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INTRODUCTION TO THE THEORY OF SINGULAR PERTURBATIONS

by
E.M. de Jager

Summary. A review of the theory of singular perturbations is presented. Essential aspects are emphasized, some methods described in detail and proofs have been given. The text is self contained.

1. Singular Perturbations.

In this review paper on singular perturbations in "classical" analysis we treat some essential aspects of the theory and we deal with the description of some methods. However, due to the diversity of problems and the vast amount of literature on the subject it is impossible to present here a complete review. The non initiated reader can find rather complete bibliographies in the references [26], [6], [31]. For recent developments of the theory and an impression of the "state of the art" we mention the proceedings of the Oberwolfach meeting in 1981, lit[45].
The term "classical" is meant to emphasize the contrast with so-called non-standard methods in singular perturbations, which are treated by other authors in this volume, a.o. R.Lutz, M.Goze and T.Sari; the reader, interested in applications of non-standard analysis in a variety of problems, is referred to lit[27].

Let there be given a family of problems P_ε depending on a small positive parameter ε. Whenever we can define a problem P_0 which is obtained from the problem P_ε by -among other things- putting $\varepsilon = 0$, the problem P_ε is called a perturbation problem with respect to the unperturbed problem P_0. In case the solution u_ε of the problem P_ε depends continuously on ε with $\varepsilon \geq 0$ and in case the problem P_0 is more easily solvable than the general problem P_ε, one may obtain an approximation u_0 of the solution u_ε, where u_0 is the solution of the problem P_0. This idea for obtaining approximate solutions for problems containing a small parameter has been applied since many years to differential equations, containing a small parameter.
As an example we consider the problem

$$\left.\begin{aligned} &\frac{du}{dx} = f(x,u;\varepsilon),\ x > 0 \\ &u(0) = g(\varepsilon), \end{aligned}\right\} P_\varepsilon \tag{1.1}$$

where u,f and g are vector functions.
If f is sufficiently regular in x and u and f and g are sufficiently smooth in ε, the problem P_0 is readily defined by simply setting ε equal to zero. The solution of P_ε, with ε sufficiently small, may be obtained by expanding f and g into a Taylor series with respect to ε and one gets an expansion of the solution $u(x;\varepsilon)$ into powers of ε; the first term of this expansion is the solution of the problem P_0.
Problems of this type are called regular perturbation problems, in contrast to the so-called singular perturbation problems, where f is no longer smooth in ε, sothat the Taylor expansion method is no longer applicable. As an example of the latter type of problems we consider the following initial value or boundary value problem

$$\varepsilon\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = 0, \quad 0 < x \quad \text{or} \quad 0 < x < 1 \tag{1.2}$$

$$\text{with} \quad u(0) = \alpha, \quad \frac{du}{dx}(0) = \gamma \quad \text{respectively} \quad u(0) = \alpha, \quad u(1) = \beta \tag{1.3}$$

Putting $u = u_1$ and $\frac{du}{dx} = u_2$ the equation (1.2) becomes

$$\left.\begin{aligned} \frac{du_1}{dx} &= u_2 \\ \frac{du_2}{dx} &= -\frac{1}{\varepsilon}\{b(x)u_1 + a(x)u_2\} \end{aligned}\right\} \tag{1.4}$$

It follows from (1.4) that the problem P_0 cannot be defined by putting simply $\varepsilon = 0$. Also the original problem (1.2)-(1.3) gives difficulties because putting $\varepsilon = 0$ reduces the order of the differential equation and so only one initial or boundary condition can be fulfilled; it is not a priori clear which condition should be chosen in the problem P_0.

The problem (1.2)-(1.3) is readily generalized by considering boundary value problems of the type

$$\varepsilon L_2[u_\varepsilon(x)] + L_1[u_\varepsilon(x)] = 0, \quad x \in \Omega \subset \mathbb{R}_n, \quad 0 < \varepsilon << 1, \tag{1.5}$$

$$B_i[u_\varepsilon(x)] = \varphi_i(x), \quad x \in \partial\Omega, \quad i = 1,2,\dots p_1, \tag{1.6}$$

where L_1 and L_2 are in principle arbitrary differential operators of orders m_1 respectively m_2 with $m_1 < m_2$, and where $B_i[u_\varepsilon] = \varphi_i$ represent the boundary condition along the boundary $\partial\Omega$ of the domain Ω; the number of boundary conditions depends of course on the type and order of the operator L_2. Further we assume that

L_1 and L_2 are independent of ε. We state now the following problem: formulate conditions for the operators L_1, L_2 and B_i such that it is possible to derive from (1.6) p_2 boundary conditions $\tilde{B}_j[\cdot] = \tilde{\varphi}_j$ with the property that the boundary value problem

$$L_1[w(x)] = 0, \qquad x \in \Omega \subset \mathbb{R}_n \tag{1.7}$$

$$\tilde{B}_j[w(x)] = \tilde{\varphi}_j(x), \quad x \in \partial\Omega_j \subset \partial\Omega, \quad j = 1,2,\dots p_2 \tag{1.8}$$

is well posed and that its solution is an "approximation" of the full problem (1.5)-(1.6). The conditions $\tilde{B}_j[w] = \tilde{\varphi}_j$ should be compatible with the operator L_1 and contain as many conditions of (1.6) as possible. Since L_1 is of lower order than L_2, the function w can only be submitted to a part of the conditions (1.6). (Loss of boundary conditions).

The word "approximation" is to be understood in the sense

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon = w \tag{1.9}$$

or more precisely

$$\|u_\varepsilon - w\| = O(\varepsilon^\nu), \quad \nu > 0, \tag{1.10}$$

where the norm $\|\cdot\|$ and the positive number ν have to be determined explicitly. Because of the loss of boundary conditions it is clear that (1.9) can only have a restricted meaning and we shall in general not obtain an approximation for u_ε and its derivatives, uniformly pointwise valid in all of $\bar{\Omega}$.

So the second question which is of importance in the theory is to construct ε dependent correction terms which allow for that part of the boundary conditions (1.6) which are not fulfilled by the function w. The ideal is to construct with the aid of w and these correction terms an approximation for u_ε which is pointwise uniformly valid in all of $\bar{\Omega}$.

In this lecture we adopt the convention that <u>singular</u> perturbation problems are boundary or initial value problems with the property that putting $\varepsilon = 0$ reduces the order of the differential equation. The boundary value problem (1.7)-(1.8) is called the <u>reduced</u> problem.

2. General Outline of the Method.

A method to investigate the questions concerning the validity of (1.9)-(1.10) consists in the construction of a so-called formal approximation for $u_\varepsilon(x)$, i.e. a function $\tilde{u}_\varepsilon(x)$ satisfying

$$\varepsilon L_2[\tilde{u}_\varepsilon(x)] + L_1[\tilde{u}_\varepsilon(x)] = O(\varepsilon^\mu), \text{ uniformly in } \Omega, \ \mu > 0 \tag{2.1}$$

$$B_i[\tilde{u}_\varepsilon(x)] - \varphi_i(x) = O(\varepsilon^{\nu_i}), \ i = 1,2,\dots p_1, \text{ uniformly on } \partial\Omega, \ \nu_i > 0 \tag{2.2}$$

This formal approximation has a composite character

$$\tilde{u}_\varepsilon(x) = W(x;\varepsilon) + V(x;\varepsilon) \tag{2.3}$$

The function $W(x;\varepsilon)$, the regular part, is obtained by substituting

$$W(x;\varepsilon) := \sum_{j=0}^{N} \varepsilon^j w_j(x) \tag{2.4}$$

into the boundary value problem

$$\varepsilon L_2[W] + L_1[W] = 0, \quad x \in \Omega \tag{2.5}$$

$$\tilde{B}_j[W] - \tilde{\varphi}_j = 0, \qquad x \in \partial\Omega_j \tag{2.6}$$

Expanding the left hand sides into powers of ε and putting all terms, except that one with the highest power, equal to zero, we obtain a system of boundary value problems for w_j and the solution gives $W(x;\varepsilon)$; $w_0(x)$ is the solution of the so-called reduced problem.

The second function $V(x;\varepsilon)$, the singular part, represents the part of the solution which accounts for the boundary conditions (1.6), which are not included in (1.8). It represents also the part of the solution where $\varepsilon L_2[V]$ is of the same order as $L_1[V]$.

The function V has in many cases the expansion

$$V(x;\varepsilon) = \sum_{j=0}^{N+M} \varepsilon^{\nu j} v_j(x;\varepsilon), \tag{2.7}$$

with ν a rational number with $0 < \nu \leq 1$. The functions v_j are obtained by a local analysis and have only a significant value in the regions where $\varepsilon L_2[v_j]$ is of the same order as $L_1[v_j]$; the correction terms $\varepsilon^{\nu j} v_j(x;\varepsilon)$ are called boundary layer terms.

The larger the values of N and M the better is the formal approximation $\tilde{u}_\varepsilon$, i.e. the larger are the numbers μ and ν_i in (2.1) and (2.2).
In order to <u>prove</u> that the formal approximation $\tilde{u}_\varepsilon$ is indeed a good approximation to the unknown solution u_ε, we put

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_\varepsilon(x) \tag{2.8}$$

Putting (2.8) into (1.5)-(1.6) and using (2.1)-(2.2) we get a boundary value problem for the remainder term $R_\varepsilon(x)$, which has to be estimated by some <u>a priori</u> estimate. Whenever it appears that $\|R_\varepsilon\| \to 0$ for $\varepsilon \to 0$ one has shown that $\tilde{u}_\varepsilon(x)$ is a good approximation for $u_\varepsilon(x)$.

From this procedure it follows that a priori estimates for solutions of differential equations are an important tool in singular perturbation theory. These a priori estimates depend on the type of differential equation and they involve in general rather intricate applications of functional analysis.
Because the differential operators L_1 and L_2 may be independently of each other of elliptic, hyperbolic or parabolic type, linear or non linear it is understood that the theory of singular perturbations is a broad field for applications of results from the theory of differential equations.
Apart from its interest for mathematical models in the physical sciences this accounts for the huge amount of literature on singular perturbations.

In the following sections we illustrate the theory by means of some interesting and representative examples.

3. Linear Equations.

3.1. A linear initial value problem.

We consider the initial value problem

$$L_\varepsilon[u_\varepsilon] = \varepsilon\frac{d^2u_\varepsilon}{dx^2} + a(x)\frac{du_\varepsilon}{dx} + b(x)u_\varepsilon = f(x), \quad x > 0 \tag{3.1}$$

$$u_\varepsilon(0) = \alpha, \qquad \frac{du_\varepsilon}{dx}(0) = \beta \tag{3.2}$$

with a,b and f belonging to $C^1[0,\infty)$, $a(x) \geq a_0 > 0$ and $0 < \varepsilon << 1$.
In order to construct a formal approximation we use the reduced equation $(\varepsilon=0)$

$$a(x)\frac{dw_0}{dx} + b(x)w_0 = f(x) \tag{3.3}$$

with $w_0(0) = \alpha$.
Because the second initial condition is in general not satisfied by $w_0(x)$ the derivative of w_0 cannot be a good approximation for the unknown $\frac{du_\varepsilon}{dx}$. Therefore

we set

$$u_\varepsilon(x) = w_0(x) + v(x;\varepsilon), \tag{3.4}$$

with $v(x;\varepsilon)$ a correction term which accounts for the second initial condition. Substitution of (3.4) into (3.1)-(3.2) yields for v the initial value problem

$$\varepsilon v''(x;\varepsilon) + a(x)v'(x;\varepsilon) + b(x)v(x;\varepsilon) = -\varepsilon w_0''(x) \tag{3.5}$$

with

$$v(0;\varepsilon) = 0 \quad \text{and} \quad v'(0;\varepsilon) = \beta - w_0'(0) \tag{3.6}.$$

In order to study what happens at $x = 0$ we use a "microscope" $x = \varepsilon^\nu\xi$ with $\nu > 0$ and this gives

$$\varepsilon^{1-2\nu}\frac{d^2v}{d\xi^2} + a(\varepsilon^\nu\xi)\varepsilon^{-\nu}\frac{dv}{d\xi} + b(\varepsilon^\nu\xi)v = -\varepsilon w_0''(\varepsilon^\nu\xi)$$

with $\quad v(0;\varepsilon) = 0 \quad \text{and} \quad \frac{dv}{d\xi}(0;\varepsilon) = \varepsilon^\nu(\beta - w_0'(0)).$

The first term in the differential equation is of the same order as the second one for $\nu = 1$; focussing the microscope at $\nu = 1$, i.e. $x = \varepsilon\xi$, and putting

$$v(x;\varepsilon) = \varepsilon\bar{v}\left(\frac{x}{\varepsilon}\right) = \varepsilon\bar{v}(\xi) \tag{3.7}$$

we obtain for $\bar{v}(\xi)$ the initial value problem

$$\frac{d^2\bar{v}}{d\xi^2} + a(\varepsilon\xi)\frac{d\bar{v}}{d\xi} + \varepsilon b(\varepsilon\xi)\bar{v} = -\varepsilon w_0''(\varepsilon\xi) \tag{3.5*}$$

$$\bar{v}(0) = 0 \quad \text{and} \quad \frac{d\bar{v}}{d\xi}(0) = \beta - w_0'(0). \tag{3.6*}$$

Remembering the regularity of the coefficients a, b and f, and hence also of w_0, we consider instead of $\bar{v}$ the function v_1 which satisfies the differential equation

$$\frac{d^2v_1}{d\xi^2} + a(0)\frac{dv_1}{d\xi} = 0, \quad \xi > 0 \tag{3.8}$$

and the boundary condition

$$\frac{dv_1}{d\xi}(0) = \beta - w_0'(0). \tag{3.9}$$

The function $w_0(x) + \varepsilon v_1(\xi) = w_0(x) + \varepsilon v_1(\frac{x}{\varepsilon})$ satisfies the first initial condition up to $O(\varepsilon)$ and exactly the second initial condition (3.2); so we have still the freedom to choose a second boundary condition for $v_1(\xi)$.
Requiring that v_1 has only a significant value in a right neighbourhood of $x = 0$ we choose as the second boundary condition for v_1:

$$\lim_{\xi\to\infty} v_1(\xi) = 0 \qquad (3.10).$$

The equations (3.8), (3.9) and (3.10) yield immediately

$$v_1(\xi) = v_1(x,\varepsilon) = \frac{w_0'(0)-\beta}{a(0)} \exp - [a(0)\frac{x}{\varepsilon}]. \qquad (3.11)$$

This function has due to the fact $a(0) \geq a_0 > 0$ the property that it is asymptotically zero in any interval $[\delta,\infty)$ with δ arbitrarily small positive, but fixed and independent of ε.
v_1 has a "boundary layer" character and the "width" of the boundary layer is $O(\varepsilon)$.
Finally we put

$$\tilde{u}_\varepsilon(x) = w_0(x) + \varepsilon v_1(\frac{x}{\varepsilon}), \qquad (3.12)$$

and substitution of (3.12) in the differential equation gives with the aid of (3.3) and (3.8)

$$L_\varepsilon[\tilde{u}_\varepsilon] = \varepsilon\frac{d^2w_0}{dx^2} + f(x) + \varepsilon(a(x) - a(0))\frac{d}{dx}v_1(\frac{x}{\varepsilon}) + \varepsilon b(x)v_1(\frac{x}{\varepsilon}).$$

Using again the assumed regularity of a,b,f and w_0 we get

$$L_\varepsilon[\tilde{u}_\varepsilon] = f(x) + O(\varepsilon), \qquad (3.13)$$

uniformly in any bounded segment $[0,\ell]$ with ℓ independent of ε. Further it follows from (3.3) and (3.9) that

$$\tilde{u}_\varepsilon(0) = \alpha + \varepsilon v_1(0) = \alpha + O(\varepsilon) \quad \text{and} \quad \frac{d\tilde{u}_\varepsilon}{dx}(0) = \beta \qquad (3.14)$$

and so $\tilde{u}_\varepsilon(x)$ is a formal approximation of the unknown function $u_\varepsilon(x)$.
In order to prove that $\tilde{u}_\varepsilon(x)$ is indeed a good approximation for the solution $u_\varepsilon(x)$ we set

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_\varepsilon(x) \qquad (3.15)$$

and we have to estimate $R_\varepsilon(x)$.
According to (3.1), (3.2), (3.13) and (3.14) the remainder $R_\varepsilon(x)$ satisfies the

initial value problem:

$$\varepsilon \frac{d^2R_\varepsilon}{dx^2} + a(x)\frac{dR_\varepsilon}{dx} + b(x)R_\varepsilon(x) = O(\varepsilon), \text{ uniformly in any bdd segment } [0,\ell],$$

$$R_\varepsilon(0) = O(\varepsilon), \quad R_\varepsilon'(0) = 0.$$

Instead of estimating $R_\varepsilon(x)$ we may as well estimate

$$\bar{R}_\varepsilon(x) = R_\varepsilon(x) - R_\varepsilon(0), \tag{3.16}$$

and $\bar{R}_\varepsilon$ is the solution of the initial value problem

$$\varepsilon\bar{R}_\varepsilon''(x) + a(x)\bar{R}_\varepsilon' + b(x)\bar{R}_\varepsilon = g(x;\varepsilon), \quad \bar{R}_\varepsilon(0) = \bar{R}_\varepsilon'(0) = 0, \tag{3.17}$$

with $g(x,\varepsilon) = O(\varepsilon)$, uniformly in any bounded segment $[0,\ell]$. The function $\bar{R}_\varepsilon(x)$ is now estimated by means of "energy" integrals. After multiplication of (3.17) with $2\bar{R}_\varepsilon$ and with $2a\,\bar{R}_\varepsilon'$ and addition of the results we obtain after a small calculation

$$\frac{d}{dx}[a\bar{R}_\varepsilon^2 + 2\varepsilon\bar{R}_\varepsilon\bar{R}_\varepsilon' + \varepsilon a\bar{R}_\varepsilon'^2] =$$

$$2g\bar{R}_\varepsilon + \varepsilon(2+a')\bar{R}_\varepsilon'^2 - 2a^2\bar{R}_\varepsilon'^2 - 2ab\bar{R}_\varepsilon\bar{R}_\varepsilon' + 2ag\bar{R}_\varepsilon' + (a'-2b)\bar{R}_\varepsilon^2$$

$$\leq (1+a'+b^2-2b)\bar{R}_\varepsilon^2 + \varepsilon(2+a')\bar{R}_\varepsilon'^2 + 2g^2, \tag{3.18}$$

uniformly valid for all ε and any segment $[0,\ell]$.
Since $a \geq a_0 > 0$ it follows that for $0 \leq x \leq \ell$ and ε sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$ with ε_0 a generic constant

$$a\bar{R}_\varepsilon^2 + 2\varepsilon\bar{R}_\varepsilon\bar{R}_\varepsilon' + \varepsilon a\bar{R}_\varepsilon'^2 \geq a_0\bar{R}_\varepsilon^2 + 2\varepsilon\bar{R}\,\bar{R}_\varepsilon' + \varepsilon a_0\bar{R}_\varepsilon'^2$$

$$\geq (a_0-\sqrt{\varepsilon})\bar{R}_\varepsilon^2 + \varepsilon(a_0-\sqrt{\varepsilon})\bar{R}_\varepsilon'^2 \geq m(\bar{R}_\varepsilon^2+\varepsilon\bar{R}_\varepsilon'^2),$$

with m a positive constant independent of ε. (e.g. $m = \frac{1}{2}a_0$). Integrating (3.18) and using the initial conditions for $\bar{R}_\varepsilon$ we get

$$m(\bar{R}_\varepsilon^2+\varepsilon\bar{R}_\varepsilon'^2) \leq M\int_0^x(\bar{R}_\varepsilon^2+\varepsilon\bar{R}_\varepsilon'^2)d\xi + M\|g\|^2_{[0,\ell]}$$

with $M = \max\limits_{[0,\ell]} [|2+a'|, \ |1+a'+b^2-2b|, \ 2]$ and $\|\cdot\|_{[0,\ell]}$ denoting the L^2-norm.

Finally we get with the aid of Gronwall's lemma (see e.g. [37], p.108)

$$\bar{R}_\varepsilon^2 + \varepsilon \bar{R}'^2 \leq \frac{M}{m} \| g \|^2_{[0,\ell]} \exp[\frac{Mx}{m}] \tag{3.19}$$

uniformly in $[0,\ell]$ for ε sufficiently small.
Because $g(x;\varepsilon) = O(\varepsilon)$, uniformly in $[0,\ell]$ we have

$$|\bar{R}_\varepsilon| = O(\varepsilon) \quad \text{and} \quad |\bar{R}'_\varepsilon| = O(\sqrt{\varepsilon}), \quad \text{uniformly in } [0,\ell].$$

The same estimates are valid for R_ε (see (3.16)) and so we obtain on account of (3.12) and (3.16)

$$|u_\varepsilon(x) - w_0(x) - \varepsilon v_1(\tfrac{x}{\varepsilon})| = O(\varepsilon), \quad \text{uniformly in any bounded segment } [0,\ell] \tag{3.20}$$

$$|\frac{du_\varepsilon}{dx}(x) - \frac{dw_0}{dx}(x) - \varepsilon \frac{dv_1}{dx}(\tfrac{x}{\varepsilon})| = O(\sqrt{\varepsilon}), \quad \text{" " " " " " "}$$

or

$$|u_\varepsilon(x) - w_0(x)| = O(\varepsilon), \quad \text{uniformly in } [0,\ell] \tag{3.20*}$$

and due to the boundary layer character of v_1

$$|\frac{du_\varepsilon}{dx}(x) - \frac{dw_0}{dx}(x)| = O(\sqrt{\varepsilon}), \quad \text{uniformly in } [\delta,\ell] \tag{3.21}$$

with $\delta > 0$ and δ independent of ε.

Remarks

1. In the case that the coefficient $a(x)$ is strictly negative the whole construction breaks down. It can be shown that in this case $u_\varepsilon(x)$ diverges for $\varepsilon \to 0$; the behaviour of $u_\varepsilon(x)$ as function of x and ε is described in ref[9], pp.12-17.
2. In the treatment above we have assumed a,b and f to belong to $C^1[0,\infty)$. Whenever these coefficients are submitted to stronger regularity requirements sharper estimates for u_ε can be obtained by constructing a formal approximation

$$\tilde{u}_\varepsilon = \sum_{j=0}^{N} \varepsilon^j w_j(x) + \sum_{j=1}^{N+1} \varepsilon^j v_j(\tfrac{x}{\varepsilon}) \tag{3.22}$$

Repeating the whole procedure described above we get approximations of the type

$$|u_\varepsilon - \sum_{j=0}^{N} \varepsilon^j w_j(x) - \sum_{j=1}^{N} \varepsilon^j v_j(\tfrac{x}{\varepsilon})| = O(\varepsilon^{N+1}).$$

The number N depends on the "degree" of regularity of a,b and f.
3. Using the same procedure, including the method of "energy integrals", the theory

has been generalized to initial value problems with respect to partial differential equations of hyperbolic type such as:

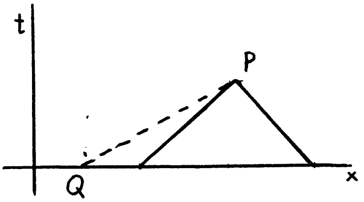
$$\varepsilon\{u_{tt} - c^2(x,t)u_{xx}\} + a(x,t)u_t + b(x,t)u_x + d(x,t)u = f(x,t), \quad -\infty < x < +\infty, \quad t > 0,$$

$$\text{with } u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad -\infty < x < +\infty. \tag{3.23}$$

Apart from regularity conditions for a,b,c,d,f,g and h one has to require again $a(x,t) \geq a_0 > 0$, $\forall t \geq 0$ and $\forall x \in \mathbb{R}$. A further requirement is that the subcharacteristics should be time like i.e.

$$\left|\frac{b}{a}\right| < c, \quad \forall x, \quad \forall t \geq 0.$$

The latter condition is clear, because in the case of $\left|\frac{b}{a}\right| > c$ the solution in a point P of the reduced problem $(\varepsilon=0)$ is completely determined by its value in the point Q (see figure). However, the solution in P of the full problem (3.23) is completely determined by the initial values along the base of the characteristic triangle and so the solution of the reduced problem has no relation with the solution of the full problem.



The interested reader is referred to [9], [21].

3.2. A linear boundary value problem.

We consider the two point boundary value problem

$$L_\varepsilon[u_\varepsilon] = \varepsilon\frac{d^2u_\varepsilon}{dx^2} + a(x)\frac{du_\varepsilon}{dx} + b(x)u_\varepsilon = f(x), \quad 0 < x < 1, \quad 0 < \varepsilon << 1, \tag{3.24}$$

with the boundary conditions

$$u(0) = \alpha \quad \text{and} \quad u(1) = \beta. \tag{3.25}$$

The coefficients a and b and the right-hand side are assumed to be continuous and further we require $a \geq a_0 > 0$ and $b \leq 0$ in $[0,1]$. The condition $a \geq a_0 > 0$ will become clear is the sequel and the condition $b \leq 0$ ensures the uniqueness of the solution; the latter can be removed by putting $u_\varepsilon = z_\varepsilon e^{\theta x}$ and choosing the constant θ properly.

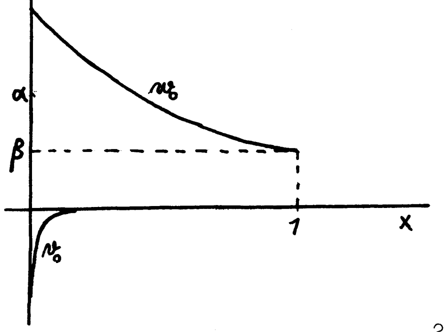
The reduced problem is defined as

$$a(x)\frac{dw_0}{dx} + b(x)w_0 = f(x) \tag{3.26}$$

with $w_0(1) = \beta$. (3.27).

The choice of the boundary condition stems from the condition $a \geq a_0 > 0$ as will be explained below.

The solution w_0 cannot be a good approximation for the function u_ε because the boundary condition at $x = 0$ is generally not satisfied by w_0.



Therefore we introduce a correction function which should account for this boundary condition.

So we get

$$u_\varepsilon(x) = w_0(x) + v(x;\varepsilon) \tag{3.28}$$

and substituting (3.28) into (3.24) we obtain for v the boundary value problem:

$$\varepsilon \frac{d^2v}{dx^2} + a(x)\frac{dv}{dx} + b(x)v = -\varepsilon \frac{d^2w_0}{dx^2} \tag{3.29}$$

with

$$v(0) = \alpha - w_0(0) \quad \text{and} \quad v(1) = 0 \tag{3.30}$$

In order to investigate what happens at $x = 0$ we use again a microscope $x = \varepsilon^\nu \xi$, $\nu > 0$ and we get

$$\varepsilon^{1-2\nu}\frac{d^2v}{d\xi^2} + a(\varepsilon^\nu\xi)\varepsilon^{-\nu}\frac{dv}{d\xi} + b(\varepsilon^\nu\xi)v = -\varepsilon\frac{d^2w_0}{dx^2}.$$

The first term is of the same order as the second one for $\nu = 1$ and we focus at $\nu = 1$. Assuming now $a \in C^2[0,1]$, $b \in C^1[0,1]$ and $f \in C^1[0,1]$ we may write

$$\varepsilon^{-1}\frac{d^2v}{d\xi^2} + \{a(0)\varepsilon^{-1} + a'(0)\xi + a''(\theta_1\varepsilon\xi)\varepsilon\xi^2\}\frac{dv}{d\xi}$$
$$+ \{b(0) + b'(\theta_2\varepsilon\xi)\varepsilon\xi\}v = -\varepsilon\frac{d^2w_0}{dx^2}(\varepsilon\xi). \tag{3.31}$$

with $v(0) = \alpha - w(0)$ and $v(\frac{1}{\varepsilon}) = 0$ (3.32)

Expanding v as

$$v(\xi) = v_0(\xi) + \varepsilon v_1(\xi) + \ldots = v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}) + \ldots$$

we obtain for v_0 and v_1 the boundary value problems

$$\frac{d^2 v_0}{d\xi^2} + a(0)\frac{dv_0}{d\xi} = 0, \quad v_0(0) = \alpha - w(0) \quad \text{and} \quad \lim_{\xi\to\infty} v_0(\xi) = 0 \tag{3.33}$$

$$\frac{d^2 v_1}{d\xi^2} + a(0)\frac{dv_1}{d\xi} = -a'(0)\xi\frac{dv_0}{d\xi} - b(0)v_0 := g(\xi)$$

$$v_1(0) = 0 \quad \text{and} \quad \lim_{\xi\to\infty} v_1(\xi) = 0. \tag{3.34}$$

Solving (3.33) and (3.34) yields immediately

$$v_0(\xi) = v_0(\tfrac{x}{\varepsilon}) = \{\alpha - w(0)\}\exp - [a(0)\tfrac{x}{\varepsilon}] \tag{3.35}$$

and

$$v_1(\xi) = v_1(\tfrac{x}{\varepsilon}) = -\frac{dv_0}{d\xi}(\xi)\left[\int_0^\xi \frac{d\xi'}{\frac{dv_0}{d\xi}(\xi')} \int_{\xi'}^\infty g(\xi'')d\xi''\right] \tag{3.36}$$

The functions v_0 and v_1 have boundary layer character: they have only significant values in an ε-neighbourhood of $x = 0$.
It follows now from the equations (3.26), (3.27), (3.31)-(3.36), that

$$\tilde{u}_\varepsilon(x) = w_0(x) + v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}) \tag{3.37}$$

is a formal approximation of u_ε; $\tilde{u}_\varepsilon$ satisfies the equations

$$L_\varepsilon[\tilde{u}_\varepsilon] = f(x) + O(\varepsilon), \quad \text{uniformly in } [0,1] \tag{3.38}$$

and

$$\tilde{u}_\varepsilon(0) = \alpha \quad \text{and} \quad \tilde{u}_\varepsilon(1) = \beta + v_0(\tfrac{1}{\varepsilon}) + \varepsilon v_1(\tfrac{1}{\varepsilon}) = \beta + O(\varepsilon^N), \tag{3.39}$$

with N arbitrarily large.
It remains to show that $\tilde{u}_\varepsilon$ is indeed a good pointwise approximation for the unknown solution u_ε. Therefore we set

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_2(x) \tag{3.40}$$

and from (3.24), (3.25), (3.38) and (3.39) it follows that the remainder term R_ε satisfies the equations

$$L_\varepsilon[R_\varepsilon] = \varepsilon\frac{d^2R_\varepsilon}{dx^2} + a(x)\frac{dR_\varepsilon}{dx} + b(x)R_\varepsilon = O(\varepsilon), \quad \text{uniformly in } [0,1] \tag{3.41}$$

$$R_\varepsilon(0) = 0 \quad \text{and} \quad R_\varepsilon(1) = O(\varepsilon^N). \tag{3.42}$$

Finally we have to estimate R_ε from (3.41), (3.42). For this purpose we apply the maximum principle for elliptic differential equations of order 2. This principle reads as follows: Let u satisfy the differential inequality

$$(L+h)[u] \geq 0 \quad \text{in a domain } D \subset \mathbb{R}_n$$

with $h \leq 0$, with L uniformly elliptic in D, and with h and the coefficients of L bounded. If u is not identically constant in D, then u can attain a non-negative maximum only at the boundary of D. A rather direct consequence of this principle is the following implication; if there exists a positive function ψ defined in D with the properties

$$|(L+h)[u(x)]| \leq (L+h)[-\psi(x)], \quad \forall x \in D$$

and

$$|u(x)| \leq \psi(x), \quad \forall x \in \partial D$$

then also

$$|u(x)| \leq \psi(x), \quad \forall x \in D.$$

The function ψ is called a barrier function for the function u and $\psi(x)$ provides an a priori estimate for the function $u(x)$. For more information concerning maximum principles we refer the reader to [35].
The construction of a barrier function for the remainder R_ε is very easy; one can take for example

$$\psi(x) = K\varepsilon\left(\frac{1-x}{a_0} + 1\right), \tag{3.43}$$

with K a constant sufficiently large positive but independent of ε.
Hence we have obtained $R_\varepsilon = O(\varepsilon)$ uniformly in $[0,1]$ and so it follows finally from (3.37) and (3.40)

$$u_\varepsilon(x) = w_0(x) + v_0\left(\frac{x}{\varepsilon}\right) + \varepsilon v_1\left(\frac{x}{\varepsilon}\right) + O(\varepsilon), \quad \text{uniformly in } [0,1],$$

or because $v_1 = O(1)$ we have also

$$\left|u_\varepsilon(x) - w_0(x) - v_0\left(\frac{x}{\varepsilon}\right)\right| = O(\varepsilon), \quad \text{uniformly in } [0,1] \tag{3.44}$$

and since $v_0(\frac{x}{\varepsilon}) \sim 0$ in $[\delta,1]$ it follows also

$$|u_\varepsilon(x) - w_0(x)| = O(\varepsilon), \quad \text{uniformly in } [\delta,1] \tag{3.45}$$

with $0 < \delta \leq 1$, and δ independent of ε.

Remarks.

1. The functions $v_0(\frac{x}{\varepsilon})$ and $v_1(\frac{x}{\varepsilon})$ are called "boundary-layer" functions; this term has been introduced in the theory of hydrodynamics of viscous media. In case of a two dimensional flow of fluid with small viscosity along a plate, the velocity field in a direction perpendicular to the plate increases rapidly from zero at the plate to some value outside the plate.
2. In case the function $a(x)$ is definite negative in $[0,1]$, the boundary layer is located at the other endpoint, viz. $x = 1$, of the segment $[0,1]$.
3. In the proof that the formal approximation is also a good approximation for the unknown solution u_ε, one needs two boundary layer functions v_0 and v_1, of which the latter has no effect in the ultimate results (3.44). So it is reasonable to ask whether the use of v_1 is really necessary to obtain (3.44); in case we might abandon the use of v_1 we can do with less regularity requirements.
4. By requiring more regularity of the coefficients a,b and the right hand side f one can obtain better results by the "Ansatz":

$$\tilde{u}_\varepsilon(x) = \sum_{i=0}^{N} \varepsilon^i w_i(x) + \sum_{i=0}^{N+1} \varepsilon^i v_i(\tfrac{x}{\varepsilon}) + R_N(x;\varepsilon).$$

The calculation, given above, can be repeated for $R_n(x;\varepsilon)$, $(n = 0,1,\ldots N)$ and the result is:

$$u_\varepsilon(x) = \sum_{i=0}^{N} \varepsilon^i w_i(x) + \sum_{i=0}^{N} \varepsilon^i v_i(\tfrac{x}{\varepsilon}) + O(\varepsilon^{N+1}), \quad \text{uniformly in } [0,1].$$

5. The theory is generalized to equations of the form

$$\varepsilon^{n-m} L_2[u_\varepsilon(x)] + L_1[u_\varepsilon(x)] = f(x), \quad 0 \leq x \leq 1, \quad 0 < \varepsilon << 1,$$

with

$$L_2 = \sum_{\nu=0}^{n} a_\nu(x) \frac{d\nu}{dx^\nu} \quad \text{and} \quad L_1 = \sum_{\nu=0}^{m} b_\nu \frac{d^\nu}{dx^\nu},$$

$n > m$, $a_n(x) \equiv 1$, $b_m(x) \neq 0$ in $[0,1]$ and with the boundary conditions

$$u^{(p_i)}(0) = \alpha_i, \quad i = 1,2,\ldots s, \quad s \leq n, \quad 0 \leq p_1 < p_2 \ldots < p_s \leq n-1,$$

$$u^{(q_i)}(1) = \beta_i, \quad i = 1,2,\ldots t, \quad t = n-s, \quad 0 \leq q_1 < q_2 \ldots < q_t \leq n-1.$$

We refer the interested reader to W.Wasow [43], [44], W.A.Harris [17], [18], R.O'Malley [33], [34] and M.I.Visik - L.A.Lyusternik [42].

6. The theory has been generalized to elliptic partial differential equations. In particular boundary value problems of the type

$$\varepsilon L_2[\Phi_\varepsilon(x,y)] + L_1[\Phi_\varepsilon(x,y)] = h(x,y), \quad (x,y) \in \Omega << \mathbb{R}_2$$

with

$$\Phi_\varepsilon(x,y)\Big|_{\partial\Omega} = \varphi(x,y)\Big|_{\partial\Omega},$$

where

$$L_2 = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x\partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + e(x,y)\frac{\partial}{\partial y} + f(x,y).$$

and $L_1 = -\frac{\partial}{\partial y} - g(x,y)$, and where Ω is a convex bounded domain, have been investigated by a.o. M.Visik and L.A.Lyusternik [42] and W.Eckhaus and E.M.de Jager [7].

Because the maximum principle is valid for elliptic equations of the second order, irrespective of the dimension of the space $\mathbb{R}_n$, the theory for this kind of singular perturbation problems is essentially the same as that given above for ordinary differential equations. Some difficulties arise at those points of $\partial\Omega$, where the tangent is vertical, see lit[12].

Other interesting phenomena appear whenever Ω is no longer convex, see lit[7]. Generalizations to higher order elliptic equations with L_2 elliptic of order $2n$ and L_1 elliptic of order $2m$ have been given by a.o. D.Huet [20] and J.G.Besjes [2]. In this case the maximum principle is no longer applicable and one has to rely on functional analytic tools yielding a priori estimates for solutions of partial differential equations of elliptic type.

3.3. <u>Turning point problems</u>.

In the preceding sections we considered differential equations of the type

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x),$$

where we assumed always $a(x) \neq 0$. This means that the reduced problem leads to a differential equation

$$a(x)w'(x) + b(x)w(x) = f(x),$$

which is <u>non</u> singular.

In case $a(x)$ is zero in some point x_0 of the segment where the differential equation is defined we can expect serious difficulties because w is in general no

longer differentiable at $x = x_0$. This point is called a turning point.
Too many technicalities are involved to describe here the theory in case of a turning point; the interested reader is referred to W.Wasow [43], R.C.Ackerberg and R.E.O'Malley [1], P.Cook and W.Eckhaus [4], P.P.N.de Groen [14], and J.Grasman and B.J.Matkovsky [13].
In the following we give an illustrative example, which has been taken from lit[43].
We consider the boundary value problem

$$\varepsilon u_\varepsilon''(x) + a(x)u_\varepsilon'(x) = f(x), \quad -1 < x < 1 \tag{3.46}$$

with $u_\varepsilon(-1) = u_\varepsilon(+1) = 0$, and where a and f are continuous functions in $|x| \leq 1$; further $a(0) = 0$, $a'(0) \neq 0$ and $a(x) \neq 0$ for $x \neq 0$.
There are two cases to be considered $a'(0) > 0$ and $a'(0) < 0$.
In case $a'(0) > 0$ we have $a(x) > 0$ for $x > 0$ and $a(x) < 0$ for $x < 0$. Using the maximum principle of section 3.2, one can show that $u_\varepsilon(x)$ is bounded, uniformly with respect to ε in any segment $[-1, -\delta] \cup [+\delta, +1]$ where δ is an arbitrarily small positive number independent of ε.
It follows now from the theory of section 3.2 that we need not to expect boundary layer behaviour in the left and right neighbourhood of $x = +1$ respectively $x = -1$ and so we have the reduced problem

$$a(x)w'(x) = f(x) \quad \text{with} \quad w(-1) = w(+1) = 0.$$

Its solution

$$w(x) = \int_{-1}^{x} \frac{f(\xi)}{a(\xi)}\, d\xi, \quad -1 \leq x < 0$$

and
$$w(x) = \int_{+1}^{x} \frac{f(\xi)}{a(\xi)}\, d\xi, \quad 0 < x \leq +1,$$

approaches for $\varepsilon \downarrow 0$ the solution of (3.46) uniformly in any segment $[-1, -\delta] \cup [+\delta, +1]$.
In case $a'(0) < 0$ it can be shown -also by barrier function technique- that $u_\varepsilon(x)$ diverges everywhere in $(-1,+1)$ whenever $\varepsilon \downarrow 0$.

4. Semi-Linear Equations.

4.1. A semi linear boundary value problem.

We consider the boundary value problem

$$\varepsilon \frac{d^2 u_\varepsilon}{dx^2} + a(x)\frac{du_\varepsilon}{dx} + b(x,u_\varepsilon) = 0, \quad 0 < x < 1 \tag{4.1}$$

with $u_\varepsilon(0) = \alpha$ and $u_\varepsilon(1) = \beta$.

We suppose $a \in C^1[0,1]$ and $b \in C^2[[0,1] \times \mathbb{R}]$; in order to have uniqueness of the solution we assume also $\frac{\partial b}{\partial u}(x,u) \le 0$ in $[0,1] \times \mathbb{R}$ and in order to avoid a turning point we take $a(x) \ge a_0 > 0$ in $[0,1]$.
The condition $\frac{\partial b}{\partial u} \le 0$ may be released by putting $u_\varepsilon = e^{\theta x} v_\varepsilon$, choosing the constant θ properly and using $a(x) \ge a_0 > 0$. The reduced boundary value problem reads as

$$a(x)\frac{dw_0}{dx} + b(x,w_0) = 0, \qquad 0 < x < 1 \tag{4.2}$$

with $w_0(1) = \beta$; the boundary condition is fulfilled at $x = 1$ because $a(x) \ge a_0 > 0$.
In order to apply our method for approximating u_ε we have to be sure that $\varepsilon\frac{d^2w_0}{dx^2} = O(\varepsilon)$, uniformly in $[0,1]$. This does not lead to serious difficulties as long as the equation is linear, but in the case of non-linear equations we have to make the extra assumption that the boundary value problem (4.2) has a solution twice continuously differentiable in $[0,1]$. For instance the case $a \equiv 1$, $b = -w_0^2$ and $w_0(1) = -2$ yields the solution $w_0(x) = -(x-\frac{1}{2})^{-1}$ and this gives troubles in $x = \frac{1}{2}$.

So we assume that the reduced problem (4.2) has a solution w_0, which belongs to $C^2[0,1]$.
A formal approximation for u_ε is now given by

$$\tilde{u}_\varepsilon(x) = w_0(x) + v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}), \tag{4.3}$$

where the boundary layer terms v_0 and v_1 satisfy the boundary value problems:

$$\begin{aligned} &\frac{d^2v_0}{d\xi^2} + a(0)\frac{dv_0}{d\xi} = 0, \qquad 0 < \xi = \frac{x}{\varepsilon} < \infty, \\ &v_0(0) = \alpha - w_0(0), \qquad \lim_{\xi\to+\infty} v_0(\xi) = 0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\frac{d^2v_1}{d\xi^2} + a(0)\frac{dv_1}{d\xi} = -\frac{da}{dx}(0)\,\xi\frac{dv_0}{d\xi} - [b\{0,w_0(0)+v_0(\xi)\} - b\{0,w_0(0)\}], \\ &\hspace{20em} 0 < \xi < \infty \\ &v_1(0) = 0, \qquad \lim_{\xi\to+\infty} v_1(\xi) = 0 \end{aligned} \tag{4.5}$$

It is useful to compare these equations with (3.26), (3.33) and (3.34); we remark that the boundary layer equations are linear and its solutions are readily given, see (3.35) and (3.36).
Putting now

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_\varepsilon(x), \tag{4.6}$$

we get after substitution of (4.6) into (4.1) for the remainder R_ε the non-linear boundary value problem:

$$\varepsilon\frac{d^2R_\varepsilon}{dx^2} + a(x)\frac{dR_\varepsilon}{dx} + \{b(x,\tilde{u}_\varepsilon+R_\varepsilon) - b(x,\tilde{u}_\varepsilon)\} = O(\varepsilon), \tag{4.7}$$

uniformly in $[0,1]$ with $R_\varepsilon(0) = 0$ and $R_\varepsilon(1)$ asymptotically equal to zero.
In contrast to the preceding section we should now concentrate on a priori estimates for non-linear boundary value problems. However, the maximum principle formulated in section (3.2) is also valid for semi-linear elliptic equations of the type:

$$u_{xx} + H(x,u,u_x) = 0,$$

with $\frac{\partial H}{\partial u} \leq 0$ (see lit[35], Chapters I, II).
Because we have assumed

$$\frac{\partial}{\partial R}[b(x,\tilde{u}_\varepsilon+R)] \leq 0$$

this maximum principle is applicable to (4.7) and after a small calculation we obtain that the function

$$\Psi(x) = \varepsilon K(\frac{1-x}{a_0} + 1), \tag{3.43}$$

with K a constant, sufficiently large positive but independent of ε, is a barrier function for R_ε.
Therefore we get finally from (4.6) and (4.7)

$$|u_\varepsilon(x) - w_0(x) - v_0(\tfrac{x}{\varepsilon})| = O(\varepsilon), \quad \text{uniformly in } [0,1] \tag{4.8}$$

and

$$|u_\varepsilon(x) - w_0(x)| = O(\varepsilon), \quad \text{uniformly in } [\delta, 1], \text{ for all } \delta \text{ independent of}$$
$$\varepsilon \text{ and with } 0 < \delta < 1.$$

Instead of the differential equation (4.1) we could have considered the more general quasi linear equation

$$\varepsilon \frac{d^2u_\varepsilon}{dx^2} + a(x,u_\varepsilon)\frac{du_\varepsilon}{dx} + b(x,u_\varepsilon) = 0.$$

The tools are essentially the same and it has been only for reasons of simplicity that we have chosen the equation (4.1) as an example. For more details and other types of elliptic equations the reader is referred to A.van Harten, lit[19].
Another way for obtaining an a priori estimate for R_ε is based on the use of functional analysis, involving a fix-point theorem. This theorem will be treated in the next section 4.2 and applications to semi-linear singular perturbation problems follow in sections 4.3 and 4.4.

Remark.
The theory of this section can be generalized to boundary value problems for second order elliptic partial differential equations of the form

$$\varepsilon L_2[\Phi_\varepsilon] + a(x,y,\Phi_\varepsilon)\frac{\partial\Phi_\varepsilon}{\partial x} + b(x,y,\Phi_\varepsilon)\frac{\partial\Phi_\varepsilon}{\partial y} + c(x,y,\Phi_\varepsilon) = 0, \tag{4.9}$$

with L_2 an arbitrary linear elliptic differential operator of second order. The tools are again the same and the interested reader may consult lit[19], pp.116-125.

4.2. A fix point theorem.

Because the maximum principle is not the most appropriate device for obtaining a priori estimates of solutions of initial value problems we shall use the above mentioned fix point theorem in singularly perturbed non-linear initial value problems. As will be shown later the theorem is also very useful for singularly perturbed non-linear boundary value problems; it provides also an estimates for the derivative of the unknown function u_ε.
The fix point theorem was first introduced and applied by A.van Harten to non-linear singular perturbation problems of elliptic type, lit[19], pp.188-213. For applications to non-linear singular perturbation problems of hyperbolic type, it has been modified later by R.Geel and E.M.de Jager, lit[9], Chs. II, IV, lit[10], [22].

Fix point Theorem.
Let N be a normal space with elements p and norm $|p|$, and B a Banach space with elements q and norm $\| q \|$.
Let F be a non-linear map $N \to B$ with $F(\emptyset) = \emptyset$ and F is assumed to be decomposable as

$$F(p) = L(p) + \Psi(p) \tag{4.10}$$

with L the linearization of F in $p = 0$.
Furthermore the operators L and Ψ are subject to the following conditions:
i) L is bijective and its inverse L^{-1} is continuous, i.e.

$$|L^{-1}(q)| \le \frac{1}{\ell} \| q \|, \quad \forall q \in B, \tag{4.11}$$

with ℓ some positive number.

ii) Let $\Omega_N(\rho)$ be the ball $\{p \mid p \in N, \ |p| \le \rho\}$

There exists a number $\bar{\rho} > 0$ with the property:

$$\| \Psi(p_1) - \Psi(p_2) \| \le m(\rho) |p_1 - p_2|, \quad \forall p_1, p_2 \in \Omega_N(\rho), \quad \forall \rho \in [0, \bar{\rho}] \tag{4.12}$$

with $m(\rho)$ decreasing for $\rho \downarrow 0$ with $\lim_{\rho \downarrow 0} m(\rho) = 0$.

When ρ_0 is defined as

$$\rho_0 = \sup\{\rho \mid 0 \le \rho \le \bar{\rho}, \quad m(\rho) \le \tfrac{1}{2}\ell\} \tag{4.13}$$

then there exists for any $f \in B$ with

$$\| f \| \le \tfrac{1}{2}\ell\rho_0 \tag{4.14}$$

a solution of the equation

$$F(p) = f \tag{4.15}$$

with $p \in N$ and

$$|p| \leqq 2\ell^{-1} \| f \| \tag{4.16}.$$

Proof.

Due to the bijectivity of the map $L(p) = q$, the equation

$$F(p) = L(p) + \Psi(p) = f, \quad p \in \Omega_N(\bar{\rho})$$

is equivalent to the equations

$$L(p) = q \quad \text{and} \quad q = f - \Psi \circ L^{-1}(q) := T(q), \quad q \in L(\Omega_N(\bar{\rho})) \tag{4.17}.$$

We consider now the ball $\Omega_B(\ell\rho) = \{q \mid q \in B, \ \| q \| \le \ell\rho\}$.

For $2\ell^{-1} \| f \| \le \rho \le \rho_0 \le \bar{\rho}$ we have on account of the Lipschitz-condition (4.12) that T maps $\Omega_B(\ell\rho)$ in $\Omega_B(\ell\rho)$ and also that T is strictly contractive on $\Omega_B(\ell\rho)$. It follows that (4.17) has a unique solution in $\Omega_B(\ell\rho)$ and therefore (4.15) possesses also a solution p in $\Omega_N(\rho)$. Choosing finally $\rho = 2\ell^{-1} \| f \|$, we get the result (4.16). □

4.3. A semi-linear initial value problem.

In this section we study the initial value problem

$$\varepsilon \frac{d^2 u_\varepsilon}{dx^2} + a(x, u_\varepsilon) \frac{du_\varepsilon}{dx} + b(x, u_\varepsilon) = 0, \quad 0 < x < \infty \tag{4.18}$$

with

$$u_\varepsilon(0) = \alpha, \quad \frac{du_\varepsilon}{dx}(0) = \beta, \tag{4.19}$$

where a and b belong to $C^2[\bar{\mathbb{R}}_+ \times \mathbb{R}]$ and $a(x,u) \geq a_0 > 0$, $\forall x \in \bar{\mathbb{R}}_+$, $\forall u \in \mathbb{R}$.
As in the linear case a formal approximation of the solution u_ε is obtained by the composite expression:

$$\tilde{u}_\varepsilon(x) = w_0(x) + \varepsilon v_1(\tfrac{x}{\varepsilon}) = w_0(x) + \varepsilon v_1(\xi), \tag{4.20}$$

with

$$a(x,w_0)\frac{dw_0}{dx} + b(x,w_0) = 0, \quad w_0(0) = \alpha \tag{4.21}$$

and

$$\frac{d^2v_1}{d\xi^2} + a(0,w_0(0))\frac{dv_1}{d\xi} = 0, \quad \frac{dv_1}{d\xi}(0) = \beta - \frac{dw_0}{dx}(0), \quad \lim_{\xi\to\infty} v_1(\xi) = 0 \tag{4.22}$$

(compare (3.3), (3.8) and (3.9)).
It follows that

$$v_1(\tfrac{x}{\varepsilon}) = \frac{w_0'(0)-\beta}{a(0,\alpha)}\exp[-a(0,\alpha)\tfrac{x}{\varepsilon}] \tag{4.23}$$

and so v_1 has the character of a boundary layer, concentrated at $x = 0$ with width $O(\varepsilon)$.
We assume now for the same reasons as explained in 4.1 that the solution w_0 of the non-linear initial value problem (4.21) is twice continuously differentiable in some segment $0 \leq x \leq X$, with X some positive number. Using finally the regularity of the coefficients a and b we obtain after substitution of (4.20) into (4.18) and (4.19)

$$\varepsilon\frac{d^2\tilde{u}_\varepsilon}{dx^2} + a(x,\tilde{u}_\varepsilon)\frac{d\tilde{u}_\varepsilon}{dx} + b(x,\tilde{u}_\varepsilon) = O(\varepsilon), \quad \text{uniformly in } [0,X],$$

and (4.24)

$$\tilde{u}_\varepsilon(0) = \alpha + \varepsilon v_1(0) = \alpha + O(\varepsilon), \quad \frac{d\tilde{u}_\varepsilon}{dx}(0) = \beta.$$

Hence $\tilde{u}_\varepsilon(x)$ is a formal approximation of the solution $u_\varepsilon(x)$ in $[0,X]$. In order to prove that $\tilde{u}_\varepsilon(x)$ is really a good approximation of $u_\varepsilon(x)$ we put

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_\varepsilon(x),$$

or equivalently

$$u_\varepsilon(x) = \bar{u}_\varepsilon(x) + \bar{R}_\varepsilon(x), \tag{4.25}$$

with $\bar{u}_\varepsilon(x) = \tilde{u}_\varepsilon(x) - \varepsilon v_1(0)$ and $\bar{R}_\varepsilon(x) = R_\varepsilon(x) + \varepsilon v_1(0)$.

It is clear that also $\bar{u}_\varepsilon(x)$ is a formal approximation and we get after substitution of (4.25) into (4.18) for the remainder term $\bar{R}_\varepsilon(x)$ the initial value problem:

$$\varepsilon\frac{d^2\bar{u}_\varepsilon}{dx^2} + \varepsilon\frac{d^2\bar{R}_\varepsilon}{dx^2} + a(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon)\left(\frac{d\bar{u}_\varepsilon}{dx}+\frac{d\bar{R}_\varepsilon}{dx}\right) + b(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon) = 0$$

$$\bar{R}_\varepsilon(0) = 0, \quad \frac{d\bar{R}_\varepsilon}{dx} = 0,$$

or after using (4.24)

$$\varepsilon\frac{d^2\bar{R}_\varepsilon}{dx^2} + a(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon)\frac{d\bar{R}_\varepsilon}{dx} + \{a(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon) - a(x,\bar{u}_\varepsilon)\}\frac{d\bar{u}_\varepsilon}{dx} +$$
$$+ \{b(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon) - b(x,\bar{u}_\varepsilon)\} = O(\varepsilon), \quad \text{uniformly in } [0,X], \tag{4.26}$$

with

$$\bar{R}_\varepsilon(0) = \frac{d\bar{R}_\varepsilon}{dx}(0) = 0.$$

(4.26) yields a non-linear initial value problem for $\bar{R}_\varepsilon$, from which we should construct an a priori estimate for $\bar{R}_\varepsilon$; the function $\bar{u}_\varepsilon$ is given by (4.20) and is considered as a known function. In order to apply the fix point theorem we define the non-linear map F by

$$F(p) = \varepsilon\frac{d^2p}{dx^2} + a(x,\bar{u}_\varepsilon+p)\frac{dp}{dx} + \{a(x,\bar{u}_\varepsilon+p) - a(x,\bar{u}_\varepsilon)\}\frac{d\bar{u}_\varepsilon}{dx} +$$
$$+ \{b(x,\bar{u}_\varepsilon+p) - b(x,\bar{u}_\varepsilon)\}. \tag{4.27}$$

The linearization in $p = 0$ yields

$$L(p) = \varepsilon\frac{d^2p}{dx^2} + a(x,\bar{u}_\varepsilon)\frac{dp}{dx} + \left\{\frac{\partial a}{\partial u}(x,\bar{u}_\varepsilon)\frac{d\bar{u}_\varepsilon}{dx} + \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)\right\}p \tag{4.28}$$

and hence

$$\Psi(p) = F(p) - L(p) = \{a(x,\bar{u}_\varepsilon+p) - a(x,\bar{u}_\varepsilon)\frac{dp}{dx} + [\{a(x,\bar{u}_\varepsilon+p) - a(x,\bar{u}_\varepsilon)$$
$$- \frac{\partial a}{\partial u}(x,\bar{u}_\varepsilon)p\}\frac{d\bar{u}_\varepsilon}{dx} + \{b(x,\bar{u}_\varepsilon+p) - b(x,\bar{u}_\varepsilon) - \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)p\}]. \tag{4.29}$$

Both operators L and Ψ are well defined on the space N, which is defined as

$$N := \{p \mid p \in C^2[0, X],\ \ p(0) = \frac{dp}{dx}(0) = 0\}, \tag{4.30}$$

and where we choose as norm

$$|p| = \max_{[0,X]} |p(x)| + \sqrt{\varepsilon} \max_{[0,X]} |\frac{dp}{dx}| . \tag{4.31}$$

The Banach space B is specified as

$$B := \{q \mid q \in C^0[0, X]\}, \text{ with } \|q\| = \max_{[0,X]} |q(x)| \tag{4.32}$$

In order to apply the fix point theorem we have to show that the conditions (4.11) and (4.12) are fulfilled.
Due to the a priori estimate (3.19) for the solution of the initial value problem (3.17) we obtain for the initial value problem $L(p) = q$, $p(0) = 0$, $p'(0) = 0$ the estimate

$$|p| = \max_{[0,X]} |p(x)| + \sqrt{\varepsilon} \max_{[0,X]} |\frac{dp}{dx}(x)| < C(X)\|q\|,$$

for ε sufficiently small; $C(X)$ is a number independent of ε and only dependent on X. The norm (4.31) has been chosen in accordance with the result (3.19), derived in section 3.1.
It follows that

$$|L^{-1}(q)| < C(X)\|q\| \tag{4.33}$$

and so the condition (4.11) is fulfilled with $\ell = \frac{1}{C(X)}$.
Henceforth the constant $C(X)$ will be used as a generic constant. The Lipschitz-condition (4.12) becomes in our case:

$$\|\Psi(p_2) - \Psi(p_1)\| = \max_{[0,X]} |\{a(x,\bar{u}_\varepsilon+p_2) - a(x,\bar{u}_\varepsilon+p_1)\}\frac{dp_2}{dx} +$$

$$+ \{a(x,\bar{u}_\varepsilon+p_1) - a(x,\bar{u}_\varepsilon)\}(\frac{dp_2}{dx} - \frac{dp_1}{dx}) + \frac{d\bar{u}_\varepsilon}{dx}\int_{p_1}^{p_2} \{\frac{\partial a}{\partial u}(x,\bar{u}_\varepsilon+q) - \frac{\partial a}{\partial u}(x,\bar{u}_\varepsilon)\}dq$$

$$+ \int_{p_1}^{p_2} \{\frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon+q) - \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)\}dq| .$$

Application of the mean value theorem and using the regularity of the coefficients a and b $(a,b \in C^2[\bar{\mathbb{R}}_+ \times \mathbb{R}])$, we get

$$\|\Psi(p_2) - \Psi(p_1)\| \le C(X)\varepsilon^{-\frac{1}{2}}\rho|p_1 - p_2|, \quad \forall p_1,p_2 \in \Omega_N(\rho), \quad \forall\rho \in [0, \bar{\rho}]; \qquad (4.34)$$

$\bar{\rho}$ may be chosen as an arbitrary constant, e.g. $\bar{\rho} = 1$ and $C(X)$ depends only on X and is independent of ε.
The function $m(\rho)$ in (4.12) is simply $C(X)\varepsilon^{-\frac{1}{2}}\rho$ and also the second condition (4.12) of the fix point theorem is fulfilled.
From (4.13) and (4.33) it follows that

$$m(\rho_0) = C(X)\varepsilon^{-\frac{1}{2}}\rho_0 = \frac{1}{2}\ell = \frac{1}{2C(X)}$$

and hence

$$\frac{1}{2}\ell\rho_0 = \frac{\varepsilon^{\frac{1}{2}}}{4\{C(X)\}^3} .$$

The right hand side f of the equation (4.26) is uniformly $O(\varepsilon)$ and so the condition (4.14) of the fix point theorem is valid for ε sufficiently small. We are now in the position to apply the theorem of section 4.2 and we obtain at last:

$$|\bar{R}_\varepsilon| \le 2\ell^{-1}\| f \| = 2C(X)\| f \| = O(\varepsilon), \text{ uniformly in } [0, X].$$

Using the definition of the norm (4.31) we have:

$$\max_{[0,X]} |u_\varepsilon(x) - \bar{u}_\varepsilon(x)| + \sqrt{\varepsilon} \max_{[0,X]} \left|\frac{du_\varepsilon}{dx} - \frac{d\bar{u}_\varepsilon}{dx}\right| = O(\varepsilon)$$

or with the aid of (4.25) and (4.20)

$$|u_\varepsilon(x) - w_0(x)| = O(\varepsilon), \quad \text{uniformly in } [0,X] \qquad (4.35)$$

and

$$\left|\frac{du_\varepsilon}{dx} - \frac{dw_0}{dx} - \varepsilon\frac{d}{dx} v_1\left(\frac{x}{\varepsilon}\right)\right| = O(\sqrt{\varepsilon}), \quad \text{uniformly in } [0,X] \qquad (4.36)$$

(4.35) and (4.36) yield a satisfactory estimate for the solution u_ε of the singular perturbation problem (4.18).

Remarks.

1. The theory of this section has been generalized by Geel and de Jager (lit[9], [10] and [22]) to singular perturbations of hyperbolic type:

$$\varepsilon\left(\frac{\partial^2 u}{\partial t^2} - c^2(x,t)\frac{\partial^2 u}{\partial x^2}\right) + a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} + d(x,t,u) = 0$$
$$-\infty < x < +\infty, \quad t > 0 \qquad (4.37)$$

with

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < +\infty \tag{4.37}$$

Besides mild conditions for the regularity of the coefficients one has again to require $b(x,t,u) > 0$ and $|a(x,t,u)| < b(x,t,u)c(x,t)$ with $c(x,t) > 0$; the latter condition means that the subcharacteristics should be timelike, (see section 3.1, remark 3). The theory is essentially quite similar to that described in this section for the initial value problem (4.18), (4.19). It is again required that the solution of the reduced problem

$$a(x,t,w)\frac{\partial w}{\partial x} + b(x,t,w)\frac{\partial w}{\partial t} + d(x,t,w) = 0, \qquad t > 0,$$

with $w(x,0) = f(x)$, is twice, continuously differentiable. This is certainly not the case whenever $w(x,t)$ is multi valued. As to singular perturbation problems of hyperbolic type we mention here also the work of J.Genet and M.Madaune, lit[11], [29], [30]. These authors consider initial boundary value problems for the equation

$$\varepsilon\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right) + b(x,t)\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x,t)\frac{\partial u}{\partial x_k} + c(x,t)u + F(u) = f(x,t) \tag{4.38}$$,

where $x = (x_1, x_2, \ldots, x_n)$ is confined to points in a compact domain of $\mathbb{R}_n$ and where the function $F(u)$ contains the non linearity.

2. As already noticed before the whole theory breaks down for $x > X$, i.e. the point where $w(x)$ becomes singular.
This happens for instance in the famous van der Pol equation:

$$\varepsilon\frac{d^2u}{dt^2} + (u^2 - 1)\frac{du}{dt} + u = 0, \tag{4.39}$$

when u becomes equal to 1.
There are numerous rather difficult papers on the asymptotics of this equation, see lit[15], [16], [5]. So called non standard methods in the asymptotic theory of differential equations with a small or large parameter have, since the appearance of the well-known book by A.Robinson in 1966 [36], been used again and with succes by several mathematicians influenced by G.Reeb, see lit[27].
Equation (4.39) is one of the first of many equations which have been investigated by the non-standard groups in Strasbourg, Mühlhouse and Oran. We refer the reader to lit[27], [40], [3], [39] and [41].

4.4. Semi-linear boundary value problems.

We consider again the boundary value problem of section 4.1:

$$\varepsilon \frac{d^2u_\varepsilon}{dx^2} + a(x)\frac{du_\varepsilon}{dx} + b(x,u_\varepsilon) = 0, \qquad 0 < x < 1. \tag{4.1}$$

$$u_\varepsilon(0) = \alpha \quad \text{and} \quad u_\varepsilon(1) = \beta,$$

with $a \in C^1([0,1])$, $b \in C^2([0,1] \times \mathbb{R})$, $\frac{\partial b}{\partial u} \leq 0$ and $a \geq a_0 > 0$.

As in section 4.1 we have

$$u_\varepsilon(x) = \tilde{u}_\varepsilon(x) + R_\varepsilon(x) = w_0(x) + v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}) + R_\varepsilon(x), \tag{4.6}$$

with w_0, v_0 and v_1 determined by (4.2), (4.4), (4.5) and where we assume explicitly $w_0 \in C^2([0,1])$. The remainder term R_ε satisfies the non-linear boundary value problem

$$\varepsilon \frac{d^2R_\varepsilon}{dx^2} + a(x)\frac{dR_\varepsilon}{dx} + \{b(x,\tilde{u}_\varepsilon+R_\varepsilon) - b(x,\tilde{u}_\varepsilon)\} = O(\varepsilon), \tag{4.7}$$

uniformly in $[0,1]$ and $R_\varepsilon(0) = 0$ and $R_\varepsilon(1)$ asymptotically zero. In this section we give an estimate of R_ε by using the fix point theorem of section 4.2. Instead of (4.6) we write:

$$\begin{aligned} u_\varepsilon(x) &= w_0(x) + v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}) - \{v_0(\tfrac{1}{\varepsilon}) + \varepsilon v_1(\tfrac{1}{\varepsilon})\}\psi(x) + \bar{R}_\varepsilon(x) \\ &= \bar{u}_\varepsilon(x) + \bar{R}_\varepsilon(x), \end{aligned} \tag{4.40}$$

where $\psi \in C^\infty([0,1])$, $\psi \equiv 1$ for $\frac{3}{4} \leq x \leq 1$, $\psi \equiv 0$ for $0 \leq x \leq \frac{1}{4}$, and ψ independent of ε.

From the boundary layer character of v_0 and v_1 and the regularity of the functions $a(x)$ and $b(x,u)$ it follows that $\bar{R}_\varepsilon(x)$ satisfies the boundary value problem:

$$\varepsilon \frac{d^2\bar{R}_\varepsilon}{dx^2} + a(x)\frac{d\bar{R}_\varepsilon}{dx} + \{b(x,\bar{u}_\varepsilon+\bar{R}_\varepsilon) - b(x,\bar{u}_\varepsilon)\} = O(\varepsilon), \tag{4.41}$$

uniformly in $[0,1]$ and $\bar{R}_\varepsilon(0) = \bar{R}_\varepsilon(1) = 0$

In order to prove that $\bar{u}_\varepsilon$, or equivalently $\tilde{u}_\varepsilon$, is a good approximation for u_ε we use now the fix point theorem and derive an a priori estimate of $\bar{R}_\varepsilon$ from (4.41). The non-linear map F reads as:

$$F(p) = \varepsilon \frac{d^2p}{dx^2} + a(x)\frac{dp}{dx} + \{b(x,\bar{u}_\varepsilon+p) - b(x,\bar{u}_\varepsilon)\} \tag{4.42}$$

The linearization in $p = 0$ gives

$$L(p) = \varepsilon \frac{d^2p}{dx^2} + a(x)\frac{dp}{dx} + \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)p \tag{4.43}$$

and hence

$$\Psi(p) = b(x,\bar{u}_\varepsilon+p) - b(x,\bar{u}_\varepsilon) - \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)p \tag{4.44}$$

The spaces N and B are defined as:

$$N := \{p \mid p \in C^2([0,1]),\ \ p(0) = p(1) = 0\} \quad \text{with} \quad |p| = \max_{[0,1]} |p(x)| \tag{4.45}$$

and

$$B := \{q \mid q \in C^0([0,1])\} \quad \text{with} \quad \| q \| = \max_{[0,1]} |q(x)| \tag{4.46}$$

With the aid of the maximum principle it is not difficult to show that

$$|L^{-1}(q)| \le \frac{1}{\ell} \| q \|, \qquad \forall q \in B, \tag{4.47}$$

where ℓ is a number independent of ε, and so the first condition (4.11) of the fix point theorem is fulfilled.
From (4.44) we get

$$\begin{aligned} \|\Psi(p_1) - \Psi(p_2)\| &= \operatorname*{Max}_{[0,1]} \left| \int_{p_1}^{p_2} \{\frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon+q) - \frac{\partial b}{\partial u}(x,\bar{u}_\varepsilon)\} dq \right| \\ &\le C\rho |p_1 - p_2|, \quad \forall p_1,p_2 \in \Omega_N(\rho) \ \text{ and } \ \forall \rho \in [0, \bar{\rho}], \end{aligned} \tag{4.48}$$

where C and $\bar{\rho}$ are positive numbers independent of ε and $\bar{\rho}$ is arbitrarily positive. It follows that also the second condition (4.12) of the fix point theorem is satisfied. Further we have

$$m(\rho) = C\rho \quad \text{and} \quad m\rho_0 = C\rho_0 = \tfrac{1}{2}\ell \quad \text{or} \quad \rho_0 = \frac{\ell}{2C}.$$

From the fix point theorem we get now the result that the equation $F(p) = f$ with

$$\| f \| \le \tfrac{1}{2}\ell\rho_0 = \frac{1}{4}\frac{\ell^2}{C} \tag{4.49}$$

has a solution $p \in N$ with $|p| \le 2\ell^{-1}\| f \|$.
Because $\bar{R}_\varepsilon$ satisfies (4.49) for ε sufficiently small we obtain finally

$$|\bar{R}_\varepsilon(x)| = O(\varepsilon), \quad \text{uniformly in } [0,1],$$

or according to (4.40)

$$|u_\varepsilon(x) - w_0(x) - v_0(\tfrac{x}{\varepsilon})| = O(\varepsilon), \quad \text{uniformly in } [0,1] \tag{4.50}$$

and

$$|u_\varepsilon(x) - w_0(x)| = O(\varepsilon), \quad \text{uniformly in } [\delta,1], \ \forall\delta \text{ with } 0 < \delta < 1.$$

It is possible to obtain with the aid of the fix point theorem a better estimate, yielding also an estimate for the derivative of u_ε. Then we need a formal approximation of the form:

$$\tilde{u}_\varepsilon(x) = w_0(x) + \varepsilon w_1(x) + v_0(\tfrac{x}{\varepsilon}) + \varepsilon v_1(\tfrac{x}{\varepsilon}) + \varepsilon^2 v_2(\tfrac{x}{\varepsilon}),$$

which requires more regularity for the functions $a(x)$, $b(x,u)$ and $w_0(x)$. The interested reader is referred to E.M.de Jager, lit 23 , where also the following semi-linear boundary value problem has been treated:

$$\varepsilon \frac{d^2u_\varepsilon}{dx^2} + a(x,u_\varepsilon)\frac{du_\varepsilon}{dx} + b(x,u_\varepsilon) = 0$$

$$u_\varepsilon(0) = \alpha, \quad u_\varepsilon(1) = \beta.$$

Remarks.

1. It is again possible to generalize the theory of this section to singular perturbations of the form (4.9), see A.van Harten, lit[19], pp.205-213 and W.Eckhaus [6], Chapters 5,7.
2. Singular perturbations of elliptic type with a zero-order degeneration, such as

$$\varepsilon \frac{d^2u}{dx^2} + b(x,u) = 0, \quad u(0) = \alpha, \quad u(1) = \beta \tag{4.51}$$

   with $\frac{\partial b}{\partial u} < 0$, and its analogue in more independent variables have been considered by A.van Harten [19], pp.111-113 and W.Eckhaus [6], section 7.3, P.C.Fife [8] and R.Lutz and T.Sari [28].
   The reduced equation is $b(x,w) = 0$ and the boundary conditions can only be met by introducing boundary layers at both ends of the segment $[0,1]$; the width of the boundary layer is $O(\sqrt{\varepsilon})$. In case the reduced equation admits several solutions these may be connected with each other by free boundary layers located in the interior of the segment $[0,1]$, see [8] and [28].
3. Then non-linear boundary value problem

$$\varepsilon \frac{d^2u}{dx^2} + u\frac{du}{dx} + u = 0, \quad u(0) = \alpha, \quad u(1) = \beta$$

has been investigated thoroughly by T.Sari in lit[38].

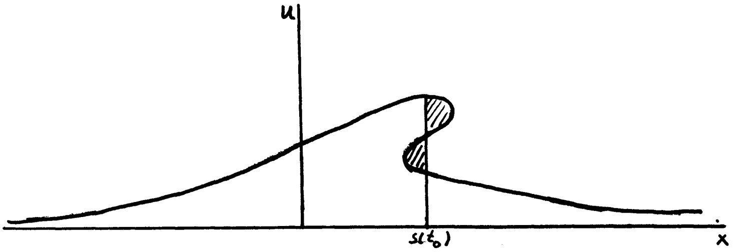
5. Singular perturbations and shock-waves.

We consider the following initial value problem for the well-known transport equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad -\infty < x < +\infty, \quad t > 0 \tag{5.1}$$

$$u(x,0) = u_0(x), \tag{5.2}$$

with $|u_0(x)|$ decreasing sufficiently fast to zero for $x \to +\infty$. Because the top of the wave travels faster than the bottom the solution will be in general multi valued which is not acceptable in physics. The solution to this anomaly is to introduce weak solutions and to allow in this way solutions which may be discontinuous across some line or across several lines in the (x,t)-plane, so-called shock-waves.



Shock Wave; conservation of momentum; $t = t_0$

In case the shock-wave is represented by $x = (s(t)$ we find by integrating (5.1) with respect to x from $-\infty$ to $+\infty$ and interchanging differentiation to t and integration:

$$\frac{d}{dt}\int_{-\infty}^{+\infty} u(x,t)dx + \frac{ds}{dt}[u] = \frac{1}{2}[u^2], \tag{5.3}$$

where $[f] := f(s(t)+0) - f(s(t)-0)$ denotes the jump discontinuity of f across the shock-wave.

We can do the same for the equation

$$u\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0$$

and we obtain similarly

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u^2dx + \frac{ds}{dt}[\frac{1}{2}u^2] = \frac{1}{3}[u^3] \tag{5.4}$$

We call $\int_{-\infty}^{+\infty} u(x,t)dx = M(t)$ the total momentum and $\int_{-\infty}^{+\infty} \frac{1}{2}u^2(x,t)dx = E(t)$ the total energy of the wave.

Conservation of momentum yields for the velocity of the shock-wave

$$\frac{ds}{dt} = \frac{\frac{1}{2}[u^2]}{[u]} \tag{5.5}$$

and conservation of energy gives for this velocity

$$\frac{ds}{dt} = \frac{\frac{1}{3}[u^3]}{\frac{1}{2}[u^2]} \tag{5.6}$$

It follows that the slope of the shock-wave and hence also the shock-wave itself depends on the conservation law, which has been imposed on the physical system a priori. In case we require conservation of momentum and so the validity of (5.5), we get from (5.4)

$$\frac{dE}{dt} + \frac{1}{2}\frac{[u^2]}{[u]}[\tfrac{1}{2}u^2] = \tfrac{1}{3}[u^3]$$

or

$$\frac{dE}{dt} = \frac{1}{12}[u]^3, \qquad 85.7)$$

which is non-zero.

Solving the initial value problem (5.1)-(5.2) we cannot satisfy at the same time conservation of momentum and conservation of energy; one has to make a choice a priori.

We consider now Burger's equation

$$\frac{\partial u_\varepsilon}{\partial t} + u\frac{\partial u_\varepsilon}{\partial x} = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0 \tag{5.8}$$

with the initial condition

$$u_\varepsilon(x,0) = u_0(x), \quad -\infty < x < +\infty. \tag{5.9}$$

This initial value problem is uniquely solvable, the solution is single-valued and we do not need the introduction of weak solutions as long as $u_0(x)$ is bounded and measurable.

Because u and $\frac{\partial u}{\partial x}$ vanish for $x \to \pm\infty$ it follows from (5.8) that u_ε satisfies conservation of momentum, but conservation of energy is in general impossible because

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\tfrac{1}{2}u_\varepsilon^2 dx = -\varepsilon\int_{-\infty}^{+\infty}\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 dx$$

Oleinik has shown in [32] that the solution of the singular perturbation problem (5.8)-(5.9) approximates for $\varepsilon\downarrow 0$ the solution of the initial value problem (5.1)-(5.2) in a weak sense and under the assumption that momentum is conserved in the latter. The singular perturbation $\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}$ works as a viscosity term and it smoothes the discontinuous transition across the shock.

P.Lax and C.Levermore [25] and the author and P.Wilders [24] have also considered the singular perturbation problem

$$\frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon\frac{\partial u_\varepsilon}{\partial x} = \varepsilon\frac{\partial^3 u_\varepsilon}{dx^3}, \quad -\infty < x < +\infty, \quad t > 0 \tag{5.10}$$

with the initial condition

$$u_\varepsilon(x,0) = u_0(x), \quad -\infty < x < +\infty \tag{5.11}$$

with $u_0(x) \to 0$ sufficiently fast for $|x| \to +\infty$.

(5.10) is known as the Korteweg-de Vries equation, which has been in the focus of much attention of many pure and applied mathematicians, as well analysts as differential geometers. It describes long waves in relatively shallow canals; u determines the height of the water above some level of equilibrium.

The equation (5.10) yields an infinite number of conservation laws for u_ε and in particular also conservation of momentum and conservation of energy.

Whenever ε approaches zero it is to be expected that the conservation laws remain valid also in the limit, but then the limit function cannot be a weak solution of the reduced problem (5.1)-(5.2), because the solution of the latter problem can sustain only <u>one</u> conservation law. It appears that the singular perturbation problem (5.10)-(5.11) is really a difficult problem and a satisfactory description of what happens when $\varepsilon \to 0$ is still an open question.

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