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STABILITY OF DEGENERATE FIXED POINTS OF ANALYTIC AREA PRESERVING MAPPINGS

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Abstract. It is well known that hyperbolic points of a germ (at 0) of a $\mathbb{R}$-analytic diffeomorphism leaving 0 fixed, preserving the Lebesgue measure and orientation (from now on analytic APM on this paper),T, are unstable. As a Corollary of Moser's twist theorem the elliptic ones are stable provided the eigenvalues λ of DT at the fixed point are not a k-th root of the unity, $k=1,2,\dots,2p+2$ (for shortness $k=1\div 2p+2$), $p\geq 1$, and any one of the first p coefficients of the Birkhoff normal form is non-zero. To end the study of the stability of fixed points we study the parabolic or degenerate case. Elliptic points for which stability can not be decided using directly Moser's theorem(specially if λ is a 3^{rd} or 4^{th} root of 1) can be reduced to the parabolic case taking a suitable power of T. The main result is that a degenerated fixed point of an analytic APM is stable if and only if the generating function of T, with the part which generates the identity suppressed, has a strict extremum at the fixed point. Some examples and comment are included.

§1. Introduction. The stability (in the sense of Lyapunov) of fixed points of analytic APM is a method usually employed for the study of the qualitative properties of periodic orbits in hamiltonian systems with two degrees of freedom. When the fixed point, that we take always as the origin, is degenerated or parabolic, i.e., the eigenvalues of the differential DT of the mapping T at the fixed point are ± 1, the stability is a more subtle question. As we shall see it is not always enough to consider only the lower degree nonlinear terms to decide about stability. If the fixed point is elliptic with eigenvalues λ, $\bar{\lambda}$, the cases $\lambda^3=1$, $\lambda^4=1$ do not allow an easy application of Moser's twist theorem. Difficulties can appear for every λ, k-th root of the unity provided that all the determined coefficients (the first $[(k-2)/2]$ ones) in the Birkhoff normal form are zero. In fact we can find examples of instability for every k (see [10] §31).

From the numerical point of view a complete survey for quadratic APM was initiated by Hénon [5] and completed in [11]. The only nontrivial maps of this family with elliptic or parabolic fixed points can be reduced to one of the following forms [11] :

a) $(x,y) \to (x\cos\alpha - (y-x^2)\sin\alpha,\ x\sin\alpha + (y-x^2)\cos\alpha)$, $\alpha \in (0,\pi)$. These maps are the composition of a de Jonquières map $(x,y) \to (x, y-x^2)$ and a rotation R_α and the origin is an elliptic point.

b) $(x,y) \to (x+y, y+(x+y)^2)$. The origin is a double fixed point of parabolic type.

c) $(x,y) \to (-(x+y), -y-(x+y)^2)$. The origin is a fixed point of parabolic type.

It follows from the figures in [5], [11] that in case a) for $\alpha = \pi/2$ one has stability and for $\alpha = 2\pi/3$ instability. For cases b) and c) we get instability and stability, respectively.

In [2] Chirikov and Izraelev study the behavior of the iterates of the map $(x,y) \to (x-y^3, x+y-y^3)$. The origin is parabolic. However it seems that there is a region of bounded motion reaching the point $(0.52\ ,\ 0)$. The parabolic point has stable character surrounded by invariant curves. The point $(1,0)$ is 6-periodic and $DT^6(1,0) = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$. By simulation it seems that there are stable islands near the orbit, despite the parabolic character of the points.

A criterion due to Levi-Civita assures that for maps $T : (x,y) \to (x+f(x,y), x+y+g(x,y))$ where f and g begin with terms of second or larger order, the origin is unstable if the coefficient of y^2 in f is non-zero [7]. However this criterion tells nothing about cases b) and c) of quadratic maps or about the Chirikov-Izraelev map, that prompt us for a theorem.

A question related to the stability of parabolic points is the study of the stability of some second order finite difference equations. Let E be the shifting operator: $Ex_n = x_{n+1}$. Then the equations $(E-2+E^{-1})z_n = f(z_n)$ and $(E+2+E^{-1})z_n = f(z_n)$ are equivalent to the maps $(x,y) \to (x+f(x+y), x+y)$ and $(x,y) \to (-x+f(x-y), x-y)$. For instance, if $T(x_n,y_n) = (x_{n+1}, y_{n+1})$, $n \in \mathbb{Z}$, for the first map we have $x_n = x_{n-1} + f(y_n)$ and $x_n = y_{n+1} - y_{n-1}$. From this the relation $y_{n+1} - 2y_n + y_{n-1} = f(y_n)$ follows. If f is an analytic function without zero or first order terms, the origin is a parabolic point.

The case λ k-th root of the unity is reduced to the parabolic one taking T^k instead of T. Without loss of generality we can suppose that in the parabolic case the eigenvalues are equal to one. (Take T^2 if necessary. This accounts also for T orientation reversing). However see later for some direct applications to the elliptic case and for the case with -1 eigenvalues.

Two cases appear : diagonal and non diagonal linear part. In [12] the following results are proven for the second case :

1.1.Lemma. *Let* $T(x,y)=(x+f(x,y),x+y+g(x,y))$ *be an analytic APM with* f,g *beginning with terms of degree at least two. Then, for each positive integer* n, *there exists a near the identity polynomial change of variables* C *such that the transformed mapping* $T^*=C^{-1}TC$ *is given by* $T^*(x,y)=(x+F_n(x+y)+O_{n+1},x+y+O_{n+1})$ *where* F_n *is a degree* n *polynomial without linear terms and* O_s *stands for a series with terms of lower degree at least* s.

1.2.Theorem. *In the hypothesis of the Lemma, let* $F_n(z)=a_m z^m+O_{m+1}$, $a_m\neq 0$. *Then the origin is stable under* T^* *(and therefore under* T*) if and only if* m *is odd and* $a_m<0$.

Our main objective is to give a theorem characterizing the stable parabolic points for the first case. Let $(x',y')=T(x,y)$ a canonical mapping (for dimension two, canonical is equivalent to APM). If $D_y y'$ is regular (that is the case if T is near the identity, i.e., $T(x,y)=(x+O_2,y+O_2)$) we can define an analytic generating function (see [1]) $\hat{G}(x,y')$ such that $\hat{G}(x,y')=xy'+G(x,y')$ and $x'=D_{y'}\hat{G}$, $y=D_x\hat{G}$. For the non-diagonal case of 1.2. we get $G(x,y')=-x^2/2+\int^{y'}F_n(u)du+O_{n+2}(x,y')$. Theorem 1.2. can be reformulated as: stability is equivalent to $G(x,y)$ having a strict extremum at the origin. We shall prove that this characterization is applicable to the diagonal case. We state the main result.

1.3.Theorem. *Let* O *be a parabolic fixed point of an analytical APM*, T, *and* $\hat{G}(x,y')=xy'+G(x,y')$ *a generating function for* T. *Then* O *is Lyapunov stable if and only if* G *has a strict extremum at* O.

We end this section giving some explicit results for the nondiagonal case with eigenvalues -1. In section 2 we prove some preliminary results concerning the conditions to be fulfilled by the Newton polygon associated to G and showing the twist character of an auxiliar mapping T_1. Section 3 is devoted to the proof of 1.3. We end with some examples with third and fourth roots of the unity as eigenvalues. Stable and unstable mappings are displayed in both cases. Some remarks are added concerning the stable and unstable invariant branches in the unstable case and a better method for obtaining T_1 which allows for more accurate estimates of the branches and for the unification of the proofs of 1.2. and 1.3.

As far as instability is concerned the results obtained here extend the ones obtained by McGehee [8] but only for the conservative case. A short account of the statements of this paper was given in [13].

Now we come directly to the case $(x,y) \to (-x+f(x,y), x-y+f(x,y))$. Using the method of [12] for the proof of Lemma 1.1. with minor modifications, we can find the following result.

1.4. Lemma. Let $T(x,y)=(-x+f(x,y), x-y+g(x,y))$ be an analytic APM with f,g beginning with terms of degree at least two. Then, for each positive integer n, there exists a near the identity polynomial change of variables C such that the transformed mapping $T^*=C^{-1}TC$ is given by $T^*(x,y)=(-x+F_n(x-y)+O_{n+1}, x-y+O_{n+1})$.

Let us suppose $F_n(x-y)=a(x-y)^r+b(x-y)^s+c(x-y)^t+\ldots$, where $2\leq r<s<t<\ldots$, $a,b,c,\ldots$ different from zero. Then, removing the terms O_{n+1} (of order as large as desired) we have $x'=-x+F_n(y')$, $y'=x-y$, $x''=-x'+F_n(y'')$, $y''=x'-y'$, where $T(x,y)=(x',y')$, $T(x',y')=(x'',y'')$. Let $z=x-y$, $z^*=x+y$. From $z^*=-z+F_n(z)$ we get the inverse function $z=-z^*+Q(z^*)$. Then the desired function G is given by $G(x,y'')=x^2-\int Q(z^*)dz^*+\int F_n(y'')dy''$. We need some approximate expression of $Q(z^*)$. For our purpose it is enough to take $Q(z^*)=a(-z^*)^r+b(-z^*)^s+(-z^*)^t+\ldots$. Therefore

$$G(x,y'')=x^2+(-1)^{r+1}a(x+y'')^{r+1}/(r+1) + (-1)^{s+1}b(x+y'')^{s+1}/(s+1) + $$
$$+ (-1)^{t+1}c(x+y'')^{t+1}/(t+1) + \ldots + ay''^{r+1}/(r+1) + by''^{s+1}/(s+1)$$
$$+ cy''^{t+1}/(t+1) + \ldots + (-1)^{2r}a^2(x+y'')^{2r}/2 + \ldots .$$

If r is odd the dominant terms in the Newton polygon of G are $x^2+2ay''^{r+1}/(r+1)$, but for even r we must take into consideration the terms $x^2-axy''^r + (1+(-1)^{s+1})by''^{s+1}/(s+1) + (1+(-1)^{t+1}cy''^{t+1}/(t+1) + + \ldots +a^2y''^{2r}/2$. We can state the following result.

1.5. Corollary. Under the hypothesis of 1.4. we have the following character concerning the stability of the origin :

- r odd : $a>0$ ($a<0$) stable (unstable).
- r even: (*) $s\geq 2r$ stable.
  - $s=2r-1$ $4b/r+a^2>0$ (<0) stable (unstable).
  - $s<2r-1$: s odd $b>0$ ($b<0$) stable (unstable).
    - s even : replace $b(x-y)^s$ by the next term and go to (*).

For instance, the map $(x,y)\to(-x+(x-y)^2, x-y)$ which is equivalent to the quadratic case c) is stable, but a modification like $(x,y)\to(-x+(x-y)^2-(x-y)^3, x-y)$ turns out to be unstable.

The missing case in 1.5., i.e., r even, $s=2r-1$, $4b/r+a^2=0$, requires a more detailed study (see §2). If there is some term $c(x-y)^t$ in F_n after the (present) term $b(x-y)^s$ such that t is odd and less than 3r-2, we take the term of lower degree with this property. Then $c>0$ ($c<0$) implies stability (instability). If there is no term with this property the fixed point is unstable. For instance, consider the mapping $T_1(x,y)=(-x,x-y)$ and the successive modifications $T_2(x,y)=(-x+(x-y)^4,x-y)$, $T_3(x,y)=(-x+(x-y)^4-(x-y)^7,x-y)$, $T_4(x,y)=(-x+(x-y)^4-(x-y)^7+(x-y)^9,x-y)$. Under T_i, $i=1,2,3,4$ the origin is respectively unstable, stable, unstable and stable. The addition of new terms of higher order does not modify the character of the origin.

§2.Preliminary results. Let us suppose that a real analytic function G of two variables $G(x,y)$ has a strict minimum at the origin and $G(0,0)=0$ (the case of a strict minimum being similar). It is clear that $G(x,0)$ and $G(0,y)$ can not be identically zero, i.e., if $G(x,y)=\sum a(m,n)x^m y^n$, there are terms $a(m,0)$, $a(0,n)$ different from zero. There are additional conditions on the Newton polygon and on the coefficients of G (see [15], Ch.IV, § 3).

2.1.Proposition. A necessary and sufficient condition in order that $G(x,y)$ has a minimum at the origin is that the Newton polygon of G satisfies :

a) Every vertex has even coordinates. The associated coefficient is positive.
b) Let (m_k,n_k), $m_k=m+kr$, $n_k=n-ks$, $r,s\in\mathbb{Z}_+$, g.c.d.$(r,s)=1$, $k=0\div 2q$ be the points on one of the sides of the polygon. Then the function $g(t)=\sum a(m_k,n_k)t^k$ has no real zeros of odd multiplicity.
c) If g has a zero of even multiplicity, three cases are possible, associated to each one of such zeros: $y=O(x)$; $x=O(y^p)$, $p>1$; $y=O(x^p)$, $p>1$. The first and second cases can be reduced to the third one through a rotation or a relabelling of the axes, respectively. Therefore, we can suppose $y=mx^{i/j}+\dots$, $i/j>1$. Introducing $x=u^j$, $y=mu^i+z$ we get a new Newton polygon and we start again the process of checking the conditions for the terms of the form $z=O(u^t)$, $t>j$.

Proof. If we have a vertex in the Newton polygon of G with at least one odd coordinate, selecting the quadrant in a suitable form we have some curve $y=mx^{i/j}$ such that the dominant term on it is negative. If both coordinates are even but the coefficient is negative than G is locally negative along the curves $y=mx^{i/j}$ for which this term is dominant. This ends a).

Now we go to the sides of the polygon. If for one side g(t) (see condition b)) has a zero of odd multiplicity, as we have that the dominant terms are found on curves of the type $y=zx^{r/s}$, they are $x^{(sm+rn)/s} z^n g(t)$, with $t=z^{-s}$. Then for curves whose t is near the odd zero of g, G(x,y) changes sign, against the hypothesis of being a strict minimum. It is clear that if g has a zero of even multiplicity we must check the subdominant terms as explained in condition c).

In order to prove the sufficiency let us suppose that for each side of the Newton polygon there are no real zeros of the associated function g(t). (If there are zeros of even multiplicity but in some of the next steps we have condition a) satisfied and no real zeros on the new sides, the proof can be obtained through an easy modification of the following argument). Then the curves $y=mx^q$, $q\in Q$, $m\in R$ and x=0 cover all the possible approaches to the origin (in fact all but a finite number of q are associated to the verteces of the polygon). Over each one of those curves G is positive definite. Therefore, given a value of h small enough, there is locally one point on each one of the curves for which the value of G equals h. The curve formed with all the points obtained in this way encloses the origin. This is true for h in a set $(0,h_o)$. Therefore the origin is a strict minimum. By the way this proves that G(x,y)=h, $0<h<h_o$ defines a closed curve and there is only one component near 0.

Remark 1. If G changes sign in a neighborhood of the origin, the branches of G(x,y)=0 can be easely obtained studying the odd zeros of the sides of the Newton polygon (or the possible odd zeros of the new polygons if the original one has zeros of even multiplicity). ([6] ChX,1.2)

Remark 2. The procedure a),b),c) of 2.1. has cases without stop. They correspond to a non strict minimum but to a minimum. For instance, if f(x) is a real analytic function with f(0)=0, take $G(x,y)=(y-f(x))^2$. Then 0 is not an isolated singularity. If G has an isolated singularity at the origin in $\mathbb{C}^2$ then we claim that the fact that 0 is a strict minimum can be decided in a finite number of steps. To prove this claim it is enough to use the Weierstrass preparation theorem [6,Ch.I, 14.1]. Then, having an isolated singularity at 0, $G(x,y)=(y^p+a_1(x)y^{p-1}+\dots+a_p(x))E(x,y)$, where $a_i(x)$ is a polynomial in x and E(x,y) is a unit (E(0,0)=0). To decide about the strict minimum character it is enough to study $y^p+\dots a_p(x)$ and this has a finite number of a(m,n)=0.

Remark 3. In [8] McGehee considers maps $(x,y)\to(x-x^n+yP_{n-1}+O_{n+1}, y+nyx^{n-1}+y^2Q_{n-2}+O_{n+1})$ that for the conservative case give $G(x,y')=-x^ny'+y'^2R_{n-1}+O_{n+2}$ where P,Q,R are polynomials of the showed degree and $n\geq 2$. The vertex (n,1) in the Newton polygon implies instability according to 1.3. and 2.1. There are several cases of instability for which McGehee criterion does not apply and we must use 1.3. For instance, take $T(x,y)=(x-4(y-6x^5)^3, y-6x^5)$, where $G(x,y')=x^6-y'^4$. There are two tangent manifolds $y=x^{3/2}+\ldots$ (stable) and $y=-x^{3/2}+\ldots$ (unstable).

Let us suppose now that $G(x,y)=h$ defines closed curves around the origin for small values of h (of some sign, for instance, positive). Let T_1 be the time unit flow associated to the hamiltonian system with hamiltonian G : $T_1(x,y)=(\bar{x},\bar{y})$. Let U be a neighborhood of the origin. We intend to use T_1 as an approximation of T in U. In $U-\{0\}$ we define $r=G(x,y)$, $s=2\pi t/T(r)$, where T(r) is the period of the flow of hamiltonian G along the closed curve $w=\{G(x,y)=r\}$. Here t stands for the time interval in going from $(x_o,0)$ to (x,y) along w, with $x_o>0$. In the (r,s) variables one has $T_1(r,s)=(r,s+2\pi/T(r))$. Now we consider T_1 associated to G as defined in §1 for T parabolic with linear diagonal terms at the origin.

2.2.Lemma. The mapping T_1 is a twist.

Proof. The only thing to prove is $dT(r)/dr\neq 0$ if r is small enough. We claim that the curves $G(x,y)=r$ are star shaped with respect to the curves $y=zx^{u/v}$ if r is small. We can select values of z, $u/v\geq 1$ such that on the curves $y=zx^{u/v}$ the dominant term of G is of the form $x^{(mv+un)/v}z^n g(z^{-v})$ (see 2.1.b)) when this curve cuts w if r is small. Therefore we have $ax^b+O(x^c)=r$ with $b<c$ and $a>0$. A similar expression is obtained if for some z the function g is zero and we must use subdominant terms. Then we get locally only one value of x (other values are relatively as far as desired if r is small). In fact we have $x=O((r/a)^{1/b})$ with $b\geq 4$. We can exchange x and y if $u/v<1$. Using compactness we get a value r_o of r such that for all $r\leq r_o$ the claim is true.

Let S(r) the area enclosed by the curve w. For the period T(r) one has $T(r)=dS/dr$. One needs d^2S/dr^2. We compute S using slices of the type regions comprised between curves $zx^{u/v}$ and $(z+\Delta z)x^{u/v}$. For instance, for the sector where $y=O(x)$ one has $\rho=O(r^{1/b})$ where $\rho^2=$

x^2+y^2. The contribution of this sector is $\int_{\theta_1}^{\theta_2} \rho^2(\theta)\,d\theta = O(r^{2/b})$. For the other sectors we get similar terms with a larger value of b. Through integration we obtain $S=mr^n(1+o(1))$ with $n \leq 1/2$. Therefore $d^2S/dr^2 = O(r^{n-2})$ and in fact it is negative, showing that $T(r)$ goes to infinity when r goes to zero and proving the Lemma.

<u>§3. Proof of the main theorem</u>. In order to compare T and T_1 we obtain an approximate expression for T_1. We have $\bar{x}=x+\dot{x}+\ddot{x}/2+\ldots$, $\bar{y}=y+\dot{y}+\ddot{y}/2+\ldots$ and $\dot{x}=G_2$, $\dot{y}=-G_1$, $\ddot{x}=G_{12}G_2-G_{22}G_1$, $\ddot{y}=-G_{11}G_2+G_{12}G_1$, where G_i is the partial derivative of G w.r.t. the i-th argument and G_{ij}, G_{ijk},... the second, third,...partial derivatives.

Then

$$\bar{x}=x+G_2+(G_{12}G_2-G_{22}G_1)/2+\ldots \quad , \quad x'=x+G_2-G_{22}G_1+\ldots$$

$$\bar{y}=y-G_1+(G_{12}G_1-G_{11}G_2)/2+\ldots \quad , \quad y'=y-G_1+G_{12}G_1+\ldots \quad , \text{ and therefore}$$

$$\Delta = \begin{pmatrix} \bar{x}-x' \\ \bar{y}-y' \end{pmatrix} = \begin{pmatrix} (G_{12}G_2+G_{22}G_1)/2+\ldots \\ (-G_{11}G_2-G_{12}G_1)/2+\ldots \end{pmatrix} = \begin{pmatrix} O(G_{12}G_2, G_{22}G_1) \\ O(G_{11}G_2, G_{12}G_1) \end{pmatrix} .$$

Let us compute the difference between T_1 and T in the r,s coordinates.

$$r'=G(x',y')=G(\bar{x},\bar{y})+G_1(x'-\bar{x})+G_2(y'-\bar{y})+\ldots=$$

$$=r+G_1(-G_{22}G_1-G_{12}G_2)/2+G_2(G_{11}G_2+G_{12}G_1)/2+\ldots=r+O(r^2),$$

because from terms x^my^n, $m,n>0$, m,n even, we get perturbations of the order $x^{3m-2}y^{3n-2}$.

In order to see the relative variation s_Δ of s with respect to s due to the difference between the images of (x,y) under T_1 and T, we introduce the velocity $v=(G_2,-G_1)^T$. Then we have $s_\Delta=(\Delta,v)/(v,v)$ where (,) denotes the inner product on the plane. By substitution we have

$$s_\Delta = \tfrac{1}{2}(G_{12}(G_2^2+G_1^2)+G_1G_2(G_{22}+G_{11}))(1+o(1))/(G_1^2+G_2^2) .$$

If the current dominant terms are x^my^n, $m,n>0$, m,n even, we obtain $s = O(x^{m-1}y^{n-1}) = (r^d)$, $d \geq 1/2$. Therefore T is a relatively small perturbation of a twist. Then we can use the Moser's perturbed twist theorem (see [10], §32). This produces the existence of invariant curves and hence the stability.

If G has not a strict minimum at the origin there are curves reaching the origin. If the multiplicity of these curves is one we have hyperbolic sectors (see [6] ChX) and therefore instability for T_1 and hence for T. If there are branches with multiplicity larger than one (then the origin is not an isolated singular point of the hamiltonian field associated to G) on the left and right neighborhoods of these branches the flow approaches or leaves any (sufficiently small) neighborhood of the origin on both sides (even multiplicity) or approaches on one side and leaves on the other (odd multiplicity). In any case we get instability. This ends the proof of 1.3.

Remark 1. A better choice of the hamiltonian can produce maps T_1 more faithful to T. For instance we can take

$$H=G-\frac{1}{2}G_1G_2+\frac{1}{12}(G_{11}G_2^2+4G_{12}G_1G_2+G_{22}G_1^2)-\frac{1}{2}(G_{112}G_1G_2^2+G_{122}G_2G_1^2+G_{11}G_{12}G_2^2+$$
$$+G_{22}G_{12}G_1^2+G_{11}G_{22}G_1G_2+3G_{12}^2G_1G_2)+\dots\ .$$

Taking the first two terms of H we can unify the proofs of 1.2. and 1.3.

Let us check the given expression of H up to the second term displayed. We have $x'=x+G_2(x,y')$, $y'=y-G_1(x,y')$. Using the implicit function theorem we get

$$x'=x+G_2-G_{22}G_1+\dots\quad ,\qquad y'=y-G_1+G_{12}G_1+\dots\ .$$

From $x=H_2$, $y=-H_1$ follows

$$\bar{x}=x+x+x/2+\dots=x+H_2+(H_{21}H_2-H_{22}H_1)/2+\dots=x+G_2-G_{21}G_2/2-G_1G_{22}/2+$$
$$+(G_{12}G_2-G_{22}G_1)/2+\dots=x+G_2-G_{22}G_1+\dots$$
$$\bar{y}=y+y+y/2+\dots=y-H_1+(-H_{11}H_2+H_{12}H_1)/2+\dots=y-G_1+G_{11}G_2/2+G_1G_{12}/2+$$
$$+(-G_{11}G_2+G_{12}G_1)/2+\dots=y-G_1+G_{12}G_1+\dots\quad ,$$

as desired.

Remark 2. Between the invariant curves of the stable case there are (at least generically) elliptic and hyperbolic periodic points. The period of these points increases while the rotation number tends to zero as r goes to zero. This can be seen because the rotation number of T on the invariant curves behaves like the rotation number of T_1. Then the argument of Poincaré-Birkhoff can be applied.

§4.Some examples and applications. Take a map $(x,y)\to(x,y+\sum_{j\geq k}a_jx^j)$, $a_k\neq 0$. Through composition with a rotation $R_{2\pi/3}$ we have an elliptic fixed point whose eigenvalue λ is a cubic root of the unity. Let $(x',y')=T^3(x,y)$. It is convenient to use the map T in complex form : $z\to\lambda(z+\sum_{j\geq k}a_j\,2^{-j}(z+\bar{z})^j)$. Then if k is odd we obtain the function

$$G(x,y')=\frac{a_k}{k+1}\left(x^{k+1}+2^{-k}\sum_{r\geq 0}\binom{k+1}{2r}3^r x^{k+1-2r}y'^{2r}\right)+O_{2k}\ .$$

If k is even we must change the sign in the first term x^{k+1}. Hence we have the following result.

4.1.Corollary. *Let* $T(x,y)=R_{2\pi/3}(x,y+\sum_{j\geq k}a_jx^j)$ *with* $a_k\neq 0$. *If* k *is even (odd) the origin is unstable (stable)*.

In a similar but longer way we can study the case of a quartic root of the unity.

4.2.Corollary. Let $T(x,y)=R_{\pi/2}(x,y+\sum_{j\geq k} a_j x^j)$ with $a_k\neq 0$. If k is odd the origin is stable (the dominant terms in G are $a(x^{k+1}+y'^{k+1})$).If k is even we can scale and suppose $a_k=1$. Then if for all odd i k' we have $a_i=0$, the origin is stable. Otherwise let a_j be the first non zero term of odd index.

If $j<2k-1$ one has stability.

If $j=2k-1$ the origin is unstable (stable) when a_j belongs (does not belong)to $(0,k/2]$.

If $2k-1<j\leq k(k+1)-2$ the origin is unstable (stable) when $a_j>0$ $(a_j<0)$.

If $j=k(k+1)-1$ the origin is unstable (stable) when $a_j\geq k/2$ $(a_j<k/2)$.

If $j\geq k(k+1)$ the origin is always stable.

Some results related to 4.2. can be found in [9] .

Another interesting application concerns the stability of the Lagrangian equilibrium poits L_4, L_5 for the restricted problem of three bodies when the mass ratio of the primaries equals one of the critical masses of Routh. In [3] it is shown stability except for the critical values μ_1, μ_2, μ_3 (and another exceptional value that has been shown to be stable after). These exceptional cases are related to 1:1, 2:1 and 3:1 resonances and can be studied using a suitable Poincaré map or its power. See [14] for the details.

As a last application we mention that also for the RTBP, G.Gómez [4] has used those methods to show the stability of some families of periodic orbits at the bifurcation point.(RTBP=Restricted Three Body P roblem).

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