Astérisque

JEFF CHEEGER

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Astérisque, tome 101-102 (1983), p. 118-134
<http://www.numdam.org/item?id=AST_1983__101-102__118_0>

Hodge Theory of Complex Cones

by

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0. Introduction

In this note, we ammend and continue the final portion of the discussion of [2]. In the initial paragraph of Section 7 of [2], we stated that for "admissible" riemannian pseudomanifolds (in the sense of that paper) for which the metric is Kähler, the standard consequences of Hodge theory (the "Kähler package") follow from the Strong Hodge Theorem. In particular, it was asserted that the properties of the Kähler metric on which these results depend are purely local on the nonsingular part. However, as we realized shortly after the publication of [2], and pointed out in [5] (see pp. 307 and 317) this assertion is incorrect. Let J be the almost complex structure and Δ the Laplacian. Then the relation $\Delta J = \Delta J$ holds locally; see [6]. Since J is an isometry, it follows that $h \in L^2$ implies $Jh \in L^2$ and $\Delta h=0$ implies $Jh=0$. However, in the present context (of incomplete riemannian manifolds) the global relations $Jh \in L^2$, $\Delta Jh=0$ do not in general imply $dJh=\delta Jh=0$, even if $dh=\delta h=0$. This is the additional fact which is actually required for the applications to Kähler geometry; see [6], pp. 109-111. Of course it fails locally, even in $\mathbb{C}^N$. More generally however, let X be a suitable space with singularities Σ. Then there does exist a criterion which is local near Σ, and which implies that if $h \in L^2$, $dh=\delta h=0$ on $X \setminus \Sigma$, then $dJh=\delta Jh=0$. Consider the Laplacian Δ_G, such that a smooth form θ is in dom Δ_G if $\theta, d\theta, \delta\theta, d\delta\theta, \delta d\theta \in L^2$. For spaces with conical singularities, $\overline{d}^* = \overline{\delta}$ (apart from the case of ideal boundary conditions) and we assume this from now on; see [2]. Then

Δ_G is essentially self adjoint and its kernel contains precisely those $h \in L^2$ with $dh=\delta h=0$. If one assumes that away from the singular set, X is a complete manifold without boundary, and $\theta \in L^2$ is smooth, then to check if $\theta \in \operatorname{dom} \Delta_G$, it suffices to check the above conditions near Σ.

The purpose of the present paper is to show that with suitable extra hypotheses, J does preserve dom Δ_G, and hence ker Δ_G as well. Our argument is computational and requires rather strong assumptions. However, it does apply in some cases where Δ_0, the Laplacian on forms of compact support, is not essentially self adjoint. This possibility, which is not an invariant of quasi-isometry type, is the essential complication in the general problem (which is still open).

In Section 1, we consider a metric cone $C(N^m)$, and a bounded map $J: \Lambda^i(C(N)) \to \Lambda^i(C(N))$, such that

1) J is linear with respect to functions of r,

2) J commutes with dilations,

3) $\Delta J = J\Delta$.

We call such a map J <u>conical</u>. We show that subject to these assumptions, J preserves dom Δ_G except for a small number of possibilities. By 1) and 3) it is easy to see that these can occur only in dimensions $i = \frac{m+1}{2}$, $\frac{m+1}{2} \pm 1$, ($m+1$ even), or $i = \frac{m+1}{2} \pm \frac{1}{2}$, $\frac{m+1}{2} \pm \frac{3}{2}$, ($m+1$ odd). Using the strong hypothesis 2), we show that in fact, only $i = \frac{m+1}{2}$ or $i = \frac{m+1}{2} \pm \frac{1}{2}$ are possible.

In Section 2, we consider an arbitrary Kähler manifold Y^{m+1} with the property that an L^2-harmonic form is in dom Δ_G except possibly when $i = \frac{m+1}{2}$, $\frac{m+1}{2} \pm 1$. By an argument which fails for $i = \frac{m+1}{2} \pm 1$, we rule out the case $i = \frac{m+1}{2}$.

As announced in [5], the proof can then be generalized to case of piecewise flat spaces, but we will give the details of this elsewhere (see Theorem 2.3 for the general result proved here).

We mention that the exceptional cases of Section 1 actually occur as formal possibilities for the double cover of the punctured complex plane (but, as allowed in Section 1, J is not an almost complex structure). This space also provides a simple example for which Δ_0 fails to be essentially self adjoint.

We conclude this section by observing that for spaces with (not necessarily isolated) conical singularities, the "Kähler package" can fail to hold if the singularities are not complex analytic. According to [1], an arbitrary orientable manifold X^{2n}, can be realized as an r-fold branched cover of the standard p.l. sphere, S^{2n}. By forming the connected sum $S^{2n}\#\mathbb{C}P(n)$ (removing a ball which is disjoint from the branch locus L) it follows that $X'=X\#rCP(n)$ is an r-fold branched cover of $\mathbb{C}P(n)$. By pulling back the metric on $\mathbb{C}P(n)$, one obtains a Kähler metric on X' which has conical singularities. The Mayer-Vietoris sequence shows $b_1(X)=b_1(X')$. In particular, if $b_1(X)$ is odd, $H^1(X',\mathbb{C})$ does not have a Hodge structure. Of course, metrics constructed via branched covers are of considerable interest for L^2-cohomology. But in the present context one should require that the branch locus be a complex subvariety.

1. $\Delta J = J\Delta$ on cones

In this section we consider the metric cone $C(N^m)$ on a compact smooth manifold N^m with empty boundary (see [2], [4] for background and notation). We begin by deriving the formula for the Laplacian of an i-form,

(1.1) $\theta = g(r)\phi(x) + f(r)dr\wedge\omega(x)$,

on $C(N^m)$. Intrinsic operations on N^m are denoted by adding a tilda. Let $\Omega, dr\wedge\Omega$ denote the volume forms on $N^m, C(N^m)$ respectively.

(1.2) $d(g\phi) = gd\phi + g'dr\wedge\phi$,

(1.3) $*d(g\phi) = r^{m-2i}g'\tilde{*}\phi + (-1)^{i+1}r^{m-2i-2}gdr\wedge\tilde{*}d\phi$,

(1.4) $$d*d(g\phi) = (r^{m-2i}g'' + (m-2i)r^{m-2i-1}g')dr\wedge\tilde{*}\phi + r^{m-2i}g'd\tilde{*}\phi \\ + (-1)^i r^{m-2i-2}g\ dr\wedge d\tilde{*}d\phi$$

(1.5) $$*d*d(g\phi) = (-1)^{i(m-i)}(g'' + (m-2i)r^{-1}g')\phi \\ + (-1)^{i+1}r^{-2}g\tilde{*}d\tilde{*}d\phi + (-1)^{m-i+1}r^{-2}g'dr\wedge\tilde{*}d\tilde{*}\phi,$$

(1.6) $\delta d(g\phi) = (-g'' - (m-2i)r^{-1}g')\phi + r^{-2}g\delta d\phi - r^{-2}g'dr\wedge\delta\phi$,

Similarly,

(1.7) $*(g\phi) = (-1)^i r^{m-2i}gdr\wedge\tilde{*}\phi$,

(1.8) $d*(g\phi) = (-1)^{i+1}r^{m-2i}gdr\wedge d\tilde{*}\phi$,

(1.9) $*d*(g\phi) = (-1)^{i+1}r^{-2}g\tilde{*}d\tilde{*}\phi$,

(1.10) $d*d*(g\phi) = (-1)^{i+1}r^{-2}gd\tilde{*}d\tilde{*}\phi + (-1)^{i+1}(-2r^{-3}+r^{-2}g')dr\wedge\tilde{*}d\tilde{*}\phi$,

(1.11) $d\delta(g\phi) = (-2r^{-3}g + r^{-2}g')dr\wedge\tilde{\delta}\phi + r^{-2}gd\tilde{\delta}\phi$.

Also,

(1.12) $d(fdr\wedge\omega) = -fdr\wedge d\omega,$

(1.13) $*d(fdr\wedge\omega) = -r^{m-2i}f\tilde{*}d\omega,$

(1.14) $d*d(fdr\wedge\omega) = (-r^{m-2i}f'-(m-2i)r^{m-2i-1}f)dr\wedge\tilde{*}d\omega$
$- r^{m-2i}fd\tilde{*}d\omega,$

(1.15) $*d*d(fdr\wedge\omega) = (-1)^{m-i}r^{-2}fdr\wedge\tilde{*}d\tilde{*}d\omega + (-1)^{i(m-i)+1}(f'+(m-2i)r^{-1}f)d\omega,$

(1.16) $\delta d(fdr\wedge\omega) = r^{-2}fdr\wedge\tilde{\delta}d\omega + ((m-2i)r^{-1}f+f')d\omega,$

and,

(1.17) $*(fdr\wedge\omega) = r^{m-2i+2}f\tilde{*}\omega,$

(1.18) $d*(fdr\wedge\omega) = (r^{m-2i+2}f'+(m-2i+2)r^{m-2i+1}f)dr\wedge\tilde{*}\omega + r^{m-2i+2}fd\tilde{*}\omega,$

(1.19) $*d*(fdr\wedge\omega) = (-1)^{m-i+1}r^{-2}fdr\wedge\tilde{*}d\tilde{*}\omega$
$+ (-1)^{(m-i+1)(i-1)}(f'+(m-2i+2)r^{-1}f)\omega,$

(1.20) $d*d*(fdr\wedge\omega) = (-1)^{(m-i+1)(i-1)}(f''+(m-2i+2)r^{-1}f'-(m-2i+2)r^{-2}f)dr\wedge\omega$
$+ (-1)^{m-i}r^{-2}fdr\wedge d\tilde{*}d\tilde{*}\omega$
$+ (-1)^{(m-i+1)(i-1)}(f'+(m-2i+2)r^{-1}f)d\omega,$

(1.21) $d\delta(\omega) = (-f''-(m-2i+2)r^{-1}f + (m-2i+2)r^{-2}f)dr\wedge\omega$
$+ r^{-2}fdr\wedge d\tilde{\delta}\omega + (-f'-(m-2i+2)r^{-1}d\omega.$

Thus, on i-forms,

(1.22) $\Delta(g\phi+fdr\wedge\omega) = (-g''-(m-2i)r^{-1}g')\phi + r^{-2}g\tilde{\Delta}\phi$
$- 2r^{-3}gdr\wedge\tilde{\delta}\phi + (-f''-(m-2i+2)r^{-1}f' + (m-2i+2)r^{-2} f)dr\wedge\omega$
$+ r^{-2}fdr\wedge\tilde{\Delta}\omega-2r^{-1}fd\omega.$

Following [3], for $\mu\geq 0$, we set

(1.23) $\alpha(i) = \frac{1+2i-m}{2}$

(1.24) $\nu(i,\mu) = \sqrt{\alpha^2(i)+\mu}$

(1.25) $a^{\pm}(i,\mu) = \alpha(i)\pm\nu(i,\mu)$

Note that $a^+(i,\mu) \geq 0$, $a^-(i,\mu) \leq 0$. Let $\phi(x)$ be a coclosed i-form and ψ a closed (i+1)-form, with

(1.26) $\tilde{\Delta}\phi = \mu\phi$,

(1.27) $\tilde{\Delta}\psi = \mu\psi$,

(where the μ's in (1.26), (1.27) may be distinct). Then as in [4], for $0<r_1<r_2<\infty$, an harmonic form on $\overline{C_{r_1,r_2}(N^m)}$ can be written as a convergent series on $C_{r_1,r_2}(N^m)$ of forms of the following four types (the forms in (1.30) were missed in [3]; see [4] for corrections).

(1.28) $r^{a^{\pm}(i)}\phi$,

(1.29) $r^{a^{\pm}(i)}d\phi + a^{\pm}(i)r^{a^{\pm}(i)-1}dr\wedge\phi$,

(1.30) $r^{a^{\pm}(i)+2}d\phi + a^{\mp}(i)r^{a^{\pm}(i)+1}dr\wedge\phi$,

(1.31) $r^{a^{\pm}(i)+1}dr\wedge\psi$.

In (1.29), (1.30) ϕ can be taken coexact. If $a^{\pm}(i,\mu) = 0$, we must also introduce -( ) solutions $\log r\, h_{\frac{m-1}{2}}$, $r \log r\, dr\wedge h_{\frac{m+1}{2}}$, where $\tilde{\Delta}h_j=0$.

We say that the forms in (1.28)-(1.30) are of types $\pm(1)$, $\pm(2+3)$, $\pm(2-3)$, $\pm(4)$ respectively. Those of type (1.28), (1.30) are coclosed and closed respectively. Those of type (1.29) are exact and coexact while those of type (1.30) are neither closed nor coclosed for $\nu(i)\neq 1$. For $\nu(i)=1$, the +(2+3) and -(2-3) solutions coincide.

As noted in [3] (since $\bar{d}^*=\bar{\delta}$) <u>the sums of +( ) solutions are precisely those harmonic forms which are locally in dom Δ_G</u> near the vertex. Specifically,

A) Type -(1) $\in L^2(C_{0,1}(N^m))$, means $2\alpha(i)-2\nu(i)-2i+m>-1$, or

(1.32) $\nu(i) < 1$.

If we apply d to type -(1), the resulting type -(2+3) form is never in $L^2(C_{0,1}(N^m))$, (unless we have nonempty ideal boundary).

B) Type -(2+3) is never in $L^2(_{0,1}(N^m))$.

C) Type -(2-3) is in $L^2(C_{0,1}(N^m))$ means $2\alpha(i)-2\nu(i)+2-2i+m>-1$, or

(1.33) $\nu(i)<2$.

If d,δ of type -(2-3) (types -(1), -(4) respectively) are in L^2

(1.34) $\nu(i)<1$.

D) Type -(4) is in $L^2(C_{0,1}(N^m))$ means

(1.35) $\nu(i)<1$.

Now assume that we are given J satisfying 1)-3) of Section 0. In particular, J preserves homogeneity, ϕ,ω have homogeneity 0 and dr has homogeneity 1. Thus there exist linear maps

$J_+\colon \Lambda^i(N^m) \to \Lambda^{i+1}(N^m)$,

$J_0\colon \Lambda^i(N^m) \to \Lambda^i(N^m)$,

$\mathcal{J}_0\colon \Lambda^i(N^m) \to \Lambda^i(N^m)$,

$\mathcal{J}_-\colon \Lambda^i(N^m) \to \Lambda^{i-1}(N^m)$,

such that if we set

(1.36) $J(\phi(r,x)) + dr\wedge\omega(r,x)) = rJ_+(\omega(r,x))$
$\qquad + J_0(\phi(r,x) + dr\wedge\mathcal{J}_0(\omega(r,x))$
$\qquad + r^{-1}dr\wedge\mathcal{J}_-(\phi(r,x))$,

then we have

(1.37) $\Delta J=J\Delta$.

Of course, we are motivated by the case in which J is a conical almost complex structure with respect to which the metric is Kähler. But we do not assume this yet.

In view of (1.36), (1.37), J takes harmonic forms to harmonic forms, <u>preserving homogeneity with respect to r</u>. We consider then the possibility that the expansion in terms of harmonic forms of $J(\theta)$ contains a term of the form θ_2 where $J(\theta_1)$ and θ_2 are of two particular pure types $\pm(1)$, $\pm(2+3)$, $\pm(2-3)$, $\pm(4)$. This leads to the equations

$$(1.38) \quad a^{\pm}(i,\mu_{\alpha}^{\pm}) = a^{\pm}(i-1),\mu_{\beta}^{\pm}) = a^{\pm}(i-1),\mu_{\gamma}^{\pm}) + 2 = a^{\pm}(i-2,\mu_{\delta}^{\pm}) + 2.$$

For each particular pair of choices, say $(+,\beta)$ and $(-,\gamma)$, we can solve uniquely for say μ_{γ}^{-} in terms of μ_{β}^{+}. This illuminates to some extent the action of J on harmonic forms for the special case, $C(S_1^{2n-1}) = \mathbb{C}^n$. Of course, this action can be calculated explicitly by other methods.

We now make explicit the possible solutions of (1.38) corresponding to pairs containing a + sign and a - sign. The other cases, which are straightforward will be omitted. View each of the expressions in (1.38) as a function of μ with α held fixed. In the first and fourth cases we have $|\alpha|+\mu>0$ while in the middle two cases $\mu>0$. Observe that in all cases $\mu=0$ is a minimum for functions corresponding to a + sign and a maximum for those corresponding to a - sign. Since $+(\ )\to+\infty$, and $-(\ )\to-\infty$ as $\mu\to\infty$, it follows by inspection that the only possibilities for the degree of homogeneity of a $+(\ )$ solution to equal that of a $-(\ )$ solution are

$$(1.39) \quad \nu(i,\mu_{\alpha}^{-}) + \nu(i-1,\mu_{\beta}^{+}) = 1,$$

(1.40) $\nu(i-2,\mu_\delta^-) + \nu(i-1,\mu_\beta^+) = 1,$

(1.41) $\nu(i-1,\mu_\gamma^-) + \nu(i,\mu_\alpha^+) = 1,$

(1.42) $\nu(i-1,\mu_\gamma^-) + \nu(i-2,\mu_\delta^+) = 1,$

(1.43) $\nu(i-1,\mu_\gamma^-) + \nu(i-1,\mu_\beta^+) = 2.$

The possibilities (1.39)-(1.42) occur if respectively

(1.44) $i = \frac{m}{2},\ i-1 = \frac{m}{2} - 1,$

(1.45) $i-1 = \frac{m}{2},\ i-2 = \frac{m}{2} - 1,$

(1.46) $i = \frac{m}{2},\ i-1 = \frac{m}{2} - 1,$

(1.47) $i-1 = \frac{m}{2},\ i-2 = \frac{m}{2} - 1.$

If m+1 is odd, (1.43) corresponds to

(1.48) $i-1 = \frac{m+1}{2} - \frac{1}{2},\ \frac{m+1}{2} - \frac{3}{2}\ .$

If m+1 is even, (1.43) corresponds to

(1.49) $i-1 = \frac{m-1}{2}.$

It will be important to note that in either of these cases, if

(1.50) $$J(r^{a^+(i-1,\mu)}d\phi + a^+(i-1,\mu)r^{a^+(i-1,\mu)-1}dr\wedge\phi)$$
$$= r^{a^-(i-1,\lambda)+2}d\eta + a^+(i-1,\lambda)r^{a^-(i-1,\lambda)+1}dr\wedge\eta + h,$$

then multiplying through by $r^{2-2\nu(i-1,\mu)} = r^{2\nu(i-1,\lambda)-2}$; gives also

(1.51) $$J(r^{a^-(i-1),\mu)+2}d\phi + a^+(i-1,\mu)r^{a^-(i-1,\mu)+1}dr\wedge\phi)$$
$$= r^{a^+(i-1,\lambda)}d\eta + a^+(i-1,\lambda)r^{a^+(i-1,\lambda)-1}dr\wedge\eta + r^{2-2\nu(i-1,\mu)}h$$

If we now restrict attention to the case m+1 even we have

<u>Proposition 1.1</u>. Let J on $C(N^m)$ be conical. Let the i-forms θ_1^+ and θ_2^- be +( ) and -( ) solutions as in (1.28)-(1.31). If the expansion of $J(\theta_1^+)$ contains θ_2^-, then

(1.52) $\theta_1^+ = r^{a^+(i-1,\mu)}d\phi^+ + r^{a^+(i-1),\mu)-1}dr\wedge\phi_+^+$,

(1.53) $\theta_2^- = r^{a^-(i-1,\lambda)+2}d\phi^- + r^{a^-(i-1,\lambda)+1}dr\wedge\phi^-$,

(1.54) $\nu(i-1,\mu) + \nu(i-1,\lambda) = 2$,

(1.55) $i = \frac{m+1}{2}$.

Example 1.1 Consider the cone on a circle of length 4π, the double cover of the complex plane minus the origin. Then $\nu(0,\mu) = 0, \frac{1}{2}, 1, \frac{3}{2}$, - where the nonzero values have multiplicity 2. Then we have the possibilities $\frac{1}{2} + \frac{3}{2} = 2$, $1+1 = 2$.

We now consider briefly Δ_F, the self adjoint Friedrichs extension of Δ_0. Recall that $\theta \in \operatorname{dom} \Delta_F$ if $\theta, \Delta\theta \in L^2$ and there exist compactly supported forms $\{\theta_j\}$, such that $\theta_j \to \theta$, $d\theta_j \to d\theta$, $\delta\theta_j \to \delta\theta$. Whereas for Δ_G we had $d\delta\theta$, $\delta d\theta \in L^2$, this is not required for Δ_F. However, for Δ_G there need only exist possibly distinct sequences of compactly supported forms $\theta_j \to \theta$, $\bar{\theta}_j \to \theta$, with $d\theta_j \to d\theta$, $\delta\bar{\theta}_j \to \delta\theta$. It follows easily that $\theta \in \ker \Delta_F$ implies $d\theta = \delta\theta = 0$ (but not conversely). It is also easy to check (with the help of a cut-off function) that for $\nu(i-1,\mu) < 1$ the +(2+3) solution is in $\operatorname{dom} \Delta_G \setminus \operatorname{dom} \Delta_F$ and the -(2-3) solution is in $\operatorname{dom} \Delta_F \setminus \operatorname{dom} \Delta_G$. For $\nu(i-1,\mu) > 1$, Δ_F and Δ_G coincide. But if $\nu(i-1,\mu) < 2$, Δ_0 is still not essentially self adjoint (this is the "limit circle" case for forms of type -(2-3)).

$\Delta J = J\Delta$ on Complex Cones

In the previous section we saw that for even dimensional cones and J as in Proposition 1.1, the possibility that J takes +( ) solutions to -( ) solutions could be ruled out unless $i = \frac{m+1}{2}$. In this section we want to show that in case J is the almost complex structure associated to a Kähler metric with conical singularities then this possibility can be ruled out as well.

Let Y^{m+1} be a (possibly incomplete) Kähler manifold and suppose that with the possible exception of dimensions $\frac{m+1}{2}$, $\frac{m+1}{2} \pm 1$, every L^2-harmonic form is in dom Δ_G (see A)-D) of Section 2).

Let $h \in \Lambda^{(m+1)/2}(Y^{m+1}) \cap L^2$, $\Delta h=0$ and put

(2.1) $h = Lx+p$

where p is primitive. If $\Lambda^2 y=x$, then also

(2.2) $h = L\Lambda^2 y+p = \Lambda L\Lambda y+p$.

As is well known, it follows that $x,y,p \in L^2$; see [6]. Since Jx is an $(\frac{m+1}{2} - 2)$-form, $Jx \in \text{dom}\, \Delta_G$ and

(2.3) $dJLx = dLJx$
$\qquad\qquad = LdJx \in L^2$

Similarly,

(2.4) $\delta J\Lambda L\Lambda y = \delta\Lambda JL\Lambda y$
$\qquad\qquad = \Lambda\delta JL\Lambda y \in L^2$

Thus $dJ(h-p)$, $\delta J(h-p) \in L^2$ and in the same way $d(h-p), \delta(h-p) \in L^2$. Thus, if we also assume $dh, \delta h \in L^2$, then $dp, \delta p \in L^2$. Since $Lp=\Lambda p=0$, by [6], p. 109, we have

(2.5) $L^2 \ni (d\Lambda - \Lambda d)p$

$= -\delta^c p,$

(2.6) $L^2 \ni (\delta L - L\delta)p$

$= d^c p.$

Thus $dJp, \delta Jp \in L^2$. This, together with (2.3), (2.4) implies

(2.7) $dJh, \delta Jh \in L^2$.

Essentially the same argument shows that if we make the (global) hypothesis that L^2-harmonic forms on Y^{m+1} are closed and coclosed, except perhaps in dimensions $\frac{m+1}{2}$, $\frac{m+1}{2} \pm 1$, then in fact the same holds in dimension $\frac{m+1}{2}$. But for primitive $(\frac{m+1}{2} - 1)$-forms a similar argument shows only $dJp \in L^2$ and not $\delta Jp \in L^2$.

Now in the previous section, we saw that if the metric and J are conical, then J carries +( ) solutions to +( ) solutions with the only possible exception corresponding to (1.43), (1.49). Moreover, if this exception occurs, we can assume that J of a +(2+3) solution has a component corresponding to a -(2-3) solution, d and δ of which are not in L^2 on $C_{0,\epsilon}(N^m)$. Indeed, by (1.43), (1.50), (1.51) we can choose the -(2-3) solution to correspond to $\nu \leq 1$, and our claim then follows from A), D) of Section 1. Thus, applying (2.7), we see that <u>J carries +( ) harmonic forms to +( ) harmonic forms in all cases</u>. This is the crux of what we wanted to prove.

We now show that subject to the same assumptions, J actually preserves dom Δ_G. We recall the easily proved statement from [3] that an arbitrary eigenform of Δ which is in L^2 $(C_{0,\epsilon}(N^m))$, admits a convergent series representation, which can be grouped so that the first term is a harmonic form h. Also, the notions of +( )

and -( ) solutions are defined for arbitrary eigenforms. Then our previous analysis immediately implies that for arbitrary λ, J as above takes +( ) eigenforms to +( ) eigenforms. However, the +( ) solutions are precisely the eigenforms in the domain of the Laplacian Δ_G. Since whether a form is in dom Δ_G can be checked by examining its spectral representation, we now have the following result.

Theorem 2.1. Let Y be a complete Kähler manifold with isolated metrically conical singularities, and suppose that near the singularities the almost complex structure J is also conical. Then J preserves dom Δ_G. In particular, J preserves ker Δ_G, the space of closed and coclosed harmonic L^2-forms.

Finally, we observe that the conclusions of Theorem 2.1 remain true, if the hypotheses are only satisfied up to sufficiently high order. In the proposition that follows, we will assume for simplicity that $\overline{d}^* = \overline{\delta}$. Clearly it would suffice to only assume this near the singularities, and the case of ideal boundary conditions could also be included.

Proposition 2.2 Let X^{m+1} have isolated metrically conical singularities, with respect to the metric g. Let g' be a second metric on X^{m+1} for which, in polar coordinates, we have

i) $\qquad \|g-g'\| = O(r^2)$,

ii) $\qquad \|\nabla(g-g')\| = O(r)$,

iii) $\qquad \|\nabla^2(g-g')\| = O(1)$,

where ∇, denotes covariant differentiation with respect to g.

Assume that $\overline{d^*}=\overline{\delta}$ and let Δ_G, Δ'_G denote the corresponding Laplacians. Then dom Δ_G = dom Δ'_G.

Proof: We can assume that near some $p\in\Sigma$, g is of the form $C_{0,2}(N^m)$. Set

$$(2.10)\quad U_j = C_{2^{-(j+2)},2^{-(j-1)}}(N^m),$$

$$(2.11)\quad V_j = C_{2^{-(j+1)},2^{-j}}(N^m).$$

It follows from the usual elliptic estimate and an obvious scaling argument, that for $k=0,1,2$,

$$(2.12)\quad \|\nabla^k\theta\|_{V_j} \le c(2^j)^k(\|\theta\|_{U_j}+\|\Delta\theta\|_{U_j})$$

Also, it is easy to check that

$$(2.13)\quad \|(d\delta-d\delta')\theta\|_{V_j} \le c_2(\|\theta\|_{V_j}+2^{-j}\|\nabla\theta\|_{V_j}+2^{-2j}\|\nabla^2\theta\|_{V_j})$$

with similar estimates, for $\|(\delta-\delta')\theta\|$, $\|(\delta d-\delta' d)\theta\|$.

Thus,

$$\begin{aligned}(2.14)\quad \|d\delta'\theta\|_{C_{0,1}(N^m)} &\le \|d\delta\theta\|_{C_{0,1}(N^m)} + \|(d\delta-d\delta')\theta\|_{C_{0,1}(N^m)},\\ &= \|d\delta\theta\|_{C_{0,1}(N^m)} + \sum_{j=1}^{\infty}\|(d\delta-d\delta')\theta\|_{V_j},\\ &\le \|d\delta\theta\|_{C_{0,1}(N^m)} + 9c_1c_2(\|\theta\|_{C_{0,2}(N^m)} + \|\Delta\theta\|_{C_{0,2}(N^m)})\end{aligned}$$

where the last inequality follows from (2.12), (2.13). This together with the corresponding arguments for $(\delta-\delta')$, $(\delta d-\delta' d)$ shows that dom $\Delta'_G \subseteq$ dom Δ_G. However, it is straightforward to check that conditions i)-iii) imply the same conditions with the roles of g,g' interchanged. Then in the same way one shows that dom $\Delta_G \subseteq$ dom Δ'_G

In the same way (using [2]) we have

Theorem 2.3 Let Y be a Kähler manifold with isolated singularities. Let g' denote the metric and J' the almost complex structure. Assume that near the singularities there exist a conical metric g and almost complex structure J such the g,g' are related as in Proposition 2.2 and

i) $\|J-J'\| = O(r^2)$,

ii) $\|\nabla(J-J')\| = O(r)$,

iii) $\|\nabla^2(J-J')\| = O(1)$.

Then J' preserves dom Δ'_G.

It is not difficult to check that Theorem 2.3 applies to a complex submanifold Y of a Kähler manifold X such that the singularities of Y are isolated and metrically conical up to higher order terms in local normal coordinates (for example complex projective cones with nonsingular base). It then follows from [2] that for such Y, the "Kähler package" holds for the middle intersection cohomology groups, $IH^*(Y)$. Of course, by now there has been considerable progress on the conjecture that this holds for arbitrary singular projective complex algebraic varieties; see [2] p. 137 and [5]. However, our purpose here was to show that at least in the simplest cases, this conjecture can be explained by considerations based on the induced metric, and that even for these cases the situation is quite delicate; compare [5], section 4, Conjectures B and C. In fact, the existence of a Hodge theory for the induced metric in the conical case was the original motivation for the conjecture that IH^* satisfies the Kähler package in general.

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J.CHEEGER

State University of New-York
at Stony Brook
Department of Mathematics
STONY BROOK, N.Y.11794

U.S.A.