int_0^1 \frac{\partial M_2}{\partial y}(rs, rv(s))dr,$$

$$V_3^1(s,t) = \int_0^1 \frac{\partial M_2}{\partial y}(s, rst + v(s))\ dr .$$

Define

$$U_1 = c\ V^1(s) + V_1^1(s) + V^1(s)V_2^1(s) + t\ V_3^1(s,t) ,$$

Which is certainly of class C^1 in a neighborhood of $(0,0)$.

Analogous definition leads to

$$U_2 = 2bV^1(s) + V_1^2(s) + V^1(s) .\ V_2^2(s) + V_2^2(s) + tV_3^2(s,t) ,$$

where V_i^2 is obtained by the same integral expressions as V_i^1, substituting M_2 by M_3.

Let $\tilde{Z}$ be defined by $\tilde{Z} = w(s,t)\ Z = \tilde{S} \frac{\partial}{\partial s} + \tilde{T} \frac{\partial}{\partial t}$, where

$$w(s,t) = \frac{1}{ts} [- tS - v'(s)S + S_1 + \rho \sqrt{S_1^2 + SS_2}]$$

It follows that

$$\tilde{S} = S\, w(s,t)$$

(3)

$$\tilde{T} = t\left(\frac{S}{s}\right)^2 + (v'(s))^2\left(\frac{S}{st}\right)\frac{S}{s} + 2v'(s)\left(\frac{S}{s}\right)^2 - 2\left(\frac{S}{s}\right)\left(\frac{S_1}{s}\right) -$$

$$- 2\, v'(s)\left(\frac{S}{st}\right)\left(\frac{S_1}{s}\right) - \left(\frac{S}{st}\right)\left(\frac{S_2}{s}\right)$$

Using the expressions (2) it follows that $\tilde{S}$ and $\tilde{T}$ are actually restrictions of class C^1 functions in a neighborhood of (0,0), which are denoted by the same symbols. In fact,

$$\tilde{S} = 2(b+U(s,t))[-tS - v'(s)S + S_1 + (b-a)s\, R(s,t)]$$

where R(s,t) is of classe C^1 since it is defined by

$$R(s,t) = \sqrt{\left(1+\frac{c}{b-a}\, t + \frac{U_1}{b-a}\right)^2 + \frac{2t}{(b-a)^2}\,(b+U)\,(2bt + U_2)}$$

The extension of $\tilde{T}$ is given by

$$\begin{aligned}\tilde{T} &= t(2(b+U)t)^2 + (v'(s))^2(2(b+U))^2\, t\, + \\ &+ 2v'(s)\, t^2(2(b+U))^2 - 2t(2(b+U))((b-a)+ct+U_1) - \\ &- 2v'(s)(2(b+U))((b-a)+ct+U_1) - (2(b+U))\,(2bt+U_2).\end{aligned}$$

The first partial derivates of $\tilde{S}$ and $\tilde{T}$ are given by

$$\frac{\partial\tilde{S}}{\partial s}(0,0) = 4(b-a)b \quad , \quad \frac{\partial\tilde{S}}{\partial t}(0,0) = 0$$

$$\frac{\partial\tilde{T}}{\partial s}(0,0) = -\, 2v''(0)(2b)(b-a) - 2b\,\frac{\partial U_2}{\partial s}(0,0)$$

$$\frac{\partial\tilde{T}}{\partial t}(0,0) = -\, 4b(b-a) - 4b^2.$$

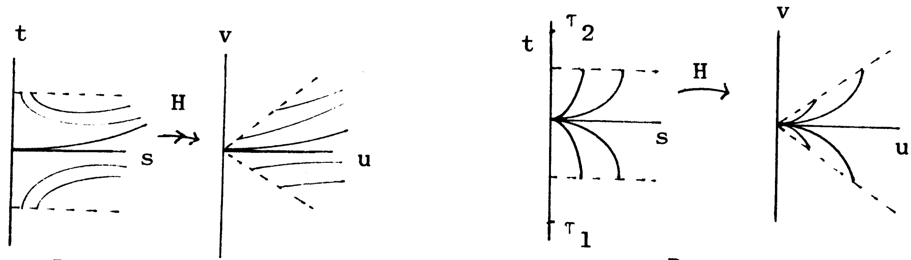
The Jacobian matrix of $\tilde{Z}$ at (0,0) is therefore given by

$$D\,\tilde{Z}(0,0) = \begin{pmatrix} 4(b-a)b & 0 \\ * & -4b(2b-a) \end{pmatrix},$$

which is hyperbolic, provided condition D is assumed.

In cases D_1 , D_3 the origin is always a saddle point of $\tilde{Z}$. In fact, the determinant of $D\tilde{Z}(0,0)$ is $\Delta = -16b^3(1-\frac{a}{b})(2-\frac{a}{b})$, which is negative since in case D_1 (resp. D_3) the two factors are always negative (resp. positive).

In case D_2, Δ is negative (resp. positive) if, with the notation of 1.c), $\tau_1 . \tau_2 > 0$ (resp. $\tau_1 . \tau_2 < 0$). This situation corresponds to taking the u-axis along one of the rays ℓ_1 or ℓ_2 (resp. along the ray ℓ_0).



D_1, D_3 D_2

Fig. 3.2.

Applying H to the orbits of $\tilde{Z}$, it follows that there is a unique principal line of class C^1 which approaches the umbilical point and is tangent to each one of lines which pass through 0 and solve (3). This is the case for types D_1 and D_3. This unique principal line is the umbilical separatrix; it is the image by H of the saddle separatrix of $\tilde{Z}$. In case D_2 the unicity holds only for the separatrices tangent to the rays ℓ_1, ℓ_2, when ℓ_o is contained in the sector determined by these rays. In fact ℓ_1 and ℓ_2 correspond by H to saddles of $\tilde{Z}$, while ℓ_o corresponds to a node. In this case there are infinitely many principal lines approaching the umbilical point and tangent to ℓ_o. See Fig. 3.2. To complete the analysis of the principal configurations under condition D, it must be proved that the only lines of principal curvature which tend to 0 are those which are tangent to one of lines which solve (3). This can be seen using the blowing up u=s, v=ts. It follows that the blown up vector field does not have singularities outside $0,\tau_1,\tau_2$. This implies that no principal line can approach the umbilical point in a direction different from $0,\tau_1,\tau_2$.

This finishes the discussion of the local principal configuration around an umbilical point which satisfies condition D and justifies Fig.2.1.

3. <u>Proof of (2.2.1.)</u>

The interpretation of condition τ) of 2.2.1 as a transversality condition together with the openness of each one of the conditions D_i, i=i,2,3, imply that the umbilical point of type D_i depends continuosly on C^3-small perturbations of immersions $\alpha \in \mathcal{J}^r$, $r \geq 4$. Also, the umbilical separatrices depend continuosly on

α, as follows from their interpretation in terms of blowing up in 2). The topological equivalence is constructed in a more global context in section 5.

4. Remarks on the literature

No proof is given in any of references given in the introduction of the unicity of umbilical separatrices for C^r immersions. The existence of such separatrices is proved in [7]. The unicity is claimed in [3] for analytic immersions; the proof is however very hard to follow.

4. PRINCIPAL CYCLES

Let $\alpha \in \mathcal{J}^r$, a compact integral curve c of $\mathcal{f}_\alpha$ (resp. $\mathcal{F}_\alpha$) is called a minimal (resp. maximal) principal cycle. When it is not relevant the minimal or maximal character of a cycle, it will be refered to simply as a principal cycle.

When the cycle c is oriented by a periodic regular parametrization $u \to c(u)$, it will be denoted by [c]. When the orientation is not relevant for an assertion the cycle will be simple denoted by c.

4.1. Proposition

Let $\alpha \in \mathcal{J}^{r \geq 4}$, and c be a minimal (resp. maximal) principal cycle of α. The following assertions are equivalent

H_1) $$\int_c \frac{d\mathcal{H}_\alpha}{\sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}} \neq 0$$

H_2) c is a hyperbolic minimal (resp. maximal) principal cycle of α.

H_3) $$\int_c \frac{dk_\alpha}{K_\alpha - k_\alpha} \neq 0 \quad \left(\text{resp. } \int_c \frac{dk_\alpha}{k_\alpha - K_\alpha} \neq 0\right)$$

The proof will follow from lemmas (4.2), (4.3) and (4.4) bellow.

4.2. Lemma:

Let $\alpha \in \mathcal{J}^r$, $r \geq 3$, for any principal cycle c of α, it holds that

$$\int_{[c]} \frac{dk_\alpha}{K_\alpha - k_\alpha} = \int_{[c]} \frac{dK_\alpha}{K_\alpha - k_\alpha} = \frac{1}{2} \int_{[c]} \frac{d\mathcal{H}_\alpha}{\sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}}$$

Proof. Follows from definition 2.1. that

$$(1) \qquad \frac{d\mathcal{H}_\alpha}{\sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}} = \frac{dk_\alpha}{K_\alpha - k_\alpha} + \frac{dk_\alpha}{K_\alpha - k_\alpha}$$

But

$$(2)\qquad \frac{dK_\alpha}{K_\alpha - k_\alpha} = d[\log(K_\alpha - k_\alpha)] + \frac{dk_\alpha}{K_\alpha - k_\alpha}$$

Therefore, for an oriented principal cycle [c],

$$(3)\qquad \int_{[c]} \frac{dK_\alpha}{K_\alpha - k_\alpha} = \int_{[c]} \frac{dk_\alpha}{K_\alpha - k_\alpha}$$

This is the first equality of the lemma. The second equality follows integrating (1).

4.3. Lemma

Let (u,v) be a chart such that $\{v=0\}$ is the minimal principal cycle c of α and u is the arc lenght parametrization of c. The derivative of the first return map h of c is given by

$$h'(0) = \exp\left\{\int_0^{\ell} \left. \frac{-\frac{\partial}{\partial v}[E_\alpha f_\alpha - F_\alpha e_\alpha]}{E_\alpha g_\alpha - G_\alpha e_\alpha}\right|_{v=0} du\right\}$$

where $E_\alpha, F_\alpha, G_\alpha$ and $e_\alpha . f_\alpha, g_\alpha$ are respectively the coefficients of the first and second fundamental forms of α in the chart (u,v) and ℓ is lenght of c.

Proof. The differential equation of the principal lines is given according to [12], by:

$$(4)\qquad \begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E_\alpha & F_\alpha & G_\alpha \\ e_\alpha & f_\alpha & g_\alpha \end{vmatrix} = 0$$

The solution $v = v(u,v_o)$ of (4) with $v(0,v_o)=v_o$ and $v(u,0)=0$, defines, making $u = \ell$ = length c, the first return map $h(v_o)=v(\ell,v_o)$ of the cycle [c].

The derivative $\eta(u) = \frac{\partial v}{\partial v_o}(u,v_o)|_{v_o=0}$ satisfies the differential equation obtained substituting $v(u,v_o)$ in (4) and derivating with respect to v_o. As follows

$$2\,\frac{\partial}{\partial v_o}\left(\frac{\partial v}{\partial u}(u,v_o)\right)\frac{\partial v}{\partial u}(u,v_o)[F_\alpha g_\alpha - G_\alpha f_\alpha] + \left(\frac{\partial v}{\partial u}(u,v_o)\right)^2 . \frac{\partial}{\partial v_o}[F_\alpha g_\alpha - G_\alpha f_\alpha] +$$

$$\frac{\partial}{\partial v_o}\left(\frac{\partial v}{\partial u}(u,v_o)\right)[E_\alpha g_\alpha - G_\alpha e_\alpha] + \frac{\partial[E_\alpha f_\alpha - F_\alpha e_\alpha]}{\partial v}\ \frac{\partial v}{\partial v_o}(u,v_o) = o$$

substituting $v_o=0$ and using the invariance of the mixed derivatives on the order of derivation, it follows that

$$\frac{d\eta}{du}[E_\alpha g_\alpha - G_\alpha e_\alpha](u,0) + \eta\ \frac{\partial}{\partial v}[E_\alpha\ f_\alpha - F_\alpha e_\alpha](u,0) = 0.$$

The solution $\eta(u)$ of this equation, with initial condition $\eta(0)=1$, gives the expression in the conclusion of the lemma, when evaluated at $u=\ell$.
Obviously $\eta(\ell) = h'(0)$.

4.3' Remark: Notice that arc lenght parametrization in 4.3 is not essential for the validity of the expression for $h'(o)$.

4.4. Lemma:

Let u be the arc length parametrization of a minimal principal curve c of $\alpha \in \mathcal{J}^r$, $r \geq 4$.

Denote by $T_\alpha(u) = D\alpha(c(u))\ c'(u)$ the unitary tangent vector of the curve $u \to \alpha(c(u))$.

The expression.

$$(5)\quad \alpha(u,v) = \alpha(c(u)) + vN_\alpha(c(u)) \wedge T_\alpha(u) + v^2\left[\frac{K_\alpha(c(u))}{2} + \right.$$

$$\left. + A(u,v)\right] N_\alpha(c(u)),$$

with $A(u,0) = 0$, defines a chart (u,v) of class C^{r-2} around c. Then,

$$(6)\quad \frac{\partial}{\partial v}[E_\alpha f_\alpha - F_\alpha e_\alpha](u,0) = \frac{d(K_\alpha \circ C)}{du}(u)$$

$$(7)\quad [E_\alpha g_\alpha - G_\alpha e_\alpha](u,0) = (K_\alpha - k_\alpha)(c(u)),$$

where E_α ,$_\alpha$,G_α and e_α, f_α, g_α are the coefficients of the first and second fundamental forms of α in the chart defined by (5).

Proof. Clearly the map

$$g:(u,v,w) \to \alpha(c(u)) + vT_\alpha(u) \wedge N_\alpha(c(u)) + wN_\alpha(c(u))$$

is a C^{r-2} local diffeomorphism onto a tube around $\alpha(c)$. The v-coordinate of the required chart is obtained composing α with a local inverse of g. The w-coordinate of this composition is precisely the coefficient of $N_\alpha(c(u))$ in (5). Therefore, $v^2A(u,v)$ is of class C^{r-2}.

The equalities (6) and (7) follow from the calculation below.

$$E_\alpha(u,0) = \langle \alpha_u, \alpha_u \rangle = 1$$

$$F_\alpha(u,v) = \langle \alpha_u, \alpha_v \rangle = \left(\frac{v^2}{2}\ \frac{d(K_\alpha \circ c)}{du}(u)\right) + \frac{\partial(v^2 A)}{\partial u}).(vK_\alpha(c(u)) + \frac{\partial(v^2A)}{\partial v}$$

$$G_\alpha(u,0) = \langle \alpha_v, \alpha_v \rangle = 1$$

$$e_\alpha(u,0) = \langle \alpha_{uu}, N_\alpha(c(u)) \rangle = k_\alpha(c(u))$$

$$f_\alpha(u,v) = \langle \alpha_{uv}, N_\alpha(c(u)) \rangle = v \frac{d(K_\alpha \circ c)}{du}(u) + \frac{\partial^2(v^2 A(u,v))}{\partial u \partial v}$$

$$g_\alpha(u,0) = \langle \alpha_{vv}, N_\alpha(c(u)) \rangle = K_\alpha(c(u)).$$

Therefore,

$$\frac{\partial}{\partial v}[E_\alpha f_\alpha - F_\alpha e_\alpha]\ (u,0) = \frac{d}{du}(K_\alpha \circ c)(u)\ ,$$

$$[E_\alpha g_\alpha - G_\alpha e_\alpha](u,0) = K_\alpha(c(u)) - k_\alpha(c(u))\ .$$

This ends the proof of the lemma.

4.5. Proof of (2.2.2.)

The hyperbolicity of the minimal principal cycle implies the local stability of ℓ_α in c, under small C^3-perturbations of α. The construction of the simultaneous topological equivalence between $\mathcal{F}_\alpha, \mathcal{f}_\alpha$ and the foliations corresponding to the perturved immersion offers no difficulty. It is done in a more global context in section 5 The same works for maximal principal cycles.

5. PROOF OF THE MAIN THEOREM

The openness of S^r in $\mathcal{J}^{r,3}$, $r \geq 4$, follows from the local stability of umbilical points and the continuity, on compact parts, under small C^3-perturbations, of the umbilical separatrices (section 3) together with the local stability of hyperbolic principal cycles (section 4). Conditions c) and d) of 2.2.3. are also essential, in this part; in fact, they make possible to reduce the analysis of the openess to the local study of umbilical points, umbilical separatrices and principal cycles.

A maximal (resp. minimal) canonical region of $\alpha \in S^r$, $r \geq 4$, is a connected component of the complement of umbilical points and maximal (resp. minimal) principal cycles and umbilical separatrices. The canonical regions can be parallel or cylindrical. In the first case the principal line field restricted to the region is topologically equivalent to $\frac{\partial}{\partial u}$ in R^2, in the second one it is topologically equivalent to $u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$ in $R^2-\{0\}$. Figures 5.1. and 5.2. show some typical examples of canonical regions.

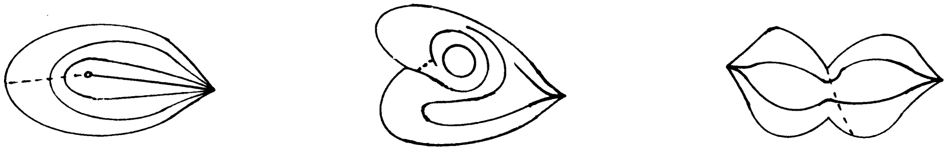


Fig. 5.1.; Parallel regions

Fig. 5.2.; Cylindrical regions

The dotted lines in the pictures of the canonical regions are cross sections of foliations in the region. For parallel regions they can be taken as umbilical separatrices of the other foliation. For a cylindrical region it can be that either all the orbits of the other foliation cross the region, like in case a), or the region contains at least one principal cycle of the other foliation like in case b). The minimal (resp. maximal) cylindrical regions of case a) are called irreducible transversal minimal (resp. maximal) canonical regions; those of case b) are decomposed into the union of a finite number of irreducible transversal maximal (resp. minimal) canonical regions and two irreducible semi-transversal minimal regions, like those in which Fig. 5.2.b. is divided by the maximal principal cycle in dotted lines. The boundary of an irreducible semi-transversal minimal (resp. maximal) canonical regions is the union of one minimal cycle to which the minimal principal cycles tend and one maximal principal cycle, to which the minimal foliation is transversal. In Fig. 5.2.b) appears a cylindrical region decomposed into one irreducible transversal and two semi-transversal canonical regions.

It can be found a neighborhood $\vartheta(\alpha)$ in $\mathcal{J}^{r,3}$ such that $\vartheta(\alpha) \subset S^r$, $r \geq 4$, and such that along a continuous arc α_t, $t \in [0,1]$, in $\vartheta(\alpha)$ joining $\alpha = \alpha_o$ to $\beta = \alpha_1$, there is a unique way to continue the umbilical points, principal cycles and umbilical separatrices as well as their intersections (of maximal with minimal elements) in such a way that there is a natural unique continuation of the canonical regions of α_o into those of α_t, which defines a one-to-one correspondence between the canonical regions of α and those of $\beta \in v(\alpha)$. Such a correspondence preserves the type of the canonical regions.

The continuation procedure defines uniquely a partial topological equivalence h_t between the umbilical points of α and α_t and the set of points which are simultaneously on a maximal and minimal separatrix or principal cycle of α with the similar set of

α_t, $t \in [0,1]$.

At this point it is possible, using the "method of canonical regions" [9] to produce different topological equivalences $(\mathcal{U}_\alpha, \mathcal{F}_\alpha) \to (\mathcal{U}_{\alpha_t}, \mathcal{F}_{\alpha_t})$ and $(\mathcal{U}_\alpha, f_\alpha) \to (\mathcal{U}_{\alpha_t}, f_{\alpha_t})$, which extend h_t. Below it is indicated how to proceed in order to extend h_t to a topological equivalence $\bar{h}_t$ between P_{α_o} and P_{α_t}. This extension is obtained by means of a sequence of partial extensions.

1) Definition of $\bar{h}_t$ on the intersection of maximal and minimal parallel canonical regions. On each minimal parallel canonical region A of α take a cross section S, which can be chosen on a maximal umbilical separatrix. The extremes, a and b, of S have natural continuations $h_t(a)$, $h_t(b)$; these points define the extremes of the natural continuation S_t of S; S_t is on the maximal umbilical separatrix of α_t which is the continuation of the separatrix of α which contains S. Define $\bar{h}_t : S \to S_t$ to be any extension of h_t : $\{a,b\} \to \{h_t(a), h_t(b)\}$.

Analogous definition of $\bar{h}_t : T \to T_t$ is made on segments T on minimal separatrices which are cross sections of maximal parallel regions B of α.

On each connected component C of the intersection $A \cap B$, define $\bar{h}_t : C \to C_t$, where C_t is the natural continuation of C. Notice that h_t is defined in the corners of C which are either umbilical points or intersections of maximal and minimal separatrices of α. On a point p of C define $\bar{h}_t(p)$ as the point in C_t which is on the intersection of the minimal principal line of α_t which passes through $h_t(\sigma(p)) \in S_t$ with the maximal principal line which passes through $\bar{h}_t(\tau(p)) \in T_t$. Here $\sigma(p)$ (resp. $\tau(p)$) is the point of intersection of the minimal (resp. maximal) principal line through p writh S (resp. T). This procedure defines a homeomorphism which is a topological equivalence between P restricted to the union of the closures of the intersections of parallel minimal regions with parallel maximal regions of α, and the correspondent objects of α_t which are their natural continuations.

Notice that the extension $\bar{h}_t$ defined above is already a topological equivalence between P_α and P_{α_t} in the case in which $\alpha \in S^r$ does not have principal cycles.

2) Definition of $\bar{h}_t$ on maximal (resp. minimal) cylindrical canonical regions of type a) which are contained in the union of minimal (resp. maximal) parallel regions.

Let R be such a region (Fig.5.2., a). Notice that $\mathfrak{h}_t$ is already defined on ∂R by the procedure described in 1), using $\mathfrak{h}_t$ on the segments S. This also defines a one-to-one correspondence between the lines ℓ of $f_\alpha|R$ and those lines ℓ_t of $f_{\alpha_t}|R_t$, where R_t is the natural continuation of R. Take an arc $\mathcal{T}_o$ of $f_\alpha|R$ and its correspondent $\mathcal{T}_t$ of $f_{\alpha_t}|R_t$. The foliation $\mathfrak{F}_{\alpha_t}|\bar{R}_t$ defines a Poincaré map $\pi_t:\mathcal{T}_t \to \mathcal{T}_t$, with only two fixed points, one attractor and one repellor, in the extremes of $\mathcal{T}_t$. Take a topological conjugation $\theta_t:\mathcal{T}_o \to \mathcal{T}_t$, between π_o and π_t, that is $\theta_t^{-1}\circ\pi_t\circ\theta_t = \pi_o$ Define $\mathfrak{h}_t|\mathcal{T}_o = \theta_t$. Extend $\mathfrak{h}_t$ to $R-\{\mathcal{T}_o\}$ so that an arc $A_o=\overrightarrow{a,\pi_o(a)}$ of $\mathfrak{F}_\alpha|(R-\mathcal{T}_o)$ is mapped onto the arc $A_t=\theta_t(a),\theta_t\circ\pi_o(a)$ of $\mathfrak{F}_{\alpha_t}|(R_t-\mathcal{T}_t)$. The map on the arc itself is given by the construction of 1) on the minimal parallel canonical regions.

3) Definition of $\mathfrak{h}_t$ on the maximal (resp. minimal) cylindrical canonical regions of type a) which are contained in minimal (resp. maximal) cylindrical canonical regions of type b).

Take a minimal cylindrical canonical region R_o of type b) for α_o and a homeomorphism $\theta_t:C_o \to C_t$, where C_o is a minimal principal cycle of α_o contained in R_o and C_t is its natural continuation. The homeomorphism θ_t defines a one-to-one correspondence between the lines ℓ_o of $f_\alpha|R_o$ and the lines ℓ_t of $f_\alpha|R_t$, where R_t is the natural continuation of R_o. Now define $\mathfrak{h}_t$ on the maximal cylindrical canonical regions of type a) contained in R_o following the procedure defined in 2), conjugating first the Poincaré maps.

4) Definition of $\mathfrak{h}_t$ on the irreducible semi-transversal regions which are contained in minimal (resp. maximal) cylindrical canonical regions of type b).

Let R_o be such a semi-transversal region contained in a minimal cylindrical region of type b). Notice that the minimal principal cycle of ∂R_o is contained either in the union of maximal parallel regions or in a maximal cylindrical canonical region of type b). This implies that $\mathfrak{h}_t$ by construction in 2) and 3) must be already defined in ∂R_o. Let p_o (resp. P_o) be a point on the minimal (resp. maximal) principal cycle of ∂R_o. It may certainly be assumed that $\mathfrak{h}_t(p_o)=p_t$ (resp. $\mathfrak{h}_t(P_o)=P_t$) depends continuously on (t,p_o) (resp. on (t,P_o)). Let ℓ_t (resp. L_t) denote the element of f_{α_t} (resp. $\mathfrak{F}_{\alpha_t}$) passing through p_t (resp. P_t), then for any $s_o \in \ell_o \cap L_o$ there is a unique point $s_t \in \ell_t \cap L_t$ which is the natural continuation of s_o. Define $h_t(s_o) = s_t$.

5) Definition of h_t on the maximal (resp. minimal) cylindrical canonical regions which intersect minimal (resp. maximal) principal cycles.

Let R_o be such a region, for α_o, which intersects the minimal (resp. maximal) principal cycle C_o. Take a connected component $\mathcal{I}_o$ of $C_o \cap R_o$. Define $h_t|\mathcal{I}_o$ as in 2), using the Poincaré map. That is, if R_t and $\mathcal{I}_t$ are the corresponding natural continuations of R_o and $\mathcal{I}_o$. The foliation $\mathcal{F}_{\alpha_t}|R_t$(resp. $f_{\alpha_t}|R_t$) defines a Poincaré map $\pi_t = \mathcal{I}_t \to \mathcal{I}_t$. Take a topological conjugation $\theta_t: \mathcal{I}_o \to \mathcal{I}_t$ between π_o and π_t and define $h_t|\mathcal{I}_o = \theta_t$.

Now, take a maximal cylindrical canonical region R_o. Notice that R_o can intersect only minimal cylindrical canonical regions which are of the type being treated now. Define h_t on a connected component of the intersection of R_o with a minimal parallel canonical region by the same procedure defined in 2). Now, let R_o^* be a minimal cylindrical canonical region which intersects R_o. Denote by ℓ_o^* the global cross section to the minimal foliation of R_o^* on which h_t has already been defined. Given $p \in (R_o - \ell_o) \cap (R_o^* - \ell_o^*)$ there exists a unique arc A_p (resp. a_p) of $\mathcal{F}_\alpha|R_o$ (resp. $f_\alpha|R_o^*$) with endpoints in ℓ_o (resp. ℓ_o^*) containing p. The conjugations $\theta_t: \ell_o \to \ell_t$ and $\theta_o^*: \ell_o^* \to \ell_t^*$ determine uniquely the images $h_t(a_p)$ and $h_t(A_p)$. Define $h_t(p) = h_t(a_p) \cap h_t(A_p) \cap S_t$, where S_t denotes the natural continuation of the connected component S_o of $(R_o - \ell_o) \cap (R_o^* - \ell_o^*)$ which contains p.

This ends the proof of Theorem 2.4.

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