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Hodge algebras

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Let k be a field and let A be a commutative k-algebra generated by some finite set $H = \{x_1,\ldots,x_n\} \subset A$. As a vectorspace, A will of course admit as basis a set of monomials in the $x_1,\ldots,x_n$, though this set is far from unique. It is even possible to arrange H in such a way that this basis set can be taken to be the complement of an ideal Σ of monomials (see section 1 for a precise definition), so that A has a basis in 1-1 correspondence with the natural monomial basis of a ring of the form

$$A_0 = k[x_1,\ldots,x_n]/I,$$

where I is generated by the monomials Σ in the variables $x_1,\ldots,x_n$, but in general the relation between A and A_0 will be very slight.

In this paper we consider an additional condition on Σ, H, and A, which slightly limits the multiplication in A, in terms of a partial order on H (Section 1); if the condition is satisfied, we say that A is a Hodge Algebra, governed by Σ. If this condition is satisfied, then (among other things) the relation between A and A_0 becomes very precise: A is, in a very special way, the "general fiber" of a flat deformation whose special fiber is A_0, so that many properties of A_0 may be transferred to A (Section 3). Thus when A is a Hodge algebra governed by some "good" Σ, many properties of A can be read off directly.

Many interesting examples turn out to be Hodge algebras governed by "good" ideals Σ, so results of the above type may be used to unify and extend a large amount of information about difficult concrete examples, such as coordinate rings of Grassmannians and certain generalized Grassmannians, flag manifolds, Schubert varieties, determinantal and Pfaffian varieties, varieties of minimal degree, and varieties of complexes.

History

The first explicit description of an interesting algebra via a basis of monomials and relations of our type that we know of is Hodge's study of the Grassmann variety and Schubert cycles [Hodge], undertaken with a view to obtaining explicit "postulation formulas" (in modern terms, the Hilbert functions of the homogeneous coordinate rings); the results are presented in a relatively readable way at the end of volume 2 of [Hodge-Pedoe]. It is because of this, following a suggestion of Laksov, that we have called the algebras here "Hodge Algebras". Igusa, in [Igusa (1)] also exploited what is in fact a Hodge-algebra structure in proving the projective normality of the Grassmann variety. The next occurrence we know of is the "straightening law" of [Doubilet - Rota-Stein]; this was re-proved in [De Concini-Procesi] and [De Concini-Eisenbud-Procesi], where it is shown that this "straightening law" may be deduced in a simple way from the Hodge structure on the coordinate ring of the Grassmann variety. After [De Concini-Eisenbud-Procesi] was written, we made, at the suggestion of David Buchsbaum, a study of the relation between the Doubilet-Rota-Stein "straightening law" and the proof of the Cohen-Macaulayness of the Schubert cycles and determinantal varieties found in [Musili]; it was from this that the axioms for a Hodge algebra, in the special case called an "ordinal Hodge algebra" below, emerged (Musili's motivating proof, in our axiomatic form, is given in Section 8). This material was worked out by us in 1978, and a manuscript was then circulated; it is summarized in [Eisenbud].

Shortly after, Baclawski slightly reformulated the notion of straightening law, and gave direct combinatorial proofs of some of the results we obtained by deformation arguments [Baclawski]. His definition made an important step in that it disconnected the order structure on the set of generators for the algebra from the description of the ideal of monomials Σ. Seeing his definition, and considering some other generalizations that we had worked out in order to deal with certain flag manifolds and varieties of complexes, we were led to the definition of Hodge Algebra given below.

Because the current definition is so broad, interesting properties which can be deduced from a Hodge algebra structure are principally derived from interesting properties of the corresponding ideal of monomials Σ, and our emphasis is on exploiting these.

Contents

We now turn to a detailed sketch of the contents of this work. We have eliminated much explanatory material from the body of the paper in favor of putting it together, here.

For the deformation arguments we make we have developed the theory of Hodge algebras over a (nearly) arbitrary commutative ground ring R; however our interest centers on the cases where R is a field or, at worst, the integers; and even for the deformation arguments, at this level of speciality, it is enough to think of R as a polynomial ring over a field or over the integers.

Chapter 1 is foundational. Section 1 begins with the definitions of Hodge algebra, ordinal Hodge algebra (where Σ is determined by the order on H), square free Hodge algebra (where Σ is generated by square-free monomials) and discrete Hodge algebra (a polynomial ring on the elements of H, modulo the ideal Σ). It turns out to be crucial for inductive arguments to be able to measure the difference between an arbitrary Hodge algebra A and the "corresponding" discrete algebra A_0; the correct measure seems to be the "indiscrete part" of A, a certain subset of the set H of generators.

In addition to the definitions, section 1 contains two criteria for graded algebras to be Hodge which are useful in concrete examples (Propositions 1.1 and 1.3) and the central consequences (Proposition 1.2) of the definition of Hodge algebra which we usually employ.

The next step, accomplished in Section 3, is to show how to "simplify" a Hodge algebra by passing to a certain associated graded algebra which is again Hodge, but with a smaller indiscrete part. Since there are several different examples in which associated graded Hodge algebras (and the corresponding Rees algebras) arise, we pass by way of a general treatment of "standard filtrations" of Hodge algebras in Section 2; here a standard filtration is essentially just one for which the graded ring inherits a Hodge algebra structure.

With this "simplification process" prepared we are ready to derive properties of a Hodge algebra A from those of the corresponding discrete algebras, and this is the program of Chapter 2.

To make the simplification process worthwhile, a lot must be known about discrete Hodge algebras, so for motivation, we have collected some results of this sort in section 4. These results are for the most part not logically necessary for the development, and the reader may wish to refer to them only as needed. We lean heavily on work of Hochster, Munkres, Reisner, Stanley, and Taylor, among others. (Though the results are all more or less well known, they do not seem to be centrally available.) Section 5 uses the simplification procedure to show that if A is a Hodge algebra with corresponding discrete algebra A_0, then A has no nilpotents (that is, A is reduced) when this holds for A_0. More generally, the disposition of the associated primes of 0 in A is studied, and a method of constructing "interesting" non-zerodivisors in A is formulated; this method works best in the case of ordinal Hodge algebras, and lends some credibility to the conjecture that Hodge algebras which are domains tend to be normal if Σ is nice.

In Section 6 we show how to construct systems of parameters in certain Hodge algebras, and make explicit the result $\dim A = \dim A_0$, which follows easily from the deformation argument.

In Section 7 we give a method for studying the Homological properties (depth, gorensteiness, etc.) of A and deducing them from information about A_0.

In Section 8 we give our axiomatization of Musili's proof of the Cohen-Macaulayness of Schubert cycles; it is a special but useful condition on H which implies the Cohen-Macaulayness of an ordinal Hodge algebra governed by H.

In Section 9 we go even further into the roots of the subject, and remark on the way in which the Hilbert function and certain properties of a graded Hodge algebra can be deduced. The material of this section is largely a summary of relevant results of Richard Stanley.

The third chapter contains a list of examples of Hodge algebras which, we hope, is sufficiently rich and varied to justify the preceding material.

In Section 10 we review several processes for deriving new Hodge algebras from old by factoring out certain ideals, or passing to associated graded rings or Rees algebras or (in some cases) restricted Rees algebras.

Section 11 exhibits, in our language, the original examples of Hodge, the Hodge structures on the homogeneous coordinate rings of Grassmann and Schubert varieties. Section 12 shows off the example of Doubillet-Rota-Stein (determinantal varieties) and a related one from [De Concini-Procesi]. We have also included here some new results on the graded rings with respect to symbolic powers of a determinantal ideal.

We then turn in Section 13 to questions about the equations of reduced varieties of minimal degree in $\mathbb{P}_n$; their homogeneous coordinate rings are ordinal Hodge algebras.

Though there is a good classification of such varieties (due to Del Pezzo and Bertini in the irreducible case, and [Xambò] in general) we stop short of giving a normal form for the equations of each type.

Next we look at examples of small dimension. We show that all 1-dimensional square-free Hodge algebras are discrete in Section 14. In Section 15 we show that 2-dimensional square-free Hodge algebras that are domains are very special: for example, if A is such an algebra, and A is the homogeneous coordinate ring of a projective variety V, then V is a rational normal curve! (This result has been generalized by Watanabe to the case of two-dimensional graded Hodge domains generated by forms of various degrees.) We give an extended analysis of the equations of a Hodge algebra of this type, and show that its associated graded algebra with respect to the powers of the "obvious" maximal ideal inherits a Hodge structure.

It seems reasonable to conjecture that such an algebra (over a field of characteristic 0, say) must have a rational singularity.

The Hodge algebras above are essentially all ordinal Hodge algebras. In Sections 16 and 17 we treat two interesting examples which are square-free but not ordinal; they were our main motivation for extending the definition beyond the ordinal case.

The examples of Section 16 are the varieties of complexes studied by [Kempf], [Huneke], [De Concini-Strickland] and others. They are the coordinate rings for the varieties (really: affine schemes) parameterizing the complexes of the form

$$0 \longrightarrow A^{n_m} \xrightarrow{\phi_m} A^{n_{m-1}} \longrightarrow \dots \longrightarrow A^{n_1} \xrightarrow{\phi_1} A^{n_0},$$

possibly with supplementary conditions on the ranks of the maps ϕ_i. Our treatment follows closely that of [De Concini-Strickland], though some change in details is necessary to conform to our current notions, and these changes allow a slightly simplified treatment, since it becomes enough to treat the case where no rank conditions are imposed.

The second non-ordinal example is that of the Schubert cycles in flag varieties in the multi-homogeneous embeddings. Here the multihomogeneous coordinate ring of a flag manifold which is the ring generated by certain minors of a generic matrix, is an ordinal Hodge algebra (the poset may be taken to be a subset of the one used in the Doubilet-Rota-Stein example, but the natural Hodge structure on the factor rings corresponding to the Schubert cycles is not ordinal [Lakshmibai-Musili-Seshadri].

In the final section, 18, we discuss a generalization of the notion of Hodge algebras which allows one to treat the determinantal ideals of symmetric matrices and certain generalized Grassmann varieties.

Further examples may be found in [Strickland 1, 2].

Related techniques are also used in [Igusa 2].

Buchweitz has recently used Hodge algebra techniques in proving the rigidity of a number of algebras; see [Buchweitz]. Eisenbud and Harris have used them to analyse degenerations of intersections of Schubert varieties in a new approach to Brill-Noether theory. See [Eisenbud-Harris].

A speculation

It seems worth remarking that there seems to be a connection, at least on the level of examples, between 3 interesting classes of algebras: square-free Hodge algebras, algebras with rational singularities, at least in characteristic 0, and algebras of "F-pure type" in the sense of [Hochster-Roberts]. The known theoretical connections are rather slim: It is known that complete, 1-dimensional F-pure algebras are exactly the completions of the 1-dimensional square-free Hodge algebras of Section 14 ([Goto-Watanabe]). It is also known that a square-free Hodge algebra which is a graded Gorenstein domain is F-pure [Hochster , unpublished] (both these results work over any field of nonzero characteristic); this gives the only known proof that the coordinate rings of Grassmann varieties are F-pure (in a strong sense).

The apparent connection of rational singularities and F-pure algebras was noted by Hochster.

As for the connection of Hodge algebras to algebras with rational singularities, one may remark first that all the main examples of square-free Hodge algebras given in Chapter VI, below, are known to have rational singularities at the ideals generated by H; of course, for a ring to have rational singularities it must be a Cohen-Macaulay domain, so presumably one should only look at square-free Hodge algebras which are domains and which are governed by "perfect" ideals of monomials (that is, A_0 should be Cohen-Macaulay; A_{HA} is then automatically Cohen-Macaulay too).

A well-known theorem of [Elkik] asserts that if the special fiber in a flat deformation (of local rings, say) has a rational singularity, then so does the general fiber. It would be nice to use that theorem on the deformation "from A_0 to A" in the language above; but although A_0 has a nice resolution of singularities, it is essentially never a domain, and so does not have a rational singularity in the usual sense. Perhaps there is some sense in which, if Σ is sufficiently nice, A_0 does have a rational singularity, so that Elkik's ideas could be used, at least, say, on "deformations through Hodge algebras."

Using recent results of Flenner and Watanabe, Buchweitz has been able to explain virtually all the known examples of Hodge algebras with rational singularity by proving:

Theorem [Buchweitz]: Let A be a graded square-free Hodge algebra of dimension ≥ 2 over a field of characteristic 0 with generators H in positive degrees. Suppose that for each prime $P \subset A$ not containing the maximal homogeneous ideal, the localization A_p has only rational singularities. If the discrete algebra A_0 is Cohen-Macaulay then A has only rational singularities.

Other interesting applications and examples can be found in [Baclawski 2] and [Baclawski-Garsia].

An acknowledgement

We would like to thank David Buchsbaum and Richard Stanley for their help and patience with the material of this paper.

Some preliminaries

We now collect a few pieces of terminology concerning partially ordered sets which we will use:

We write $\# X$ for the number of elements of a set X.

We often write poset for "partially ordered set". A clutter is a poset in which no two elements are comparable. A chain X is a totally ordered set; its length (or dimension) is $(\# X) - 1$. The dimension of a poset is the supremum of the lengths of chains it contains. The height of an element in a poset is the supremum of lengths of chains descending from that element.

An ideal in a poset H is a subset I such that $x \in I$, $y \in H$, and $y \leqslant x$ together imply $y \in I$. For any $x \in H$ we write $I_x = \{y \in H \mid y \not\geqslant x\}$, and call it the ideal cogenerated by x.

We write $\mathbb{N}$ for the set of non-negative integers. If H is a set, then $\mathbb{N}^H$ is the set of functions from H to $\mathbb{N}$.

I. Definitions and basic methods

1) Hodge algebras and ideals

Let H be a finite set. A monomial on H is an element of $\mathbb{N}^H$. If M and N are monomials, then their product is defined by $(MN)(x) = M(x) + N(x)$ for $x \in H$. We say that N divides M if $N(x) \leq M(x)$ for all $x \in H$. The support of M is the set $\operatorname{supp} M := \{x \in H \mid M(x) \neq 0\}$.

An ideal of monomials is a subset $\Sigma \subset \mathbb{N}^H$ such that $M \in \Sigma$ and $N \in \mathbb{N}^H$ imply $MN \in \Sigma$. A monomial M is called standard with respect to Σ if $M \notin \Sigma$. A generator of an ideal Σ is an element of Σ which is not divisible by any other element of Σ. The set of generators of Σ is finite, as can be seen, for example, from the Hilbert Basis Theorem by regarding Σ as a set of multiindices, defining an ideal generated by ordinary monomials in a polynomial ring. If I is a subset of H, we define Σ/I to be the ideal of $\mathbb{N}^{H-I}$ obtained by restricting elements of Σ which vanish on I.

If A is a commutative ring and an injection $\phi : H \longrightarrow A$ is given, then to each monomial M on H we may associate $\phi(M) := \prod_{x \in H} \phi(x)^{M(x)} \in A$. We will usually identify H with $\phi(H)$ and write $M \in A$ for $\phi(M) \in A$; it will be clear from the context whether an abstract monomial or an element of A is intended. If Λ is a subset of $\mathbb{N}^H$, we may write ΛA for the ideal generated by the elements $\phi(M)$ for M in Λ.

Now let R be a commutative ring and let A be a commutative R-algebra. Suppose that H is a finite partially ordered set with an injection $\phi : H \longrightarrow A$, and that Σ is an ideal of monomials on H.

We call A a Hodge algebra (or algebra with straigtening law) governed by Σ and generated by $\phi(H)$ if the following axioms are satisfied:

Hodge-1. A is a free R-module admitting the set of standard monomials (with respect to Σ) as basis.

Hodge-2. If $N \in \Sigma$ is a generator and

*) $$N = \sum_i r_{N,i} M_{N,i}; \qquad 0 \neq r_{N,i} \in R,$$

is the unique expression for $N \in A$ as a linear combination of distinct standard monomials guaranteed by Hodge-1, then for each $x \in H$ which divides N and each $M_{N,i}$ there is a $y_{N,i} \in H$ which divides $M_{N,i}$ and satisfies $y_{N,i} < x$.

The relations *) are called the straightening relaticns for A. Note that the right-hand side of a straightening relation can be the empty sum (= 0) but that, though 1 is a standard monomial, no $M_{N,i}$ can be 1.

If the right-hand sides of all the straightening relations are 0 -- that is, if $N = 0$ in A for all $N \in \Sigma$ - then we say that A is discrete. If we write R[H] for the polynomial ring over R whose indeterminates are the elements of H, then $R[H]/\Sigma\, R[H]$ is a discrete Hodge algebra, and any other discrete Hodge algebra governed by Σ is isomorphic to it.

Two special types of Hodge structures are of such importance as to deserve names: We will say that an ideal Σ of monomials, or a Hodge algebra A it governs, is square-free if Σ is generated by square-free monomiials. We will say that Σ or A is ordinal if Σ is generated by the products of the pairs of elements which are incomparable in the partial order on H; Σ then consists of all monomials whose supports are not totally ordered, and is, of course, square free.

If A is suitably graded, as in the important examples, we can relax the axiom Hodge-1 slightly, and also show that the straightening relations give a presentation:

Proposition 1.1: Suppose that A is a graded commutative R-algebra, generated as an R-algebra by a finite partially ordered subset H of homogeneous elements of degree > 0. Let $\Sigma \subset \mathbf{N}^H$ be an ideal of monomials. If the standard monomials with respect to Σ are linearly independent in A and if, for each generator N of Σ, a relation of the form *) holds, then A is a Hodge algebra governed by Σ. Further, the straightening relations *) give a presentation of A.

Proof: It suffices to show that any nonstandard monomial M can be expressed as a linear combination of standard monomials by using the straightening relations; this will prove Hodge-1 directly, and it follows that the straightening relations *) generate all the relations because the standard monomials are linearly independent.

Let d be the maximum number of factors in a generator of Σ. Define the weight of a monomial M to be the number

$$\sum_{x \in H} M(x)(d+1)^{\dim x},$$

where $\dim x$ is the maximum length of a chain of elements in H ascending from x. If we choose any generator N dividing M and replace N by its expression *, we see that M can be expressed as a linear combination of monomials of the same degree but of strictly greater weight. If any of the resulting monomials is non-standard, we repeat this process, but we must terminate eventually because there are only finitely many monomials of a given degree. //

Despite a remark in [Eisenbud] to the contrary, we do not know whether the result of Proposition 1.1 holds in general.

We now adopt the following notation, to be held throughout this paper:

Notation: A will henceforth denote a Hodge algebra, governed by an ideal $\Sigma \subset \mathbb{N}^H$, generated by $\phi(H)$, which we will identify with H, and whose straightening relations are given by *).

We wish to have a measure of the difference between A and the discrete algebra $R[H]/\Sigma\, R[H]$. We define the indiscrete part $\operatorname{Ind} A \subset H$ of A to be the set

$$\operatorname{Ind} A = \cup \operatorname{supp} M_{N,i},$$

the union running over all $M_{N,i}$ appearing on the right-hand sides of straightening relations for A.

We can now give the form of Hodge-2 that we will generally use:

Proposition 1.2. a) If M is a standard monomial none of whose factors is $>$ any element of $\operatorname{Ind} A$, and N is any standard monomial, then MN is standard or is 0 in A.

b) If $I \subset H$ is such that $I \ni x > y \in \operatorname{Ind} A$ implies $y \in I$, then IA is a free R-module admitting the standard monomials divisible by elements of I as a basis, and A/IA is a Hodge algebra governed by Σ/I with generators $H - I$.

Proof: a) If MN is not standard then, since N is standard we may write $M = M'M''$ and $N = N'N''$ with $M'' \neq 1$ and $M''N''$ a generator of Σ. It follows from the definition of $\operatorname{Ind} A$ that $M''N'' = 0$ in A.

b) Let $x \in I$ be a maximal element of I and set $J = I - \{x\}$. By induction, we may assume the result for JA and A/JA. By a), the ideal xA/JA of A/JA has a basis consisting of the standard monomials divisible by x, and it follows that IA has a basis of the given form. That A/IA is a Hodge algebra as claimed is now immediate. //

The following result, abstracted from [Seshadri], often gives a useful method for verifying the Hodge algebra axioms in the ordinal case.

Proposition 1.3. Let V be a projective variety over R with (reduced) homogeneous coordinate ring A. Let $H \subset A$ be a finite set of homogeneous elements, partially ordered in some way.

If, for each $x \in H$ there is a reduced irreducible subvariety V_x of V such that

1) $x \leqslant y$ implies $V_x \supseteq V_y$,

2) If $y \in H$ and $y \not\geqslant x$ then y vanishes on V_x,

3) x does not vanish identically on V_x,

then, for each x, the standard monomials on $H - I_x = \left\{ y \in H \mid y \geqslant x \right\}$ are linearly independent in the homogeneous coordinate ring A_x of V_x, and the ordinal standard monomials on H are linearly independent in A.

Further, if relations of the form * hold in A for the ordinal ideal Σ associated to H, then A is a Hodge Algebra governed by Σ, and for each x, $I_x = \{y \in H \mid y \not\geqslant x\}$ is the homogeneous ideal of V_x in A.

Proof: Suppose $\Sigma M_\alpha = 0$ is a nontrivial relation among ordinally standard monomials M_α on $H - I_x$, holding in A_x. Since x is a nonzero-divisor on A_x, we may assume that not every M_α is divisible by x. Restricting the relation to a suitable $V_y \subset V_x$ we obtain a non-trivial relation in A_y among ordinally standard monomials on $H - I_y$. This is impossible by induction. Thus the ordinally standard monomials on $H - I_x$ are independent in A_x.

If now relations of the form * hold, then A is clearly a Hodge algebra as claimed by Proposition 1.1.

By Proposition 1.2, A/I_xA is a Hodge algebra on $H - I_x$, so the natural surjection $A/I_xA \longrightarrow A_x$ is an isomorphism. //

2) Standard Filtrations and their Rees algebras

Consider a filtration

$$I : A = I_0 \supset I_1 \supset \ldots$$

of A by ideals which is multiplicative in the sense that $I_p I_q \subset I_{p+q}$ for all $p, q \geqslant 0$, and such that $R \cap I_1 = \{0\}$. For any $x \in A$, we define the order of x (with respect to I) to be

$$\operatorname{ord} x = \sup \{j \mid x \in I_j\}.$$

We say that I is standard if for each $p \geqslant 0$, the ideal I_p is spanned by the standard monomials M such that $\sum_{x \in H} M(x) \operatorname{ord} x \geqslant p$. It follows at once that $\bigcap_p I_p = 0$.

If I is any multiplicative filtration, we define the (extended) Rees algebra to be

$$R(I,A) = \ldots \oplus At^k \oplus \cdots \oplus At \oplus A \oplus I_1 t^{-1} \oplus \cdots \oplus I_k t^{-k} \oplus \cdots \subset A[t,t^{-1}],$$

where t is a new indeterminate. We regard $R(I,A)$ as an $R[t]$-algebra, and we regard $A = At^0$ as a subring. In particular, if M is a monomial, we continue to write M for $\phi(M) \in A \subset R(I,A)$.

Theorem 2.1: If I is a standard filtration of A, then $R(I,A)$ is a Hodge algebra governed by Σ over $R[t]$ with injection $\tilde{\phi} : H \longrightarrow A$ defined by $\tilde{\phi}(x) = xt^{-\operatorname{ord} x}$ for $x \in H$. Further, the straightening relations for $\mathcal{R}(I,A)$ have the form

$$\tilde{\phi}(N) = \sum_i r_{N,i} t^{e(N,i)} \tilde{\phi}(M_{N,i}),$$

where $e(N,i) \geqslant 0$ and $e(N,i) > 0$ if and only if

$$\sum_{x \in H} N(x) \operatorname{ord} x > \sum_{x \in H} M_{N,i}(x) \operatorname{ord} x$$

$$= \operatorname{ord} M_{N,i}.$$

Corollary 2.2. The filtration I is standard if and only if $\mathrm{gr}_I\, A := A/I_1 \oplus I_1/I_2 \oplus \ldots$ is a Hodge algebra on Δ over R with generators $\phi^\# : H \longrightarrow \mathrm{gr}_I\, A$ given by $\phi^\#(x) = x + I_{(\operatorname{ord} x)+1} \in I_{\operatorname{ord} x}/I_{(\operatorname{ord} x)+1}$.

Proof of Corollary 2.2.: If I is standard, then $\mathcal{R}(I,A)$ is Hodge by the theorem; since

$$\mathrm{gr}_I\, A \cong R[t]/tR[t] \otimes_{R[t]} \mathcal{R}(I,A),$$

by an isomorphism sending $\phi^\#(x)$ to $1 \otimes \tilde{\phi}(x)$, it follows that $\mathrm{gr}\, A$ is Hodge in the required way. The converse is elementary; if I is not standard, then some $a \in I_p$ can be written as

$$a = \Sigma\, r_i M_i \qquad\qquad 0 \neq r_i \in R,$$

with the M_i standard and distinct, and for at least some M_i, $\sum_{x \in H} M_i(x) \operatorname{ord} x < p$. Taking leading forms, we get a relation of linear dependence among the standard monomials in $\mathrm{gr}_I\, A$. //

Proof of 2.1: We check the Hodge axioms for $\tilde{A} := R(I,A)$:

Hodge-1; Linear independence: The $R[t,t^{-1}]$ algebra $A[t,t^{-1}] = R(I,A)[t^{-1}]$ is Hodge with generators $x \in H$, by change of base. Since t is a unit, the standard monomials in the $xt^{-\mathrm{ord}\, x}$ are also linearly independent over $R[t,t^{-1}]$, and thus over $R[t]$.

Spanning: Since I_p is spanned by standard monomials of order $\geq p$, it suffices, for each standard monomial M of A to write $Mt^{-\mathrm{ord}\, M}$ as an $R[t]$-linear combination of standard monomials in the putative generators $xt^{-\mathrm{ord}\, x}$. Since $\mathrm{ord}\, M \geq \sum_{x\in H} M(x)\, \mathrm{ord}\, x$, the expression

$$Mt^{-\mathrm{ord}\, M} = t^{\sum_{x\in H} M(x)\mathrm{ord}\, x - \mathrm{ord}\, M} \prod_{x\in H} (xt^{-\mathrm{ord}\, x})^{M(x)}$$

is the required one.

Hodge-2: If N is a generator of Σ, then setting $n = \sum_{x\in H} N(x)\mathrm{ord}\, x$ we have

$$\tilde{\phi}(N) = \prod_{x\in H} (xt^{-\mathrm{ord}\, x})^{N(x)} = t^{-n}N$$

since I is standard we have

$$n \leq \mathrm{ord}\, N \leq \sum_{x\in H} M_{N,i}(x)\mathrm{ord}\, x$$

for each i. Thus

$$\tilde{\phi}(N) = \sum_i r_{N,i} t^{(\sum_{x\in H} M_{N,i}(x)\mathrm{ord}\, x)-n} \tilde{\phi}(M_{N,i})$$

is a relation of the desired form. //

3) The simplification of Hodge algebras

The "simplest" Hodge algebra over R, governed by Σ, is surely the discrete one $R[H]/\Sigma R[H]$. The following result allows us to reduce many questions about Hodge algebras to questions about more nearly discrete, and therefore simpler Hodge algebras.

Theorem 3.1: If $x \in H$ is a minimal element of $\operatorname{Ind} A$, then

$$I = \{x^n A\} : A \supset xA \supset x^2 A \supset \ldots$$

is a standard filtration and $A^{\#} := \operatorname{gr}_I A$ is a Hodge algebra governed by Σ with

$$\operatorname{Ind} A^{\#} \subseteq \operatorname{Ind} A - \{x\}.$$

Proof: The standardness of I follows from Proposition 1.2a, so $A^{\#}$ is a Hodge algebra over R, generated by the leading forms $x^{\#}$ of the elements $x \in H$, by Corollary 2.2. The straightening relations for $A^{\#}$ are derived from those of $R(I,A)$ by setting $t = 0$; thus in the notation of Theorem 2.1 they have the form

$$\phi^{\#}(N) = \sum_{\substack{i \\ e(N,i)=0}} r_{N,i}\phi^{\#}(M_{N,i}),$$

whence $\operatorname{Ind} A^{\#} = \bigcup\limits_{\substack{N,i \\ e(N,i)=0}} \operatorname{supp} M_{N,i} \subset \operatorname{Ind} A.$

Now if $e(N,i) = 0$, then x occurs to the same power in N and $M_{N,i}$. But if x divides N, then the corresponding straightening relation is

$$N = 0$$

because of the minimality of x in Ind A, so $x \notin \text{Ind } A^{\#}$. //

Applying the above result inductively we get

Corollary 3.2. For any Hodge algebra A there is a sequence of elements $x_1,\ldots,x_n \in \text{Ind } A$ such that defining A_i inductively by $A_0 = A$, $A_i = \text{gr}_{\{x_i^n A_{i-1}\}} A_{i-1}$ we have $A_n = R[H]/\Sigma\, R[H]$. //

Using the Rees algebras, we may view this as a stepwise flat deformation, whose most special fiber is $R[H]/\Sigma\, R[H]$ and whose most general fiber is A. It is interesting to note that one can avoid this stepwise procedure and make the deformation all at once; since we will not apply the result, we leave the proof to the reader.

II. Results on Hodge algebras

We adhere to the notation established in Section 1. If $x \in \mathrm{Ind}\, A$ is minimal, we use the notation $\phi^{\#} : H \longrightarrow \mathrm{gr}_{\{x^n\}} A$ as in Sections 2 and 3. Also, we will often write

$$A_0 = R[H]/\Sigma\, R[H]$$

for the discrete Hodge algebra over R governed by Σ. Many of the results and proofs work without hypothesis on R, but we will feel free to assume when necessary that R is noetherian and, when we speak of dimension theory, "universally Catenary", things which are true for any ring finitely generated over a noetherian Cohen-Macaulay ring -- that is, any "reasonable" ring.

4) Review of the discrete case

All the results of this section are known, though perhaps some have not appeared in print before. They are summarized here for the convenience of the reader. We suppose throughout this section that R is a domain; the extension to the general case is straightforward.

Proposition 4.1. Let Σ be an ideal of monomials on H, and set $I = \Sigma\, R[H]$.

a) I is prime if and only if Σ is generated by a subset of H.

b) I is radical if and only if Σ is generated by square-free monomials.

c) I is primary if and only if, whenever $x \in H$ divides a generator of Σ, there is a generator which is a power of x.

d) The associated primes of I are all generated by subsets of H.

e) height $HR[H]/\Sigma\, R[H]$ is independent of R; it is the cardinality of H minus the cardinality of the smallest subset H' of H such that every element of Σ is divisible by some element of H'.

The proofs are easy.

Note that a decomposition of an ideal I as in 4.1 into irreducible primary components is easy to calculate: If the generators of Σ are

$$N_1, \ldots, N_k,$$

and if N_1 can be written $N_1 = N_1'N_1''$ with $\operatorname{supp} N_1' \cap \operatorname{supp} N_1'' = \emptyset$, then I is the intersection of the ideals

$$(N_1', N_2, \ldots, N_k)R[H]$$

and

$$(N_1'', N_2, \ldots, N_k)R[H].$$

Repeating the procedure, we eventually write I as the intersection of ideals generated by powers of the elements of H, and these are primary and irreducible.

For some purposes it is useful to have an explicit free resolution of $A_0 = R[H]/\Sigma\, R[H]$ as an $R[H]$-module. The following result is from [Taylor]:

Let $G = \{g_1,\ldots,g_2\}$ be the set of generators of Σ, and define a complex

$$\Lambda(\Sigma) = 0 \longrightarrow \overset{s}{\Lambda}(R[H]^G) \xrightarrow{d} \ldots \xrightarrow{d} \overset{k}{\Lambda}(R[H]^G) \xrightarrow{d} \ldots \xrightarrow{d} R[H]$$

whose differential is given by

$$d(g_{i_1}\wedge\ldots\wedge g_{i_k}) = \sum_j (-1)^j \frac{\mathrm{lcm}(g_{i_1},\ldots,g_{i_k})}{\mathrm{lcm}(g_{i_1},\ldots,\hat{g}_i,\ldots,g_{i_k})} g_{i_1} \wedge\ldots\wedge \hat{g}_{i_j} \wedge\ldots\wedge g_{i_k},$$

where lcm denotes least common multiple and we have used the symbol g_i both for an element of $G \subset R[H]$ and for the corresponding basis element of $R[H]^G$.

<u>Theorem 4.2</u>. $\Lambda(\Sigma)$ is a free resolution of A_0 over $R[H]$.

Taylor's proof of this is a rather complex induction; [Gmaeda] and Ephraim (unpublished) have given much simpler proofs by decomposing $\Lambda(\Sigma)$ as a mapping cylinder of a map from $\Lambda(\Sigma')$ to itself, where Σ' is the ideal of monomials generated by all but one of the g_i.

The following result, due independently to Weyman and Lofwal, allows one to reduce some problems about arbitrary ideals of monomials to problems about ideals generated by square-free monomials: Let Σ be an ideal of monomials on $H = \{x_1,\ldots,x_n\}$ with generators $N_1,\ldots,N_k$. Set $d_i = \max_j (N_j(x_i))$, the maximum degree to which x_i appears in one of the N_j. We introduce new variables $H' = \{x_{ij}\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d_i}$ and define monomials N_i' by replacing each factor x_i^e of N_i by $x_{i1}\ldots x_{ie}$. For example if

$H = \{x_1, x_2\}$, $N_1 = x_1^3 x_2^2$, $N_2 = x_1^2 x_2^4$, then $H' = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{24}\}$ and $N_1' = x_{11}x_{12}x_{13}x_{21}x_{22}$, $N_2' = x_{11}x_{12}x_{21}x_{22}x_{23}x_{24}$.

Proposition 4.3. With the notation above, set $A_0 = R[H]/\Sigma$ and $A_0' = R[H']/\Sigma'$. The elements of

$$K = \bigcup_i \{x_{i1} - x_{i2}, \ldots, x_{i,d_i-1} - x_{i,d_i-2}\}$$

form a regular sequence (in any order) on A_0', and $A_0 = A_0'/K$.

Proof. One sees at once that

$$R[H] \otimes_{R[H']} \Lambda(\Sigma') = \Lambda(\Sigma),$$

so that $A_0 = R[H] \otimes_{R[H']} A_0'$ and

$$\mathrm{Tor}_i^{R[H']}(R[H], A_0') = 0$$

for $i > 0$. Since $R[H']$ is graded and K is a sequence of homogeneous elements, we are done. //

Remark. It is also easy to prove that the elements of K are a regular sequence on A_0' by ideal theory, using Proposition 4.1d.

A useful way of looking at ideals generated by square-free monomials is the following, which was developed in [Reisner]:

Let Δ be a simplicial complex with vertex set H; that is, Δ is a set of subsets of H, called simplices, and containing all the singletons

$\{x\}$, for which $T \subset S \in \Delta$ implies $T \in \Delta$. Define Σ_Δ to be the set of monomials whose supports are <u>not</u> simplices. Σ_Δ is clearly an ideal, and is generated by the square-free monomials corresponding to the minimal non-simplices. Further, any ideal generated by square-free monomials can be written as Σ_Δ for suitable Δ.

Recall that the dimension of a simplex is one less than the number of its vertices, and that the dimension of a simplicial complex is the maximum of the dimensions of its simplices.

<u>Proposition 4.4</u>. Suppose Σ is an ideal generated by square free monomials, and let Δ be such that $\Sigma = \Sigma_\Delta$.

a) [Hochster]. The minimal primes of A_0 are generated by the complements of the maximal simplices of Δ. In particular, if R is noetherian then $\dim A_0 = \dim R + \dim \Delta + 1$, and height $HA_0 = \dim \Delta + 1$.

b) [Stanley 1]. For each k, $H^k A_0 / H^{k+1} A_0$ is a free R-module; let $H_{A_0}(k)$ be its rank. If f_i is the number of simplices of dimension i in Δ, then

$$H_{A_0}(k) = \sum_{i=0}^{\dim \Delta} f_i \binom{k-1}{i}.$$

For example, we see from a) or b) that if R is a field, then the projective variety corresponding to the graded ring A_0 has degree equal to the number of simplices of maximal dimension in Δ.

Finally, we quote 3 striking results on the homological properties of the ideals $\Sigma = \Sigma_\Delta$. The first was proved in [Hochster] using the methods of [Reisner]. To express it we think of $R[H]$ and A_0 as graded by the semigroup of elements of $\mathbb{N}^H$, where each monomial M in $R[H]$ has degree

M in $\mathbb{N}^H$. For $M \in \mathbb{N}^H$ we think of supp $M \subset H$ as the simplicial subcomplex of Δ whose simplices are the simplices of Δ which happen to be contained in supp M. For any $\mathbb{N}^H$-graded module M, and any $M \in \mathbb{N}^H$ we write M_M for the part of M of degree M. We set $|M| = \sum_{x \in H} M(x)$, the total degree of M.

Theorem 4.5. [Hochster, Theorem 5.2]:

$$\mathrm{Tor}_j^{R[H]}(A_0,R)_M = \begin{cases} 0 & \text{if } M \text{ is not square-free} \\ \tilde{H}^{|M|-j-1}(\text{supp } M;R). & \end{cases}$$

Here $\tilde{H}^*$ denotes reduced simplicial cohomology, with the convention

$$\tilde{H}^j(\emptyset;R) = \begin{cases} R & j = -1 \\ 0 & j \neq -1 \end{cases}$$

Remark: $\mathrm{Tor}_j^{R[H]}(A_0,R)$ may be thought of as the jth homology module of the Koszul complex over A_0 corresponding to $H \subset A_0$.

Open Problem: Construct an explicit minimal free resolution of $R[H]/\Sigma$, perhaps by combining 4.2 and 4.5.

The next result, obtained by Munkres from 4.5, shows that the depth of the ideal HA_0 depends only on the topological space which is the "geometric realization", $|\Delta|$, of Δ.

Theorem 4.6 ([Munkres; Theorem 2.1]): The depth of HA_0 is the smallest integer j for which either

$$\tilde{H}^j(|\Delta|;R) \neq 0$$

or, for some $p \in |\Delta|$,

$$H^j(|\Delta|,|\Delta|-p;R) \neq 0$$

By contrast the Gorensteinness of A_0 really does depend on the combinatorial structure of Δ. We state only one of Hochster's several results in this direction. Recall that if $\sigma \in \Lambda$ is a simplex, then $\mathrm{lk}(\sigma)$ is the complex consisting of those simplices τ for which $\sigma \cap \tau = 0$ and $\sigma \cup \tau$ is a simplex in Δ. We specialize to the case $R = Z$:

Theorem 4.7 [Hochster; Theorem 6.7]. If $R = Z$, then A_0 is Gorenstein if and only if it is Cohen-Macaulay and, for each $(\dim \Delta) - 2$ - dimensional face σ in Δ, $\mathrm{Lk}\,\sigma$ is either a line with at most 3 vertices or a circle.

Finally, if H is partially ordered, we let Σ_H be the ideal generated by all products of pairs xy such that x and y are incomparable in H. If we let $\Delta(H)$ be the "order complex" whose simplices are the totally ordered subsets of Δ, then it is easy to see that $\Sigma_H = \Sigma_{\Delta(H)}$, so that the methods above may be used conveniently in this case.

5) Nilpotents and nonzerodivisors

We first collect some elementary consequences of the simplification result of Sections 2 and 3.

Recall that a ring is said to be reduced if it has no nilpotent elements. From 4.1 it is easy to see that A_0 is reduced if and only if R is reduced and Σ is square-free.

Proposition 5.1: a) If A_0 is reduced then A is.

b) If $y \in A$ is a nonzerodivisor modulo HA, then it is a nonzerodivisor.

c) If HA_0 contains a nonzerodivisor, then HA contains a nonzerodivisor.

Proof: Let $x \in H$ be a minimal element of Ind A. If $y \in A$ is nilpotent, then so is its leading form in $\mathrm{gr}_{\{x^n\}} A$, and since $\cap (x^n A) = 0$, the leading form is nonzero. The same is true with the word "zerodivisor" substituted for "nilpotent". Thus the Proposition follows by inducation on the number of elements in Ind A, using Theorem 3.1. //

Corollary 5.2. If A is a domain, A_0 is reduced, and height $HA \geq 2$, then HA contains a regular sequence of length 2.

Proof: Let $x \in H$ be minimal. Since A is a domain, x is a nonzerodivisor, and since A_0/xA_0 is reduced, HA/xA contains a nonzerodivisor by Proposition 1.2b, 4.1a, and 5.1c. //

Corollary 5.3. a) Every associated prime of 0 in A is contained in an ideal of the form $PA + HA$, where $P \subset R$ is an associated prime of 0 in R.

b) $\bigcap_n (HA)^n = 0.$

Proof: a) is immediate from 5.1 b). It is enough to verify b) locally at some prime containing each associated prime of 0 in A, whence, by a), at primes containing HA. But at these primes the result follows from the Krull Intersection Theorem. //

In good cases, such as those encountered in the examples, it is possible to produce nonzerodivisors explicitly:

Theorem 5.4. Suppose that the generators of Σ are square free and let Δ be the simplicial complex on H with $\Sigma = \Sigma_\Delta$. If $\{x_1,\ldots,x_n\} \subset H$ is a clutter meeting every maximal simplex of Δ, and if $r_1,\ldots,r_n \in R$ are any nonzerodivisors, then

$$y = \sum_1^n r_i x_i$$

is a nonzerodivisor on A.

In particular, if $\Delta = \Delta(H)$, then the sum of the minimal elements of H is a nonzerodivisor.

Proof: If A is discrete, then the associated primes of A are of the form $PA + IA$ where $P \subset R$ is an associated prime of R and $I \subset H$ is the complement of a maximal simplex, and the assertion follows.

We now do induction on # Ind A. If there is an $x \notin \{x_1,\ldots,x_n\}$ which is minimal in Ind A, then by induction $\sum_1^n r_i\phi^{\#}(x_i)$, which is the leading form of y, is a nonzerodivisor in $\mathrm{gr}_{\{x^nA\}}$, so y is a nonzerodivisor in A. Thus we may assume that every minimal element of Ind A is in

$\{x_1,\ldots,x_n\}$ or, since $\{x_1,\ldots,x_n\}$ is a clutter, that no element of Ind A is $<$ one of the x_i. Suppose that $y\ \Sigma\ u_j M_j = 0$, where the M_j are distinct standard monomials (one of which may be the empty monomial, 1) and $0 \neq u_j \in R$. Choose i so that $\{x_i\} \cup \operatorname{supp} M_1$ is a simplex of Δ. We have, by Proposition 1.2 a),

$$0 = xy \sum u_i M_i = r_1 \sum{}' u_j M_j x_i^2,$$

where $\sum{}'$ denotes the sum over all j such that $x_i \cup \operatorname{supp} M_j$ is a simplex of Δ. This is a non-trivial linear dependence relation among standard monomials, contradicting the existence of $\sum u_j M_j$. //

6) Dimension and system of parameters.

Recall that a noetherian ring R is said to be universally catenary if, for every pair $P \subset Q$ of primes in an R-algebra of finite type, all saturated chains of primes between P and Q have the same length. Any noetherian ring which is a homomorphic image of a Cohen-Macaulay ring--and thus in practical terms every noetherian ring--has this property.

Theorem 6.1. If R is universally catenary, then

$$\dim A = \dim A_0.$$

Proof: Since R is universally catenary we have:

$$\dim A = \sup(\dim(R_P) + \dim(R_P/PR_P \otimes_R A)),$$

where the supremum is taken over all primes P of R, so it suffices to treat the case where R is a field.

Let $x \in \operatorname{Ind} A$ be minimal and let $I = \{(x^n)\}$ be the usual filtration. Since $\mathcal{R}(I,A)$ is an affine ring, any nonzerodivisor $u \in \mathcal{R}(I,A)$ satisfies $\dim \mathcal{R}(I,A)/u\mathcal{R}(I,A) = \dim \mathcal{R}(I,A) - 1$. Applying this with $u = t$ and $u = t - 1$, we get $\dim A = \dim \operatorname{Gr}_I A$, and we are done by Theorem 3.1. //

Corollary 6.2. If R is universally catenary then height HA = height HA_0.

Note that by 4.1 e), height HA_0 is independent of R.

Proof: Let P be a prime of A containing HA, and Q a minimal prime of A contained in P, such that height P/Q = height HA. Factoring out $Q \cap R$, we may assume R is a domain. But then HA is prime, so $HA = P$, and by Corollary 5.3a, HA contains every minimal prime of A. Since A is Catenary, $\dim A = \dim A/HA + \text{height } HA = \dim R + \text{height } HA$, and the corresponding fact holds for A_0. Since $\dim A = \dim A_0$, we are done. //

To get a more explicit result, we follow an idea of Richard Stanley. Recall that elements $p_1, \ldots, p_n \in HA$ are a system of parameters in HA if $n = \text{height } HA$ and HA is nilpotent modulo $(p_1, \ldots, p_n)$.

Theorem 6.3. Suppose A is ordinal. If $H = \bigcup_0^n H_i$, where each H_i is a clutter, then HA is nilpotent modulo the ideal generated by the elements

$$p_i = \sum_{x \in H_i} x \qquad\qquad i = 0, \ldots, n,$$

and $A/(p_0, \ldots, p_n)A$ is generated as an R-module by square-free standard monomials. In particular, if A is ordinal and we set $H_i = \{x \in H \mid \text{height } x = i\}$ then the p_i are a system of parameters in HA.

Proof: It suffices to prove the first statement. We do induction on $\# H$. Let x_0 be a minimal element in H, and suppose for definiteness that $x_0 \in H_0$. By Proposition 1.2 b), A/x_0A is a Hodge algebra governed by $\Sigma/(x_0)$, so by induction HA is nilpotent modulo $(p_0, \ldots, p_n, x_0)$, and $A/(p_0, \ldots, p_n, x_0)$ is generated by square free monomials not involving x_0. By Proposition 1.2 a) we have $x_0 p_0 = x_0^2$, so $(p_0, \ldots, p_n, x_0)^2 \subset (p_0, \ldots, p_n)$, and HA is nilpotent modulo $(p_0, \ldots, p_n)$ as claimed. The right-exact sequence:

$$A/(p_0,\dots,p_n,x_0) \xrightarrow{x_0} A/(p_0,\dots,p_n) \longrightarrow A/(p_0,\dots,p_n,x_0) \longrightarrow 0$$

shows that $A/(p_0,\dots,p_n)$ is generated as in R-module by square free monomials not involving x_0 together with x_0 multiplied by these. Since the product of x_0 with a standard monomial is either 0 or standard, we are done. //

7) Koszul homology and homological properties.

The deformation argument of Section 3 leads, by standard methods, to a comparison of the homological properties of the Hodge algebra A with those of the discrete algebra A_0; the latter were treated in Section 4, and we regard them as known. Here is a sketch.

Consider the Koszul complex

$$\mathrm{IK}_A : 0 \longrightarrow \overset{\#H}{\Lambda} A^H \xrightarrow{\phi} \cdots \xrightarrow{\phi} \overset{2}{\Lambda} A^H \xrightarrow{\phi} A^H \xrightarrow{\phi} A,$$

where A^H is the free A-module whose basis is H, and where the differential ϕ is given by

$$\phi(x_1 \wedge \ldots \wedge x_k) = \sum_i (-1)^i x_i (x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_k),$$

for $x_1, \ldots, x_k \in H$. Here we have written $\hat{x}_i$ in the product on the right where x_i is to be omitted, and we have used the same symbol x_i for a basis element of A^H and an element of A.

By the results of Section 3, we may choose a sequence of elements $x_1, \ldots, x_n \in H$ such that, setting $A_n = A$, and $A_{i-1} = \mathrm{gr}_{\{(x_i^n)\}} A_i$ $(i = 1, \ldots, n)$, each A_i is Hodge, governed by Σ, x_i is minimal in $\mathrm{Ind}\, A_i$, and A_0 is discrete.

A standard spectral sequence argument [Serre Ch. 1] or a simple argument with Rees algebras yields:

Theorem 7.1. For each $j = 0, \ldots, \#H$ and $i = 1, \ldots, n$, there is a filtration of $H_j(\mathrm{IK}_{A_i})$ by submodules intersecting in 0 such that

$$\operatorname{gr} H_j(IK_{A_i})$$

is isomorphic to a subquotient of $H_j(IK_{A_{i-1}})$. //

Note that Proposition 4.3 and Theorem 4.6 allow an explicit determination of $H_j(IK_{A_0}) = \operatorname{Tor}_j^{R[H]}(R,A_0)$.

Corollary 7.2. If M is a maximal ideal of A containing H and $M_0 = HA_0 + (M\cap R)A_0$, then

$$\operatorname{depth} A_M \geq \operatorname{depth} {A_0}_{M_0} .$$

Further, if ${A_0}_{M_0}$ is Gorenstein, then so is A_M. //

These results are particularly nice when A is graded, the elements of H having degree ≥ 1, for it is well-known that then A is Cohen-Macaulay, Gorenstein, or locally of depth $\geq$ a given number iff this is true locally at maximal ideals containing HA. See, for example [Matijasevic-Roberts] or [Hochster-Ratcliffe].

The converse of 7.2 is false: It is easy for A to be Gorenstein without A_0 being so, and presumably the same could happen with Cohen-Macaulayness.

The same idea can be used, in simple cases, to construct a free resolution of A as an $R[H]$-module: starting with a resolution, such as Taylor's resolution $\Lambda(\Sigma)$, of A_0 one lifts it step by step to resolutions of $A_1, A_2, \ldots$. For this it is convenient to use the expression $A_i = \mathcal{R}(\{x_{i+1}^n\}, A_{i+1})/(t)$; after "lifting" the resolution of A_i modulo t

to an $R[t][H]$-free resolution of $R(\{x^n_{i+1}\},A_{i+1})$, one sets $t = 1$ to get the desired resolution of A_{i+1}. Of course the resolution obtained for $A = A_n$ will generally not be minimal, even if the resolution for A_0 was.

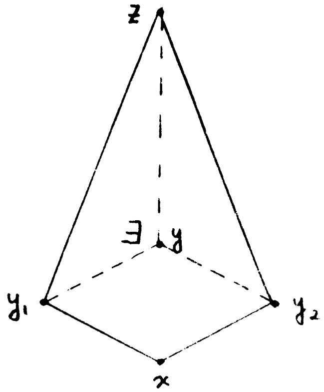
8) Wonderful Posets.

Throughout this section, we will suppose that A is an ordinal Hodge algebra, that is, we assume $\Sigma = \Sigma_H$. We give a simple condition on H which implies, by a direct argument, that the system of parameters $p_0,\ldots,p_n$ (n = height HA) given in Corollary 6.3 is a regular sequence. Of course this is practically equivalent to depth $HA \geq \dim H + 1$; but though the result of this section is far less general than that of Theorem 7.4, Corollary 7.2 and Theorem 4.5 or 4.6, it is useful in many significant examples where the condition yielded by 4.6 is not trivial to check.

The proof follows closely the ideas of [Musili].

Definitions. 1) An element $y \in H$ is a cover of an element $x \in H$ if $x < y$ and no element of H is properly between x and y.

2) H is wonderful (or locally semi-modular) if the following condition holds in the poset $H \cup \{-\infty,\infty\}$ obtained by adjoining least and greatest elements to H: If $y_1, y_2 < z$ are covers of an element x, then there is an element $y \leq z$ which is a cover of both y_1 and y_2. Pictorially:



Note in particular that a distributive lattice is a wonderful poset.

Theorem 8.1. Let A be an ordinal Hodge algebra, and set $p_i = \sum_{\substack{x \in H \\ \mathrm{ht}\, x = i}} x$. If H is wonderful, then $p_0, \ldots, p_n$ $(n = \dim H)$ is a regular sequence.

To prove the theorem, we need some elementary combinatorial properties of wonderful posets. If H is a poset, the ideal I cogenerated by a subset $\{x_1, \ldots, x_k\} \subset H$ is $\{x \in H \mid x \not\geq x_i \text{ for all } i\}$.

Lemma 8.2. Let H be a wonderful poset.

(1) If I is an ideal in H and if for any $z \in H - I$ and any two minimal elements y_1, y_2 in $H - I$ with $y_1, y_2 < z$, there is a common cover $y < z$ for y_1, y_2, then $H - I$ is wonderful.

(2) Any maximal chain in H has length equal to $\dim(H)$.

(3) If $x_1, \ldots, x_k$ are the minimal elements of H, with $k > 1$, and if $I \subset H$ is the ideal cogenerated by x_1 while $J \subset H$ is the ideal cogenerated by $x_2, \ldots, x_k$, then $I \cap J = \emptyset$ and $H - I$, $H - J$ and $H - (I \cup J)$ are wonderful, the last having dimension $\dim(H) - 1$.

Proof: 1): Immediate from the definition.

2): We use induction on the number of elements in H to prove that the lengths of two maximal chains $x_0 < x_1 \cdots$ and $y_1 < y_1 < \cdots$ are equal. We may assume $x_0 \neq y_0$; else pass to H - (the ideal cogenerated by x_1) $- \{x_1\}$. Since x_0 and y_0 both cover $-\infty$, they have a common cover z_1. Let $z_1 < z_2 < \cdots$ be a maximal chain ascending from z_1. Passing to H - (the ideal cogenerated by x_0), we see that $x_0 < x_1 < \cdots$ and $x_0 < z_1 < \cdots$ have the same length, namely one more than the length of $z_1 < z_2 < \cdots$. Repeating the argument with y_0 in place of x_0, we are done.

3): Immediate. //

<u>Proof of 8.1</u>: We do induction on $\# H$. If H has a unique minimal element x, then $x_0 = x$ is a nonzerodivisor by Theqrem 5.4, so we can factor it out and finish by induction, using Proposition 1.2 b) and Lemma 8.2 (1).

Now suppose H has minimal elements $x_1,\dots,x_k$ with $k \geq 2$. We write I, J contained in H for the ideals defined as in Lemma 8.2 (3), and we write $\bar{I}, \bar{J}$ for the ideals IA and JA of A. By induction and Proposition 1.2, it follows that $p_0,\dots,p_n$ is a regular sequence on $A/\bar{I}$ and on $A/\bar{J}$, and that $p_1,\dots,p_d$ is a regular sequence on $A/(\bar{I}+\bar{J})$, while $(p_0A) \subset (\bar{I}+\bar{J})$. Further, $0 = \bar{I} \cap \bar{J}$ by Proposition 1.2 b), so we have a short exact sequence

$$0 \longrightarrow A \longrightarrow A/\bar{I} \oplus A/\bar{J} \longrightarrow A/\bar{I} + \bar{J} \longrightarrow 0.$$

The theorem now follows from the next, well-known lemma (which was also used by Musili).

<u>Lemma 8.3</u>. Suppose $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of modules, and that $p_0,\dots,p_d$ are ring elements such that $p_0C = 0$. If $p_0,\dots,p_d$ is a regular sequence on B and $p_1,\dots,p_d$ is a regular sequence on C, then $p_0,\dots,p_d$ is a regular sequence on A.

Proof: Since $A \subseteq B$, p_0 is a nonzerodivisor in H. Modulo p_0 we obtain the exact sequence

$$0 \longrightarrow C \longrightarrow A/p_0A \longrightarrow B/p_0B \longrightarrow C \longrightarrow 0.$$

Let $K = \ker(B/p_0B \longrightarrow C)$. Since $p_1,\ldots,p_d$ is a regular sequence on B/p_0B and on C, it is regular on K as well, and repeating the argument, on A/p_0A. //

9) Graded Hodge Algebras.

Throughout this section we suppose that A is $\mathbb{N}$-graded in such a way that R has degree 0 and each $x \in H$ is homogeneous of degree > 0. We are grateful to Richard Stanley for explaining the material of this section to us. It is clear that the component A_ν of A is a free R-module over R admitting as basis the standard monomials M with $\nu = \sum_H M(x)\deg x$. Thus the Hilbert function $H_A(\nu) = \mathrm{rank}_R A_\nu$ is the same as for $R[H]/\Sigma$. This is calculated, in the case where Σ is generated by square-free monomials in Stanley's Proposition 4.4b.

In particular, this idea allows one to check whether a graded Hodge algebra which is a Cohen Macaulay domain is Gorenstein, using some other results of Stanley's:

Proposition 9.2 [Stanley 2]. If A is a graded Cohen-Macaulay domain, and $F(t) = \sum_{t=0}^{\infty} H_A(\nu)t^\nu$, then A is Gorenstein if and only if, for some integer r, F satisfies the formal identity:

$$F(1/t) = \pm t^r F(t).$$

Of course it may be combinatorially difficult to check this criterion if Σ is not simple. But in a special case which often arises in the examples, a result from Stanley's thesis, reproduced in [Stanley 2], makes it easier. In our case it says:

Proposition 9.3. Suppose R is a field and A is ordinal, and graded in such a way that the elements of H have degree 1 and the elements of R have degree 0. If A is a domain and H is a distributive lattice, then

A is Gorenstein if and only if all maximal chains of join-irreducible elements of H have the same length.

Remark: Buchweitz has recently generalized this result, dropping the assumption that all the elements of H have degree 1, and replacing it by the assumption that they are all homogeneous of positive degree, and that $\deg : H \longrightarrow Z$ is a valuation in the sense that $\deg h + \deg h' = \deg h \vee h' + \deg h \wedge h'$.

III. Examples

10) New Hodge Algebras from old.

Before beginning with the examples themselves, we collect a number of useful general construction techniques. We maintain the notation $(R,A,H,\Sigma,\ldots)$ of Section 1.

a) The simplest method of construction a new Hodge algebra from A is to factor out the elements of some ideal of the poset H as in Proposition 1.2.

b) If H has a maximal element y such that yM is standard whenever M is, then one sees immediately that A is a Hodge algebra over $R[y]$ governed by $\Sigma/(y)$. Applying change of rings, it follows that for any $r \in R$, $A/(y-r)A$ is a Hodge algebra over R on $\Sigma/(y)$.

c) Given a standard filtration I of A, we have seen in Section 2 that the Rees algebra and the associated graded algebras of A are again Hodge algebras.

Here is a special example of a standard filtration; we shall see in Section 12 that it arises naturally in the filtration of a polynomial ring by symbolic powers of a determinantal ideal. We restrict ourselves to the case where A is ordinal.

Suppose that A is graded in such a way that elements of H are homogeneous of degree $\geqslant 0$ and elements of R have degree 0, and that A satisfies:

1) If $x \leqslant y$ in H then $\deg x \geqslant \deg y$.

2) The standard monomials $M_{N,i}$ on the right-hand sides of the straightening relations

$$N = xy = \Sigma\, r_{N,i} M_{N,i}$$

all have ≤ 2 factors; that is, $\sum_{x \in H} M_{N,i}(x) \leq 2$ for all N, i.

With these assumptions, we define for each monomial M and each integer $k \geq 0$ a number

$$\gamma_k(M) = \sum_{\deg x \geq k} M(x)(\deg x - k + 1).$$

We write $I_k^{(p)}$ for the R-linear span in A of all the standard monomials with $\gamma_k(M) \geq p$.

Proposition 10.1. $I_k^{(p)}$ is an ideal of A, and it contains all monomials M with $\gamma_k(M) \geq p$. Further, the filtration

$$\mathcal{I}_k : A = I_k^{(0)} \supset I_k^{(1)} \supset \ldots$$

is a standard.

Proof: One uses the straightening algorithm in the proof of Proposition 1.1 to show that if $M = \Sigma\, r_i M_i$ is the relation expressing any monomial M as a linear combination of standard monomials, then $\gamma_k(M_i) \geq \gamma_k(M)$ for all i. The result follows at once. //

d) Blowing up. The following general construction is treated in [Eisenbud-Huneke, Theorem 2.3]. Let A be ordinal, and suppose that $I \subset H$ is an ideal in the poset A such that for any incomparable $x, y \in I$ the product $xy \in A$ is in the ideal $\{z \in H \mid z \leq x\}AI$ of A. The algebra $A[It] \subset A \oplus It \oplus \ldots \subset A[t]$ is then an ordinal Hodge algebra in a natural way on the poset $H \propto I$ defined as the disjoint union of H with a poset $I^* = \{x^* \mid x \in I\} \cong I$, the order being given by that already defined on H and an $I^* \cong I$, with

$x^* \leq y$ in $H \propto I$ if $x \leq y$ in H. Here the injection $H \propto I \longrightarrow A[It]$ is given by $x \longrightarrow x$ for $x \in H$ and $x^* \longrightarrow xt$ for $x \in I$.

e) Segre Product. If $A = \sum_{k \geq 0} A_k$ and $B = \sum_{k \geq 0} B_k$ are graded Hodge algebras generated by homogeneous elements, then the "Segre product"

$$\Sigma\, A_k \otimes B_k$$

is also a Hodge algebra, generated by the "product" poset in an obvious way.

11) <u>The Grassmannian</u>.

For definiteness, we now take R to be the ring of integers or a field. Let (X_{ij}) be a $d \times n$ matrix of indeterminates over R, and set $G_{d,n}$ equal to the subring of the polynomial ring $R[X_{ij}]$ generated by all the $d \times d$ minors of the matrix (X_{ij}). The algebra $G_{d,n}$ is the homogeneous coordinate ring of the Grassmann variety of d-planes in n-space; see [Hodge-Pedoe] or [Kleiman]. The ring $G_{d,n}$ is a Hodge algebra in the following natural way, which was first investigated in [Hodge].

Let $\{{n \atop d}\}$ be the set of symbols $[i_1,\ldots,i_d]$ with $1 \leqslant i_1 < \cdots < i_d \leqslant n$ integers. We make $\{{n \atop d}\}$ into a poset by setting

$$[i_1,\ldots,i_d] \leqslant [j_1,\ldots,j_d]$$

if

$$i_k \leqslant j_k \quad \text{for } k = 1,\ldots,d.$$

We define an injection $\{{n \atop d}\} \hookrightarrow G_{d,n}$ by taking $[i_1,\ldots,i_d]$ to the $d \times d$ minor of (X_{ij}) which is the determinant of the sub-matrix of (X_{ij}) involving columns numbered $i_1,\ldots,i_d$. We will also write $[i_1,\ldots,i_d]$ for this minor.

<u>Theorem 11.1 [Hodge]</u>. $G_{d,n}$, with the structure defined above, is an ordinal Hodge algebra.

See [DeConcini-Eisenbud-Procesi] for a modern treatment of this result.

The most interesting subvarieties of the Grassmann variety are the Schubert varieties. If V is an n-dimensional vectorspace, and if $V = V_n \supsetneq V_{n-1} \supsetneq \cdots \supsetneq V_1 \supsetneq 0$ is a complete flag of subspaces, we define the Schubert variety $\Omega_{a_1,\dots,a_d}$ to be the set of d-dimensional subspaces such that $\dim(W \cap V_{n-d+i-a_i}) \geq i$. The ideal of forms in $G_{d,n}$ vanishing on $\Omega_{a_1,\dots,a_d}$ is the ideal generated by the ideal of $\{{n \atop d}\}$ consisting of those $[i_1,\dots,i_d]$ not less than $[n - a_d + 1,\dots,n - a_1 + 1]$; thus the homogeneous coordinate ring of $\Omega_{a_1,\dots,a_d}$ is again a Hodge algebra (Proposition 1.2). (Note: the above notation for the Grassmannian and the Schubert varieties coexists in the literature with its dual!)

Since $\{{n \atop d}\}$ is a distributive lattice, it is wonderful, and it follows from the result of Section 8 that the Grassmann variety and the Schubert varieties are all projectively Cohen-Macaulay. Using 9.3 one can combinatorially decide which are Gorenstein; this has been carried out by Stanley in [Stanley 2].

One can treat in a similar way the (multi-homogeneous) coordinate ring of the Flag manifold, the Grassmannian of isotropic spaces corresponding to an orthogonal group modulo a maximal parabolic subgroup and various other reductive groups modulo maximal parabolic subgroups (see [Lakshmibai-Musili-Seshadri]).

12) Determinantal and Pfaffian varieties.

Let $A = R[X_{ij}]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d}}$, and suppose for convenience $d \leq m$. The algebra A can of course be given a discrete Hodge structure, but there is another, due to [Doubilet-Rota-Stein] which is suited to the study of determinantal varieties. It was further studied in [De Concini-Procesi] and [De Concini-Eisenbud-Procesi].

Let H be the set of all symbols of the form $(i_1,\ldots,i_r \mid j_1,\ldots,j_r)$, with $r \leq d$ and $i_1 < i_2 < \cdots < i_r$, $j_1 < j_2 < \cdots < j_r$, partially ordered by

$$(i_1,\ldots,i_r|j_1,\ldots,j_r) \leq (i_1',\ldots,i_s'|j_1',\ldots,j_s')$$

if

$$r \geq s \quad \text{and}$$

$$i_k \leq i_k', \quad j_k \leq j_k' \quad \text{for} \quad k = 1,\ldots,s.$$

We may inject H into A by sending $(i_1,\ldots,i_r|j_1,\ldots,j_r)$ to the $r \times r$ minor of (X_{ij}) involving rows $i_1,\ldots,i_r$ and columns $j_1,\ldots,j_r$.

Theorem 12.1. With A and H as above, A is an ordinal Hodge algebra on H over R.

For the original proof, see [Doubilet-Rota-Stein]. For a faster but still combinatorial treatment, see [Désarménien-Kung-Rota]. The theorem may also be deduced from the Theorem 10.1 as is done in [De Concini-Eisenbud-Procesi].

It is easy to verify that

$$R[X_{ij}]_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant d}} \cong G_{d,d+m}/([m+1,\ldots,m+d] - 1)$$

Since $[m+1,\ldots,m+d]$ is the unique maximal element of $\left\{{m+d \atop d}\right\}$, there is by section 10b an induced Hodge algebra structure on $R[X_{ij}]$. A combinatorial argument, carried out in detail in [De Concini-Eisenbud-Procesi], identifies this induced Hodge algebra structure with that of Doubilet-Rota-Stein.

In particular, $H \cong \left\{{d+m \atop d}\right\} - \{[m+1,\ldots,m+d]\}$, and this is again a distributive lattice (as may also be verified directly without difficulty.) Thus in particular H is wonderful.

Now consider, in H, the set I_k of all minors of orders $\geqslant k$. This is clearly a poset ideal, and I_kA is just the ideal generated by the $k \times k$ minors of (X_{ij}). Since $H - I_k$ is wonderful, one shows in this way that $k[X_{ij}]/I_k$ is Cohen-Macaulay, and sees that it is Gorenstein if and only if $m = d$ or $k = 1$.

This Hodge structure on $R[X_{ij}]$ satisfies the conditions of Section 10c, so we may define the ideals $I_k^{(p)}$. It is shown in [De Concini-Eisenbud-Procesi] that $I_k^{(p)}$ is precisely the p^{th} symbolic power of the ideal I_kA. Combining this with the fact that H is wonderful, we have:

<u>Theorem 12.2</u>. The "symbolic graded ring" $A^{(\#)} = A/I_k^{(1)} \oplus I_k^{(1)}/I_k^{(2)} \oplus \ldots \oplus I_k^{(p)}/I_k^{(p+1)} \oplus \ldots$ is a Hodge algebra on H. In particular, it is Cohen-Macaulay whenever R is.

This answers a question of M. Hochster; it was also proved by him (unpublished) by invariant-theoretic methods, in some special cases.

In a completely analogous way, one can treat the ideals of Pfaffians of a generic alternating matrix, using the "straightening law" of [De Concini-Procesi]. In this case one takes $A = R[X_{ij}]_{1\leqslant i<j<n}$,

$$H = \bigcup_{r\leqslant\frac{1}{2}n} \{{n \atop 2r}\}$$

partially ordered by

$$[i_1,\ldots,i_{2r}] \leqslant [j_1,\ldots,j_{2s}]$$

if $\qquad r \geqslant s \quad$ and

$$i_k \leqslant j_k \qquad \text{for} \qquad k = 1,\ldots,2s.$$

One injects H into A by sending $[i_1,\ldots,i_{2r}]$ to the Pfaffian of the submatrix obtained from the rows and columns numbered $i_1,\ldots,i_{2r}$. The example has been treated extensively in [Abeasis-DelFra].

13) Projective varieties of minimal degree.

Throughout this section we work over an algebraically closed field k, and write $\mathbb{P}_n$ for $\mathbb{P}_n(k)$. We use the word variety for reduced (but possibly reducible) projective scheme over k. A subvariety $V \subset \mathbb{P}_n$ is nondegenerate if it is contained in no hyperplane.

An elementary result of algebraic geometry asserts that an irreducible nondegenerate sub-variety of codimension c in a projective space has degree $> c$. Further, a classical result of Del Pezzo and Bertini classifies the subvarieties of minimal degree.

Theorem 13.1 (Del Pezzo-Bertini). Let V be an irreducible nondegenerate subvariety of codimension c and degree $c + 1$ in a projective space. Then either:

1) V is a rational normal scroll

or 2) $c = 1$ and V is a quadric hypersurface

3) $c = 3$ and V is a cone over the Veronese surface $\mathbb{P}_2 \hookrightarrow \mathbb{P}_5$.

A proof of this is given in the book [Bertini]. J. Harris has given a modern arrangement of the proof in [Harris] (for char $k = 0$); an improved version of the proof is in [Xambo].

Remarks.

For our purposes the "rational normal scrolls" (which we take to include, for example, the cases of $n + 1$ points in general position in $\mathbb{P}_n$) may be defined equationally: Given integers $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 0$ with $\Sigma\, a_i = c + 1$, the k-dimensional scroll $S(a_1,\ldots,a_k) \subset \mathbb{P}_{c+k}$ may be defined as the locus defined by the 2×2 minors of the matrix

$$\left(\begin{array}{ccc|ccc|c|ccc} x_0 & x_1 \cdots & x_{a_1-1} & x_{a_1+1} & \cdots & x_{a_1+a_2} & & x_{\sum\limits_{i<k}(a_i+1)} & \cdots & x_{\Sigma(a_i+1)-2} \\ & & & & & & \cdots & & & \\ x_1 & x_2 \cdots & x_{a_1} & x_{a_1+2} & \cdots & x_{a_1+a_2+1} & & x_{\sum\limits_{i<k}(a_i+1)+1} & \cdots & x_{\Sigma(a_i+1)-1} \end{array}\right).$$

This variety is ruled by $k-1$-planes, given by the vanishing of the elements of a linear combination of the two rows in the matrix above. The statement in 1) of the theorem means that V is, in suitable coordinates, some $S(a_1,\ldots,a_k)$.

Possibility 2) of the Theorem is self-explanatory. Possibility 3 may be taken to mean that in suitable coordinates V has as equations the 2×2 minors of the generic symmetric matrix

$$\begin{pmatrix} y_0 & x_1 & x_2 \\ x_1 & y_1 & x_3 \\ x_2 & x_3 & y_2 \end{pmatrix}.$$

It is an easy corollary that an irreducible subvariety of minimal degree is projectively Cohen-Macaulay (this has recently been proved directly by S. Goto and the second author).

Note that the inequality degree $\geqslant c+1$ has an analogue for local rings only under the <u>assumption</u> that the local ring is Cohen-Macaulay; in this case the multiplicity must be strictly greater than the embedding codimension.

For reduced, possibly reducible, pure dimensional subschemes $V \subset \mathbb{P}_n$ Xambo has shown that if V is assumed connected in codimension 1 (that is, removing a codimension 2 subset does not disconnect it), a necessary condition for projective Cohen-Macaulayness by a theorem of Hartshorne, then again the degree of V must be $> c$, a classification is possible, and V is projectively Cohen-Macaulay. The classification result is essentially the following:

<u>Theorem 13.2 (Xambo)</u>. Let $V \subset \mathbb{P}_n$ be reduced and pure-dimensional codimension c and degree $d+1$. If V is connected in codimension 1, $V = V_1 \cup \cdots \cup V_r$ is a decomposition of V into irreducibles, and L_j is the linear span of V_j, then each V_j is a variety of minimal degree in L_j, and (after rearranging the indexing if necessary, for $j = 2 \ldots r$ we have

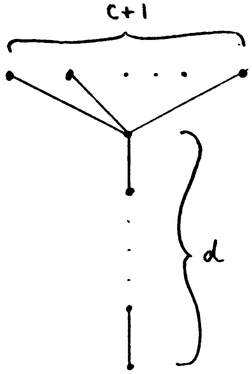
$$V_j \cap (V_1 \cup \ldots \cup V_{j-1}) = L_j \cap (L_1 + \ldots + L_{j-1})$$

is a linear space of dimension $\dim V - 1$, which is a ruling of V_j.

The precise equations satisfied by reducible subvarieties of minimal degree remain mysterious. However, we have

<u>Theorem 13.3</u>. Let V be a nondegenerate projective variety, connected in codimension 1, of dimension d and codimension c in $\mathbb{P}_{d+c}$. The homogeneous coordinate ring of V is an ordinal Hodge algebra over k on the poset:

H:



where all the elements of H correspond to linear forms.

Proof: We do induction on d, admitting the case $d = 0$ of $c + 1$ independent points in general position:

$d = 0$: It is easy to see that the equations of $c + 1$ independent points in $\mathbb{P}_c$ may be put into the form

$$(X_i X_j)_{0 \leqslant i < j \leqslant c} \,.$$

The coordinate ring is thus the (necessarily discrete) Hodge algebra on the $c + 1$-element clutter

$$\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{c+1} \qquad .$$

To complete the induction we use the following easy result:

Proposition 13.4. Let $V \subset \mathbb{P}_n$ be a subvariety of pure dimension d, and suppose the homogeneous coordinate ring of some hyperplane section $\{X_0 = 0\} \cap V \subset \{x_0 = 0\} = \mathbb{P}_{n-1}$ of dimension $d - 1$ is an ordinal Hodge algebra with poset H. The homogeneous coordinate ring of V is an ordinal Hodge algebra with poset

$$H' = \{x_0\} + H \qquad \text{(ordinal sum)}$$

if and only if the homogeneous coordinate ring of V has depth $\geqslant 2$.

Here the ordinal sum $\{x_0\} + H$ is the poset on $\{x_0\} \cup H$ where order relations within H are kept as before, and x_0 is taken $<$ every element of H.

Proof: If the homogeneous coordinate ring A of V is a Hodge algebra with poset $\{x_0\} + H$, then A has depth ≥ 2 by Proposition 5.2. Conversely, if depth $A \geq 2$, then the maximal homogeneous ideal of A is not associated to the ideal (x_0), so that $A/(x_0)$ is the homogeneous coordinate ring of $V \cap \{x_0 = 0\}$ in $\mathbb{P}_{n-1}$, and $A/(x_0)$ is Hodge on H.

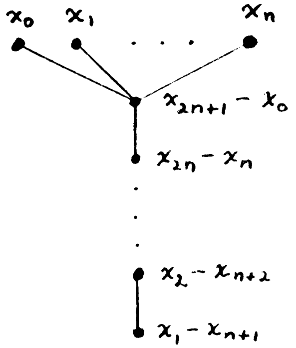
Choose any lifting $H \longrightarrow A$ of the injection $H \longrightarrow A/(x_0)$, and consider the standard monomials in the elements of A corresponding to $H' = \{x_0\} + H$. Since x_0 is a homogeneous non-zerodivisor in A, it is easy to see that A is a free module on these, so that Hodge-1 is satisfied. Further, if $x_1 \ldots x_k = \Sigma\, r_i M_i$ is the standard expression of any monomial with $x_1, \ldots, x_k \in \{x_0\} + H$, then each M_i either occurs in the standard expression for $x_1 \ldots x_k$ in $A/(x_0)$ or is divisible by x_0; Hodge-2 is satisfied by both these types. //

To finish the proof of 13.3, it remains to note only that a general hyperplane section of a non-degenerate, connected in codimension 1, variety of minimal degree will be a variety of the same type and that the homogeneous coordinate rings of these varieties will be arithmetically Cohen-Macaulay, which follows easily from Theorems 13.1 and 13.2. //

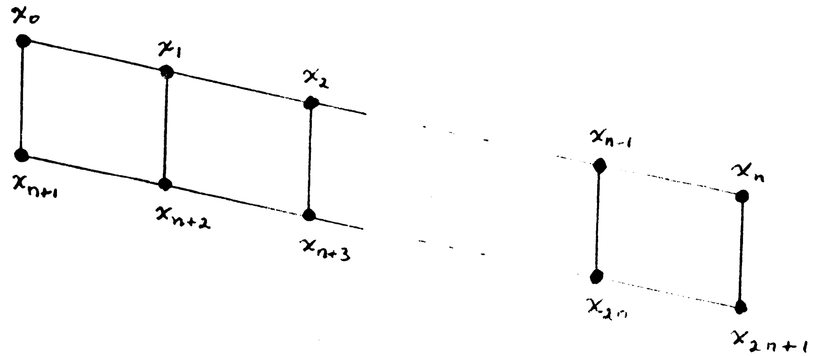
Example: Consider the Segre embedding $\mathbb{P}_1 \times \mathbb{P}_n \longrightarrow \mathbb{P}_{2n+1}$ whose image is defined by the 2×2 minors of the $2 \times n{+}1$ matrix

$$\begin{pmatrix} x_0 & \cdots & x_n \\ x_{n+1} & \cdots & x_{2n+1} \end{pmatrix}.$$

Its homogeneous coordinate ring is easily seen directly to be an ordinal Hodge algebra with poset



(Of course it is also an ordinal Hodge algebra on the poset



using the Doubilet-Rota-Stein straightening law of Section 12). //

14) One-dimensional square-free Hodge algebras.

In this and the next section we will study the Hodge algebras of low dimension governed by square-free ideals. It seems that square-free Hodge algebras, particularly if they are domains, are very special; the low dimensional cases treated here exemplify the phenomena.

Since we are interested here only in square free ideals, we may write $\Sigma = \Sigma_\Delta$ for some suitable simplicial complex Δ (see Section 4).

We assume throughout this and the next section that R is universally catenary.

Proposition 14.1. If $\dim \Delta = 0$ then A is discrete.

Proof: If not, we may choose an element x_0 which is minimal in Ind A. Since Δ has no 1-simplices, the only standard monomials are powers of the vertices of Δ, and the straightening relation in which x_0 occurs on the right may thus be written

$$xy = \sum_{x \in H} a_x(x),$$

where $a_x(x)$ is a polynomial in x, without constant term, $a_{x_0} \neq 0$, and $x, y \neq x_0$. Since x_0 annihilates any element of H except itself we may multiply this relation by x_0 to obtain

$$0 = x_0 a_{x_0}(x_0),$$

contradicting the linear independence of the powers of x_0. Thus Ind $A = \emptyset$ and A is discrete. //

<u>Corollary 14.2</u>. If A is a square-free Hodge algebra over R, then A is a 1-dimensional noetherian ring if and only if R is noetherian and either

1) $\dim R = 1$, $\Delta = \emptyset$; that is, $A = R$,

or

2) $\dim R = 0$, $\dim \Delta = 0$, and A is discrete; that is, A has the form $A = R[\{x_i\}]/(\{x_i x_j\}_{i \neq j})$.

<u>Proof</u>: Immediate from 14.1 and 6.1.

15) Some two-dimensional square-free Hodge algebras.

We retain the assumption that R is universally catenary.

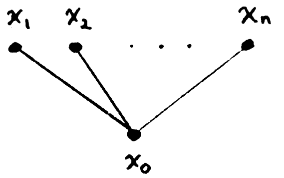
Proposition 15.1. If A is a 2-dimensional domain which is a square free Hodge algebra generated by $H \subset A$ over some ring T, then either:

a) $H = \emptyset$, and $A = R$.

b) H has one element, x, and $A = R[x]$, the polynomial ring in one variable over R; or

c) H can be re-ordered so that it has a unique minimal element x_0, and the non-minimal elements $x_1,\ldots,x_n$ are incomparable:

H:

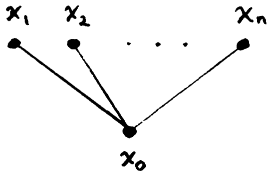


and A satisfies the axioms for an ordinal Hodge algebra generated by H.

Proof: It is enough to show that if H has at least 2 elements, then we are in case c. Write $\Sigma = \Sigma_\Delta$. Let x_0 be minimal in H, and let $x_1,\ldots,x_n$ be the other elements of H. Since A is a domain, the product of x_0 with any standard monomial must be standard by Proposition 1.2. If now some product $x_i x_j$ were standard, it would follow that $x_0 x_i x_j$ is standard, and thus $\dim \Delta \geqslant 2$, whence $\dim A \geqslant 3$. Thus the standard monomials in A are the same as for the ordinal Hodge structure associated to the partial ordering of H exhibited in c).

It remains to show that the straightening relations have the right form, and for this it suffices to show that if $1 \leqslant i < j \leqslant n$, then x divides every monomial $M_{(x_i x_j),k}$ occurring in the right-hand side of the straightening relation for $x_i x_j$. But by 1.2, A/x_0A is a Hodge algebra governed by $\Sigma_{\Delta-\{x_0\}}$. Since $\Delta - \{x_0\}$ is zero-dimensional, A/x_0A has by Proposition 14.1 the discrete Hodge structure, whence $M_{x_ix_j}$ is divisible by x_0. //

We now turn to a more detailed analysis of ordinal Hodge algebras A over a field $R = k$ whose poset H has the form

* H:  $n \geq 2$

as in Proposition 15.1 c. We fix the above notation.

Our most striking result is that if A has this form, then so does the associated graded ring of A with respect to the ideal HA. In particular $\mathrm{gr}_{HA} A$ is reduced.

Theorem 15.2. If A is an ordinal Hodge algebra over the field k on the poset H of *), then $A^{\#} = \mathrm{gr}_{HA} A$ is again an ordinal Hodge algebra governed by Σ_H with injection $H \longrightarrow A^{\#}$ given by $x_i \longrightarrow x_i^{\#}$, the leading form of $x_i \in A$. Moreover, if some $x_i \in H$ satisfies $x_i \in (HA)^2$, then $x_0 \in (HA)^2$, $n = 2$, and $\mathrm{gr}_{HA} A \cong k[x_1^{\#}, x_2^{\#}]$, the polynomial ring on 2 variables, with $x_0^{\#} = c x_1^{\#} x_2^{\#}$ for some nonzero $c \in k$.

Of course it follows from Corollary 2.2 that the first statement of the theorem is equivalent to the standardness of the filtration $\{(HA)^n\}$.

Corollary 15.3. If $X \subset \mathbb{P}_m(k)$ is a 1-dimensional, irreducible, non-degenerate subscheme whose homogeneous coordinate ring is an ordinal Hodge algebra over k with generators inside the maximal homogeneous ideal, then X is a rational normal curve of degree m.

Proof of the Corollary: If the homogeneous coordinate ring has an ordinal Hodge structure, it is reduced, and thus a domain. By 15.1 and the nondegeneracy of X the underlying poset H must have the form *) with $n \geqslant m$. By 15.2 we may define a new Hodge structure by taking the leading form $x_i^{\#}$ in place of the x_i, so we may assume that the generators are homogeneous. If $\deg x_i \geqslant 2$ for any i, then, by 15.2, $n = 2$, $m = 1$ and $X \cong \mathbb{P}_1$. Else, $n = m$ and factoring out x_0A, we get the homogeneous coordinate ring of m reduced points in $\mathbb{P}_{m-1}$, so $\deg X = m$ and m is the rational normal curve as claimed. (Alternatively, one could, as in Section 9, compute the Hilbert function of A directly). //

Proof of Theorem 15.2: Since

$$\text{**} \qquad A/x_0A = k[X_1,\ldots,X_n]/(X_iX_j)_{1\leqslant i<j\leqslant n},$$

the (discrete) ordinal Hodge algebra governed by the n-element clutter, the vectorspace $HA/(HA)^2$ has dimension n if $x_0 \in (HA)^2$ or $n+1$, if $x_0 \notin (HA)^2$. In particular no x_i other than x_0 can lie in $(HA)^2$. Set $y = \sum_1^n x_i$. Both x_0 and y are nonzerodivisors on A by Theorem 5.4, and y is a nonzerodivisor modulo x_0, as well. Further, one sees by applying the straightening relations to the expressions x_iy $(i = 1,\ldots,n)$ that $(x_0,y)A \supset (HA)^2$.

Now suppose $x_0 \in (HA)^2$, and set $B = A/yA$. We have $x_0B \supset (HB)^2$. Since the local ring A_{HA} is 2-dimensional, we have $\dim B_{HB} \geq 1$, whence $(HB)^2_{HB} \neq 0$ and $x_0 \notin (HB)^3$. Further, it follows that $x_0B_{HB} = x_ix_jB_{HB}$ for some i, j with $1 \leq i,j \leq n$.

In particular, x_i and x_j are nonzerodivisors on B_{HB}, and

$$\dim_k B/x_0B \geq \dim_k B_{HB}/x_0B_{HB}$$

$$= \dim_k B_{HB}/x_iB_{HB} + \dim_k B_{HB}/x_jB_{HB}.$$

Since $x_0A \subset x_iA$ we compute from **) that $\dim B_{HB}/x_i = n-1$ (and similarly for j) while $\dim_k B/x_0B = \dim_k A/(x_0,y) = n$. Thus $n \geq 2n-2$; we have assumed $n \geq 2$, so we get $n = 2$.

Further, the obvious epimorphism $k[x_1,x_2] \longrightarrow \mathrm{gr}_{HA} A$ sending x_i to $x_i^\#$ must be an isomorphism because $\dim \mathrm{gr}_{HA} A = 2$. Because $x_0 \in (HA)^2$ and $x_1^\#x_2^\# \neq 0$, we see that in A the straightening relation has the form $x_1x_2 = cx_0 + x_0(\text{in } HA)$, whence $x_1^\#x_2^\# = cx_0^\#$, as required.

We may now assume $x_0 \notin (HA)^2$, and it follows that rx_0 $(r \in R)$ never appears as a term on the right-hand side of a straightening relation. The form of the straightening relations now shows $(HA)^2 = (x_0,y)HA$, and it follows from the main result of [Sally] that $A^\# = \mathrm{gr}_{HA} A$ is Cohen-Macaulay, the elements $x_0^\#$, $y^\# = \sum_1^n x_i^\#$ forming a regular sequence. Further

$$A^\#/x_0^\#A^\# \cong \mathrm{gr}_{(x_1,\ldots,x_n)} A/(x_0)A$$

$$\cong k[x_1^\#,\ldots,x_n^\#]/(x_i^\#x_j^\#)_{1\leq i<j\leq n}$$

is an ordinal Hodge algebra on the n-element clutter. The result now follows from Proposition 13.4. //

We now analyze the possible straightening relations for a graded Hodge algebra on the poset *).

Definition: Let Λ be a free $k[X,Y]$-module on generators $e_0,e_1,\ldots,e_n$. A set of linear forms

$$\{\ell_{ij} = \sum_{k=0}^{n} \ell_{ijk}X_k \in k[X_0,\ldots,X_n]\}_{1\leq i<j\leq n}$$

will be called associative if, setting $\ell_{ji} := \ell_{ij}$, the following multiplication table makes Λ into an associative $k[X,Y]$-algebra:

$$e_0e_i = e_i \qquad 0 \leq i \leq n$$

$$e_ie_j = \ell_{ij0}X^2e_0 + \sum_{k=1}^{n} \ell_{ijk}Xe_k \qquad 1 \leq i \neq j \leq n$$

$$e_i^2 = \Bigg(-X^2 \sum_{\substack{j=1\\ j\neq i}}^{n} \ell_{ij0}\Bigg)e_0 + \Bigg(Y - X\sum_{\substack{j=1\\ j\neq i}}^{n} \ell_{iji}\Bigg)e_i$$

$$- X \sum_{\substack{j\neq i\\ k\neq i}} \ell_{ijk}e_k. \qquad 1 \leq i \leq n.$$

Note that the somewhat barbarous expression for e_i^2 is just $Ye_i - \sum_{j\neq i} e_ie_j$, written out.

If $\{\ell_{ij}\}$ is associative, we will write $\Lambda(\{\ell_{ij}\})$ for the $k[X,Y]$-algebra just defined.

Theorem 15.4. If A is a graded ordinal Hodge algebra over k on the poset H of *), such that the elements $x_i \in H$ all have degree 1, then for some associative set of linear forms $\{\ell_{ij}\}$

$$A \cong k[X_0,X_1,\ldots,X_n]/(X_iX_j - X_0\ell_{ij})_{1\leq i<j\leq n}$$

$$\cong \Lambda\{\ell_{ij}\}$$

the isomorphism carrying $e_i \in \Lambda\{\ell_{ij}\}$ to the class of X_i, and $X, Y \in \Lambda\{\ell_{ij}\}$ to the classes of X_0 and $\sum_1^n X_i$, respectively.

Conversely, if $\{\ell_{ij}\}$ is an associative set of linear forms, then

$$k[X_0,\ldots,X_1]/(X_iX_j - \ell_{ij})_{1\leq i<j\leq n}$$

is a Hodge algebra on the poset H of *).

Proof: Suppose that A is a Hodge algebra as in the hypothesis of the first statement. A admits a presentation of the form

$$A \cong k[X_0,\ldots,X_n]/(X_iX_j - X_0\ell_{ij})$$

by Proposition 1.1. Also, $X := X_0$, $Y := \sum_1^n x_i$ form an A-sequence by Theorem 5.4, and since everything is graded, A is a finitely generated free $k[X,Y]$-module with basis $e_0 = 1$, $e_1 = x_1,\ldots,e_n = x_n$. A trivial calculation shows that these fulfill the multiplication law defined for $\Lambda\{\ell_{ij}\}$, so the set $\{\ell_{ij}\}$ is associative and $A \cong \Lambda\{\ell_{ij}\}$ as claimed.

Conversely, suppose that $\{\ell_{ij}\}$ is associative. Since the elements $x,e_1,\ldots,e_n \in \Lambda\{\ell_{ij}\}$ fulfill the relations $e_ie_j = x\ell_{ij}(x,e_1,\ldots,e_n)$ there is an evident epimorphism

$$A := k[X_0,\ldots,X_n]/(X_iX_j - X_0\ell_{ij}) \longrightarrow \Lambda\{\ell_{ij}\}.$$

Regarding A as a graded $k[X,Y]$-module by $X \longrightarrow X_0$, $Y \longrightarrow \sum_1^n X_i$, we see that A is generated by $1,X_1,\ldots,X_n$. Since these elements go to the free generators of $\Lambda\{\ell_{ij}\}$, the map above is an isomorphism; in particular, A is a free $k[X,Y]$-module, and X_0 is a nonzerodivisor on A. Since $A/(X_0)$ is visibly a (discrete) Hodge algebra on the n-element clutter, we are done by Proposition 13.4. //

Remark. A classification up to isomorphism as algebras of the Hodge algebras of Theorem 15.4 is of course given by the results of Section 14. The case of curves needed here is actually quite elementary; see [Artin].

Explicit minimal free resolutions for the algebras in Theorem 15.4 may be constructed by the technique of [Eisenbud-Riemenschneider-Schreyer].

16) The variety of complexes.

In this section we exhibit a natural Hodge structure on the coordinate ring of the "variety of complexes". We follow [De Concini-Strickland], introducing a slight simplification, and adapting everything to our present concept of a Hodge structure. We are grateful to R.-O. Buchweitz for pointing out and correcting an error in an earlier version.

Fix a sequence of natural numbers $n_0,\ldots,n_m$. We wish to produce the homogeneous coordinate ring A of the variety parameterizing all complexes of the form

$$0 \longrightarrow A^{n_m} \xrightarrow{\phi_m} A^{n_{m-1}} \longrightarrow \cdots \longrightarrow A^{n_1} \xrightarrow{\phi_1} A^{n_0}.$$

To this end, consider the polynomial ring

$$B = R[\{\{X^k_{ij}\}_{\substack{1\leq i\leq n_{k-1}\\ 1\leq j\leq n_k}}\}_{1\leq k\leq m}]$$

in the $\sum_1^m n_k n_{k-1}$ variables X^k_{ij}. We think of X^k_{ij} as the $(i,j)\underline{^{th}}$ entry of a generic n_{i-1} by n_i matrix $\tilde{\phi}_k$, which we regard as a map $\tilde{\phi}_k : B^{n_k} \longrightarrow B^{n_{k-1}}$. Now let $C \subset B$ be the ideal generated by the entries of the m-1 matrices $\tilde{\phi}_k \circ \tilde{\phi}_{k+1}$ $(k = 1,\ldots,m-1)$, and set $A = B/C$. We write $\phi_k : A^{n_k} \longrightarrow A^{n_{k-1}}$ for the map induced by $\tilde{\phi}_k$, or for the corresponding matrix, whose $(i,j)\underline{^{th}}$ entry is the class of X^k_{ij} in A.

We will now give A a Hodge structure in such a way that the Doubilet-Rota-Stein example of Section 12 becomes the case $m = 1$.

Let $H \subset A$ be the set of minors of all orders of all the maps $\phi_1,\ldots,\phi_m$. We will write

$$(i_1,\ldots,i_s|j_1,\ldots,j_s)^{(k)},$$

with

$$1 \leqslant i_1 < \ldots < i_s \leqslant n_{k-1} \quad \text{and} \quad 1 \leqslant j_1 < \ldots < j_s \leqslant n_k$$

for the s by s minor of ϕ_k involving rows $i_1,\ldots,i_s$ and columns $j_1,\ldots,j_s$.

Set

$$x = (i_1,\ldots,i_s|j_1,\ldots,j_s)^{(k)}$$

$$x' = (i_1',\ldots,i_{s'}'|j_1',\ldots,j_{s'}')^{(k')}.$$

We partially order H by declaring x and x' to be incomparable if $k \neq k'$, while if $k = k'$ then $x \leqslant x'$ if

$$s \geqslant s' \quad \text{and}$$

$$i_t \leqslant i_t', \quad j_t \leqslant j_{t'} \quad \text{for} \quad t = 1,\ldots,s',$$

exactly as in the Doubilet-Rota-Stein Hodge structure.

Next, we define the product xx' to be standard if , possibly after interchanging x and x' to ensure $k \geqslant k'$, one of the 3 following conditions holds:

1) $k > k' + 1$

2) $k = k'$ and x and x' are comparable in the partial order on H.

3) $k = k' + 1$, $n_{k'} - s \geqslant s'$, and, writing $\ell_1 < \ldots < \ell_{n_{k'}-s}$ for the (ordered) complement of $\{i_1, \ldots, i_s\}$ in $\{1, \ldots, n_{k'}\}$, we have $j'_{s'-t+1} \leqslant \ell_{n_{k'}-s-t+1}$ for $t = 1, \ldots, s'$.

Note. If we symbolically write $x = (I|J)^{(k)}$, $x' = (I'|J')^{(k')}$, $\hat{I}$ for the sequence $[\ell_1, \ldots, \ell_{n_{k'}-s}]$, and $\overline{}$ for the operation that takes any sequence $[h_1, \ldots, h_p] \subset \{1, \ldots, n_{k'}\}$ to the sequence $[n_{k'}-h_p+1, \ldots, n_{k'}-h_1+1]$, then condition 3) could be more suggestively re-written as $k = k' + 1$, and $\overline{\hat{I}} \leqslant \overline{J}'$.

We may now define an arbitrary product of minors $x_1 \ldots x_n$ to be standard if each product $x_i x_j$ (suitably ordered) is; correspondingly, we define an ideal Σ of monomials in H as the ideal generated by the products of pairs of minors which are not standard.

Theorem 16.1. A is a Hodge algebra over R governed by Σ; in particular, A is reduced if R is.

Note. This is a mild strengthening of Theorem 2.2 of [De Concini-Strickland], which is proved simply by examining their proof carefully. However, the reader should be aware that the notion of standardness given above corresponds to that of [De Concini-Strickland] only after taking the transposes of the matrices ϕ_i of the complex; this seems necessary if one wishes that the ordering of minors should be as given above, and that the straightening relations should be generated by those given in Prop. 1.4 of [De Concini-Strickland].

Next consider a sequence $r_0, \ldots, r_{m+1}$ of integers with

$$r_0 = 0$$

$$0 \leq r_1 \leq \min(n_1, n_0)$$

$$\vdots$$

$$0 \leq r_m \leq \min(n_m, n_{m-1})$$

$$r_{m+1} = 0,$$

and let $I(r_1,\ldots,r_m) \subset H$ be the set

$$I(r_1,\ldots,r_m) = \{x \in H \mid x \text{ is a } p \times p \text{ minor of } \phi_k \text{ with } p > r_k\}.$$

This is clearly an ideal of H, so we have:

Corollary 16.2. $A/I(r_1,\ldots,r_m)A$ is a Hodge algebra governed by $\Sigma/I(r_1,\ldots,r_m)$; in particular, it is reduced if R is.

Since all the ideals $\Sigma/I(r_1,\ldots,r_m)$ are generated by square-free monomials, we may write:

$$\Sigma/I(r_1,\ldots,r_m) = \Sigma_{\Delta(r_1,\ldots,r_m)},$$

where $\Delta(r_1,\ldots,r_m)$ is a simplicial complex with vertex set $H - I(r_1,\ldots,r_m)$.

Theorem 2.4 of [De Concini-Strickland] asserts:

Theorem 16.3. If $r_k + r_{k-1} \leqslant n_k$ for $k = 0,\ldots,m+1$, then the geometric realization of $\Delta(r_1,\ldots,r_m)$ is homeomorphic to a cell; in particular, $A/I(r_1,\ldots,r_m)A$ is Cohen-Macaulay if R is.

The second statement follows from the first by the criterion given above as Theorem 4.6, together with the fact that $A/I(r_1,\ldots,r_m)A$ is suitably graded.

From this it is not too difficult to show [De Concini-Strickland, Theorem 2.11]:

Proposition 16.4. If $r_1,\ldots,r_m$ satisfies the hypothesis of Theorem 16.3, and R is a normal domain, then $A/I(r_1,\ldots,r_m)A$ is a normal domain. In particular, the ideal $I(r_1,\ldots,r_m)$ is prime.

Remark. If R is a domain and $r_1,\ldots,r_m$ does not satisfy the hypothesis of Theorem 16.3, then the radical ideal $I(r_1,\ldots,r_m)$ is an intersection of primes of the form $I(r'_1,\ldots,r'_m)$, where $(r'_1,\ldots,r'_m)$ satisfies the hypothesis of 16.3.

17) Schubert Cycles in the Flag Variety.

We follow the notation of section 12, and, for definiteness, take R to be the ring of integers or a field.

Set $d = m-1$, so that (X_{ij}) is an $(m-1) \times m$ matrix of indeterminants. Let A be the subring of $R[X_{ij}]$ generated by the minors

$$H' = \{(1,2,\ldots,e|j_1,\ldots,j_e) \mid 1 \leq e \leq m-1 \text{ and } 1 \leq j_1 < \ldots < j_e \leq m-1\},$$

that is, by the union over all e of the set of $e \times e$ minors of the submatrix consisting of the first e rows of (X_{ij}). The ring A is, in a natural way, the multi-homogeneous coordinate ring of the flag variety

$$F_m = \{(V_1,V_2,\ldots,V_{m-1}) \mid 0 \subset V_1 \subset \ldots \subset V_{m-1} \subset R^m,$$

$$\text{and } V_e \text{ is a direct summand of rank } e \text{ in } R^m\},$$

embedded in

$$\prod_{e=1}^{m-1} G_{e,m} \subset \prod_{e=1}^{m-1} \mathbf{P}(\overset{e}{\Lambda}\, R^m)$$

by sending a flag $(V_1,\ldots,V_{m-1})$ to $(\overset{1}{\Lambda}\, V_1,\ldots, \overset{m-1}{\Lambda}\, V_{m-1})$. From the form of the Doubilet-Rota-Stein straightening law given in section 12, it follows that A is an ordinal Hodge algebra on the sub-poset H' of the poset H of section 12; this fact was essentially noted in [Hodge]. If we change notation and write

$$[j_1,\ldots,j_e] \quad \text{for} \quad (1,2,\ldots,e|j_1,\ldots,j_e),$$

then we can think of H' as the poset

$$\bigcup_{e=1}^{n-1} \begin{Bmatrix} m \\ e \end{Bmatrix},$$

where $[j_1,\ldots,j_e] \leqslant [j_1',\ldots,j_{e'}']$ iff

$$e \geqslant e' \quad \text{and} \quad j_k \leqslant j_k' \quad \text{for} \quad k = 1,\ldots,e'.$$

For each permutation $\sigma : \{1,\ldots,m\} \longrightarrow \{1,\ldots,m\}$ and each flag $V = V_1 \subset V_2 \subset \ldots \subset V_{m-1} \subset R^m$ we define a <u>Schubert variety</u> $\Omega_\sigma = \Omega_\sigma(V)$ as follows:

For $1 \leqslant e \leqslant m-1$, let $\sigma^{(e)} = (\sigma^{(e)}(1),\ldots,\sigma^{(e)}(e))$ be the sequence whose elements are the elements of the set $\{\sigma(1),\ldots,\sigma(e)\}$, written in increasing order,

$$1 \leqslant \sigma^{(e)}(1) < \ldots < \sigma^{(e)}(e) \leqslant m.$$

We may regard $\sigma^{(e)}$ as an element of $\begin{Bmatrix} m \\ e \end{Bmatrix} \subset H$, and we note that $\sigma^{(m-1)} \leqslant \sigma^{(m-2)} \leqslant \ldots \leqslant \sigma^{(1)}$.

We now define the Schubert variety $\Omega_\sigma \subset F_m$ to be the set of flags $(V_1,\ldots,V_{m-1})$ such that each V_e, as an element of $G_{e,m}$, satisfies $V_e \in \Omega_{\sigma(e)_{(1)},\ldots,\sigma(e)_{(m)}}$. Let A_σ be the homogeneous coordinate ring of Ω_σ.

We may partially order the permutations of $\{1,\ldots,m\}$ by setting $\sigma \leqslant \tau$ if $\sigma^{(e)} \leqslant \tau^{(e)}$ in $\left\{\begin{matrix} m \\ e \end{matrix}\right\}$ for each $e = 1,\ldots,m-1$; this is the (reverse) "Bruhat order". We note that $\sigma \leqslant \tau$ iff $\Omega_\sigma \supseteq \Omega_\tau$.

Following the construction of [Lakshmibai-Musili-Seshadri] in this special case, we may define non-ordinal Hodge algebra structures on the A_σ as follows:

Let

$$H_\sigma = \bigcup_{e=1}^{m-1} \{h \in \left\{\begin{matrix} m \\ e \end{matrix}\right\} \mid h \geqslant \sigma^{(e)}\} \subset H'.$$

We regard H_σ as a sub-poset of $H' \subset A$, and embed it in A_σ via the natural projection $A \longrightarrow A_\sigma$.

Let Σ_σ be the ideal of monomials in H_σ generated by those square-free monomials $h_1 h_2 \ldots h_k$ such that either $\{h_1,\ldots,h_k\}$ is not totally ordered in H_σ or it is totally ordered, say

$$h_1 < \ldots < h_k,$$

but there is no sequence of permutations $\sigma_1 \leqslant \ldots \leqslant \sigma_k$ of $\{1,\ldots,m\}$ such that for some sequence of integers $e_1 \geqslant \ldots \geqslant e_k$ we have $h_i = \sigma_i^{(e_i)}$. The ideal Σ_σ is the ideal associated in the sense of [Reisner] (see section 4 above) with the simplicial complex of chains of elements of H' of the form

$$\sigma_1^{(e_1)} < \ldots < \sigma_k^{(e_k)}$$

where $\sigma_1 \leqslant \ldots \leqslant \sigma_k$ is a sequence of permutations, or, equivalently, $\Omega_{\sigma_1} \supseteq \ldots \supseteq \Omega_{\sigma_k}$ is a sequence of Schubert cycles.

Since A_σ is a factor-ring of A, we may regard H_σ as embedded in A_σ.

Theorem [Lakshmibai-Musili- Seshadri]: A_σ is a Hodge algebra generated by H_σ and governed by Σ_σ.

De Concini and Baclawski have jointly shown, by a lengthy unpublished analysis, that the simplicial complexes Δ_σ above are cells; it follows that A_σ is Cohen-Macaulay. Similar results may be proved for Schubert varieties in varieties of partial flags.

18) Dosets, Generalized Grasssmanians, and Symmetric matrices.

The following material is largely summarized from [De Concini-Lakshmibai].

Let H be a partially ordered set. Set

$$\mathcal{O}_H = \{(\alpha,\beta) \in H \times H \mid \alpha \leqslant \beta\}$$

$$\mathrm{Diag}_H = \{(\alpha,\alpha) \in H \times H\}.$$

Definition. A Doset of H is a set $D \subset H \times H$ such that

$$\mathrm{Diag}_H \subset D \subset \mathcal{O}_H$$

and such that if $\alpha \leqslant \beta \leqslant \gamma$ in H then

$$(\alpha,\gamma) \in D \Leftrightarrow (\alpha,\beta) \in D \quad \text{and} \quad (\beta,\gamma) \in D.$$

Given an injection $D \subset A$, for some ring A, we define a standard monomial to be a product of the form

$$(\alpha_1,\alpha_2)(\alpha_3,\alpha_4) \ldots (\alpha_{2n-1},\alpha_{2n})$$

where $\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \ldots \leqslant \alpha_n \in H$ and $(\alpha_{2i-1},\alpha_{2i}) \in D$ for $i = 1,\ldots,n$.

Now let R be a commutative ring, A an R-algebra, H a finite poset, and D a doset of H.

Definition. A straightening law for A on D, over R, is an injection $D \subset A$ such that:

SL 1) A is a free R-module admitting as basis the set of standard monomials in the elements of D.

SL 2) Let $M = (\alpha_1\alpha_2)\ldots(\alpha_{2k-1}\alpha_{2k})$ be any monomial in the elements of D. If

$$M = \Sigma\, r_i(\alpha_{i1},\alpha_{i2})\cdots(\alpha_{i,2k_i-1},\alpha_{i,2k_i}) \qquad 0 \neq r_i \in R$$

is the unique expression of M as a linear combination of distinct standard monomials, then for each i, we have $k_i \geqslant k$ and the sequence $\alpha_{i,1},\ldots,\alpha_{i,2k_i}$ is lexicographically earlier than $\alpha_1,\ldots,\alpha_{2k}$.

SL 3) If $\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \alpha_4 \in H$ are such that for some permutation σ of $\{1,2,3,4\}$ we have $(\alpha_{\sigma(1)}\alpha_{\sigma(2)}) \in D$ and $(\alpha_{\sigma(3)},\alpha_{\sigma(4)}) \in D$ then:

$$(\alpha_{\sigma(1)},\alpha_{\sigma(2)})(\alpha_{\sigma(3)},\alpha_{\sigma(4)}) = \pm(\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + \sum r_i M_i$$

where the M_i are standard monomials distinct from $(\alpha_1,\alpha_2)(\alpha_3,\alpha_4)$.

As in the case of Hodge algebras, we call the relations exhibited in SL2 the straightening relations; those with $k = 2$ are the quadratic straightening relations.

In analogy with Proposition 1.1 we have

Proposition 18.1. Let A be a graded R algebra, let D be a doset, and let $D \subset A$ be an injection making each element of D homogeneous of degree > 0. Suppose these data satisfy SL 1 and SL 3, and also satisfy SL 2 for $k = 2$. Then they satisfy SL 2 for all k, and thus $D \subset A$ is a straightening law. Further, all the relations on the $(\alpha,\beta) \in D$ are generated by the quadratic straightening relation.

Proof: We may define, for each $(\alpha,\beta) \in D$, an integer lex dim(α,β), as the dimension of (α,β) in the lexicographic partial order on D. The proof of 1.2, with lex dim in place of dim, yields the result. //

We say that a straightening law $D \subset A$ is discrete if for $(\alpha_1,\alpha_2) \in D$ and $(\alpha_3,\alpha_4) \in D$, we have

$$(\alpha_1,\alpha_2)(\alpha_3,\alpha_4)$$

$$= \begin{cases} \pm(\alpha_{\sigma(1)},\alpha_{\sigma(2)})(\alpha_{\sigma(3)},\alpha_{\sigma(4)}) & \text{if there is a permutation } \sigma \text{ of } \\ \quad \{1,2,3,4\} \text{ such that } \alpha_{\sigma(1)} \leqslant \alpha_{\sigma(2)} \leqslant \alpha_{\sigma(3)} \leqslant \alpha_{\sigma(4)}; & \\ 0 & \text{otherwise.} \end{cases}$$

Example 1. Let D be any doset of a finite poset H, and let $R\{D\}$ be the sub-algebra of $R\{H\}$ generated by the products $\alpha\beta$ for $(\alpha,\beta) \in D$. We call $R\{D\}$ the doset algebra of D. The injection $D \ni (\alpha,\beta) \longrightarrow \alpha\beta \in R\{D\}$ is a discrete straightening law.

Problem: Treat all Veronese embeddings of Hodge Algebras in a similar style.

In this example, $R\{D\}$ is a direct summand of $R\{H\}$ as $R\{D\}$-modules; one easily checks that the R-linear span of the standard monomials

$$\alpha_1 \ldots \alpha_k \quad \text{with} \quad \alpha_1 \leqslant \ldots \leqslant \alpha_k \in H$$

of $R\{H\}$ which are not in $R\{D\}$ is a complementary $R\{D\}$-submodule. Since $R\{\Delta_H\}$, the ring generated by all the α^2 for $\alpha \in H$, is again a doset algebra, and

$$R\{\mathrm{Diag}_H\} \subset R\{D\} \subset R\{H\}$$

we see that also $R\{\mathrm{Diag}_H\}$ is a summand of $R\{D\}$ as $R\{\mathrm{Diag}_H\}$-modules. Since $R\{\mathrm{Diag}_H\} \cong R\{H\}$ in an obvious way, this shows in particular that the depth of $HR\{H\}$ on $R\{H\}$ is the same as that of $DR\{D\}$ on $R\{D\}$.

The following construction from [De Concini-Lakshmibai] gives a similar result for all discrete doset algebras: Let $\tilde{D}$ be the set of all $(\alpha_1,\ldots,\alpha_{2k})$ such that

$$\alpha_1 \leqslant \ldots \leqslant \alpha_{2k} \in H$$

$$(\alpha_1\alpha_2),\ldots,(\alpha_{2k-1},\alpha_{2k}) \in D$$

$$(\alpha_{2i},\alpha_{2i+1}) \notin D \qquad \text{for all } i.$$

We partially order $\tilde{D}$ by setting

$$(\alpha_1,\ldots,\alpha_{2k}) \leqslant (\beta_1,\ldots,\beta_{2\ell})$$

if for each $i \leqslant \ell$ there is a $j \leqslant k$ so that

$$\alpha_{2j-1} \leqslant \beta_{2i-1} \leqslant \beta_{2i} \leqslant \alpha_{2j}.$$

Proposition 18.2. Let D be a doset and let $D \subset A$ be a discrete straightening law. The induced inclusion $\tilde{D} \subset A$ sending $(\alpha_1,\dots,\alpha_{2n}) \longrightarrow (\alpha_1,\alpha_2)\cdots(\alpha_{2k-1},\alpha_{2k})$ makes A a Hodge algebra on $\tilde{D}$. Furthermore, the simplicial complex $\Delta_{\tilde{D}}$ is a subdivision of the simplicial complex Δ_H, so that if $R\{H\}$ is Cohen-Macaulay, then A is Cohen-Macaulay.

Proof: A standard monomial in the elements of $\tilde{D}$ has an obvious expression as $\pm$ a standard monomial in the elements of D; this establishes a one-to-one correspondence between the sets of standard monomials for $\tilde{D}$ and D, as one can show by explicitly computing the inverse. One checks that this correspondence preserves the notion of discrete straightening law; see [De Concini-Lakshmibai] for details.

Example 2. Minors of a symmetric matrix: Let $A = R[X_{ij}]_{1\leq i\leq j\leq n}$ be a polynomial ring with enough indeterminates to fill a generic symmetric matrix. For $i < j$ we set $X_{ji} := X_{ij}$, and write (X) for the generic symmetric matrix. Let H be the poset of all non-empty subsets of $\{1,\dots,n\}$, with ordering $[i_1,\dots,i_k] \leq [j_1,\dots,j_\ell]$ if $k \geq \ell$ and (assuming $i_1 < i_2 < \dots < i_k$ and $j_1 < j_2 < \dots < j_\ell$) then $i_1 \leq j_1,\dots,i_\ell \leq j_\ell$. Let $D = \{([i_1,\dots,i_\ell],[j_1,\dots,j_\ell]) \in \mathcal{O}_H$, the set of comparable pairs of subsets of the same size. D is clearly a Doset.

Theorem [D-P]: The inclusion $D \longrightarrow A$ sending (α,β) to the minor of (X) with row indices β and column indices α is a straightening law on A.

As in Section 1, we may define ideals associated to an ideal of H, and we get

Corollary. If I_k is the ideal of $k \times k$ minors of (X), and D_k is the sub-doset of pairs of subsets whose common length is $\leq k$, then the induced inclusion $D_k \longrightarrow A/I_k$ is a straightening law for A/I_k.

(This example is treated in detail in [Abeasis].)

The algebras above may be thought of as coordinate rings of affine open subsets of Schubert varieties in a Grassmann associated to a symplectic group. In fact there is a general pattern, into which this and the examples of Sections 11 and 12 fit, which has emerged following work of Demazure and Lakshmibai, Musili, and Seshadri:

Example 3 [De Concini-Lakshmibai]: Let G be a semisimple algebraic group with Weyl group W, P a maximal parabolic subgroup of classical type with Weyl group $W_P \subset W$, and regard G/P as a projective variety in the standard embedding. The homogeneous coordinate ring of G/P is then an algebra with straightening law on the "doset of admissible pairs" in W/W_P, where W/W_P is endowed with the Bruhat partial order.

Here the doset of admissible pairs is the smallest doset containing the pairs (α,β) such that, the Schubert subvariety of G/P associated to α appears with multiplicity 2 in the hyperplane section of that associated to β.

As in Section 11, the homogeneous coordinate rings of the Schubert varieties in G/P inherit straightening laws, on intervals in W/W_P.

Standard filtrations and a simplification procedure analogous to those of Section I may be defined for straightening laws on Dosets: see [De Concini-Lakshmibai] for details. Taking into account the result of [Björner-Wachs] that all intervals in Coxeter groups (under the Bruhat order) are Cohen-Macaulay, one deduces that the algebras of examples 2 and 3 are all Cohen-Macaulay.

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RÉSUMÉ

Une algèbre de Hodge est une algèbre commutative dont les générateurs et les relations satisfont une condition qui permet d'écrire l'algèbre de façon particulièrement agréable comme déformation d'une algèbre dont les relations sont des monomes en les générateurs. De ce fait, beaucoup de propriétés d'une algèbre de Hodge peuvent être déduites d'une manière combinatoire simple de la forme de ses relations. Cet article contient un exposé des fondements de la théorie et présente quelques uns des principaux exemples, comme les anneaux de coordonnées de variétés déterminantielles et Pfaffiennes, des cycles de Schubert et des variétés de complexes.