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A CONSTRUCTIVE POLYNOMIAL METHOD IN THE REPRESENTATION THEORY OF SYMMETRIC GROUPS

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During the last few years polynomial rings in double-indexed indeterminates have been investigated for quite different reasons. In this note I would like to report about these polynomial rings from the viewpoint of the representation theory of symmetric groups.

§ 1 Letter Place Algebras

Let R be a commutative ring with unit element $1 = 1_R \neq 0$, and let $m,n \in \mathbb{N} := \{1,2,\dots\}$. The polynomial ring

$$R_m^{\;n} := R[X_{ij} \;/\; i=1,\dots,m;\; j=1,\dots,n]$$

in the $m \cdot n$ indeterminates $X_{ij} =: (i|j)$ – i is the <u>letter</u> and j the <u>place index</u> – is called the <u>letter place algebra</u> in m letters and n places [DKR,p.66].

The double indication of the indeterminates makes it possible to associate to a monomial

$$X_{i_1 j_1} \cdot \dots \cdot X_{i_k j_k} =: \begin{pmatrix} i_1 & j_1 \\ \vdots & \vdots \\ i_k & j_k \end{pmatrix} \qquad \text{(of total degree k)}$$

the <u>letter-content</u> $\alpha = (\alpha_1,\dots,\alpha_m)$, $\alpha_i := |\{\nu \;/\; i_\nu = i\}|$, the <u>place-content</u> $\beta = (\beta_1,\dots,\beta_n)$, $\beta_j := |\{\nu \;/\; j_\nu = j\}|$, and the <u>content</u> (α,β).

Hence the letter- (resp. place-) content of a monomial of total degree k is an <u>improper partition</u> of k, i.e. α (resp. β) is a sequence of non-negative integers which sum up to k. Let me write $\alpha \models k$ and $\beta \models k$, for short.

By homogeneity conditions with respect to monomials the letter place algebra $R_m{}^n$ can be decomposed into finite-dimensional R-subspaces $R_{\alpha\beta}$:

$$R_m{}^n = \sum_{k \geq 0}^{\oplus} \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_m)\models k \\ \beta=(\beta_1,\ldots,\beta_n)\models k}}^{\oplus} R_{\alpha\beta} ,$$

where $R_{\alpha\beta}$ is defined to be the span of the monomials of content (α,β).

(In the sequel $V = \ll B \gg_R$ means: B is an R-basis of V.)

<u>Example</u>.

$$R_{(1,2,1)(2,2)} = \ll \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} \gg_R .$$

The general linear group GL(m,R) acts from the left, and GL(n,R) acts from the right on $R_m{}^n$, and these actions induce algebra-automorphisms of $R_m{}^n$:

For all (a_{rs}) in GL(m,R), all (b_{uv}) in GL(n,R) and all monomials $\prod_{i,j}(i|j)^{c_{ij}}$ in $R_m{}^n$ put

$$(a_{rs}) \cdot \prod_{i,j} (i|j)^{c_{ij}} := \prod_{i,j} \left(\sum_k a_{ki}(k|j)\right)^{c_{ij}}$$

$$\prod_{i,j} (i|j)^{c_{ij}} \cdot (b_{uv}) := \prod_{i,j} \left(\sum_h b_{jh}(i|h)\right)^{c_{ij}} .$$

Moreover this yields a (GL(m,R),GL(n,R))-bimodule structure on ${R_m}^n$.

The symmetric group S_n is embedded into GL(n,R) via permutation matrices.

The spaces $R_{\alpha\beta}$ can be interpreted in a representation theoretical way.

Theorem [Cl I, p. 168]

$$\boxed{R_{\alpha\beta} \cong_R \mathrm{Hom}_{R[S_k]}(R[S_k]\otimes_{R[S_\alpha]}R, R[S_k]\otimes_{R[S_\beta]}R)} .$$

Here S_α (resp. S_β) denotes the Young-subgroup to $\alpha \vDash k$ (resp. $\beta \vDash k$). □

Hence the $R_{\alpha\beta}$ are intertwining spaces, and Mackey's Intertwining Number Theorem (see e.g. [CR, § 44]) suggests to deal with the question:

Are there any R-bases of $R_{\alpha\beta}$ which are of representation theoretical interest?

Before I start answering this question, let me recall some notations.

$\lambda = (\lambda_1,\ldots,\lambda_h)$ is a (proper) partition of n (for short: $\lambda \vdash n$), if λ is a non-increasing sequence of strictly positive integers which sum up to n. $\lambda' = (\lambda'_1,\lambda'_2,\ldots)$, the associated partition to λ, is defined by $\lambda'_i := |\{j/\lambda_j \geq i\}|$. [It is well-known that the (proper) partitions of n parametrize the conjugacy classes of S_n as well as the classes of ordinary irreducible representations of S_n.]

$(\lambda) := \bigcup_{i=1}^{h} \{(i,1),\ldots,(i,\lambda_i)\}$ is the Young-diagram associated to the partition $\lambda = (\lambda_1,\ldots,\lambda_h)$. A λ-tableau T (or a tableau of shape λ) is a mapping $T : (\lambda) \longrightarrow \mathbb{N}$.

One can illustrate such a tableau T in the following way

$$T = \begin{matrix} t_{11} & t_{12} & \cdots & \cdots & t_{1\lambda_1} \\ t_{21} & t_{22} & \cdots & \cdots & t_{2\lambda_2} \\ \vdots & & & & \\ t_{h1} & t_{h2} & \cdots & t_{h\lambda_h} & \end{matrix} \qquad (t_{ij} := T((i,j))),$$

and it is clear how to define the i-th row and the j-th column of a tableau $T = (t_{ij})$.

T is said to be standard if the elements in each row of T are strictly increasing from left to right and are non--decreasing down the columns.

$c(T) := (c_1(T), c_2(T),\ldots)$, $c_k(T) := |\{(i,j) \,/\, t_{ij}=k\}|$, is the content of T.

Example. The tableau $T = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & \\ 2 & & \\ 4 & & \end{matrix}$ is standard and

$c(T) = (2,3,1,1,0,\ldots)$.

□

Let $ST^{\lambda}(\alpha)$ denote the set of all Standard Tableaux of shape λ and of content α.

To get a representation theoretical description of the R-dimension $(R_{\alpha\beta}:R)$ of $R_{\alpha\beta}$ let me remind you of

Young's Rule.

The multiplicity of the irreducible representation $[\lambda]$ of S_k, $\lambda \vdash k$, in $\mathbb{C}[S_k] \otimes_{\mathbb{C}[S_\alpha]} \mathbb{C}$ is just $|ST^{\lambda'}(\alpha)|$, i.e.:

$$\boxed{\mathbb{C}[S_k] \otimes_{\mathbb{C}[S_\alpha]} \mathbb{C} \sim \sum_{\lambda \vdash k}^{\oplus} |ST^{\lambda'}(\alpha)| \cdot [\lambda]} \; .$$

□

As $(R_{\alpha\beta} : R) = (\mathbb{C}_{\alpha\beta} : \mathbb{C})$ (monomials!), one gets

$$\begin{aligned}(R_{\alpha\beta} : R) &= i\Big(\sum_{\lambda \vdash k}^{\oplus} |ST^{\lambda'}(\alpha)| \cdot [\lambda], \; \sum_{\mu \vdash k}^{\oplus} |ST^{\mu'}(\beta)| \cdot [\mu] \Big) \\ &= \sum_{\lambda, \mu} |ST^{\lambda'}(\alpha)| \cdot |ST^{\mu'}(\beta)| \cdot \delta_{\lambda\mu} \\ &= \sum_{\lambda \vdash k} |ST^{\lambda'}(\alpha) \times ST^{\lambda'}(\beta)| \; .\end{aligned}$$

($i(D_1, D_2)$ denotes the intertwining number of the two representations D_1 and D_2.)

Thus pairs of tableaux of the same shape will be of importance.

A bitableau is a pair (S,T) of two tableaux of the same shape. If S (resp. T) has content α (resp. β) then (S,T) is said to have content (α,β). (S,T) is standard if both S and T are standard.

Let $BT(\alpha,\beta)$ (resp. $SBT(\alpha,\beta)$) denote the set of all (resp. all standard) bitableaux of content (α,β).

So the main problem of this section will be to determine "natural" functions

$$F : BT(\alpha,\beta) \longrightarrow R_{\alpha\beta}$$

such that $SBT(\alpha,\beta)$ is mapped by F onto an R-basis of $R_{\alpha\beta}$.

One can look upon these functions as a kind of alternation or symmetrization process, or suitable combinations of these processes. Let me begin with a pure alternation process.

I. Bideterminants.

Call

$$\left(\begin{array}{ccc|ccc} s_{11} \cdots & & \cdots s_{1\lambda_1} & t_{11} \cdots & & \cdots t_{1\lambda_1} \\ \vdots & & & \vdots & & \\ s_{i1} \cdots & & \cdots s_{i\lambda_i} & t_{i1} \cdots & & \cdots t_{i\lambda_i} \\ \vdots & & & \vdots & & \\ s_{h1} \cdots & & \cdot s_{h\lambda_h} & t_{h1} \cdots & & \cdot t_{h\lambda_h} \end{array}\right) = (S|T) :=$$

$$\prod_{i=1}^{h} \det \begin{pmatrix} (s_{i1}|t_{i1}) & \cdots & (s_{i1}|t_{i\lambda_i}) \\ \vdots & & \vdots \\ (s_{i\lambda_i}|t_{i1}) & \cdots & (s_{i\lambda_i}|t_{i\lambda_i}) \end{pmatrix}$$

the bideterminant associated to the λ-bitableau (S,T).
If $SBD(\alpha,\beta)$ denotes the set of all bideterminants which correspond to the standard bitableaux of content (α,β), the following theorem holds.

Theorem (Doubilet, Rota, Stein)
Let R be any commutative ring with unit element $1_R \neq 0$. Then

$$\boxed{R_{\alpha\beta} = \langle\langle SBD(\alpha,\beta)\rangle\rangle_R} .$$

(See [DRS, DKR, CP, CEP, Cl III].) □

The fact that $SBD(\alpha,\beta)$ spans $R_{\alpha\beta}$ follows from a straightening algorithm, based on a generalized Laplace expansion. The linear independence of $SBD(\alpha,\beta)$ results from certain nice properties of the so-called Capelli operators. These Capelli operators are suitable products of (set) polarization operators.

After this pure alternation process I now mention a pure symmetrization process.

II. Bipermanents.

To every λ-bitableau (S,T) of content (α,β) corresponds the following element of $R_{\alpha\beta}$, which I would like to call the <u>bipermanent</u> to (S,T):

$$\left(\begin{array}{ccccc|ccccc} s_{11} & \cdots & s_{1j} & \cdots & s_{1\lambda_1} & t_{11} & \cdots & t_{1j} & \cdots & t_{1\lambda_1} \\ s_{21} & \cdots & s_{2j} & \cdots & s_{2\lambda_2} & t_{21} & \cdots & t_{2j} & \cdots & t_{2\lambda_2} \\ & & \vdots & & & & & \vdots & & \\ s_{h1} & \cdots & & \cdots & s_{h\lambda_h} & t_{h1} & \cdots & & \cdots & t_{h\lambda_h} \end{array}\right)^{\#} = (S|T)^{\#} :=$$

$$\prod_{j=1}^{\lambda_1} \mathrm{per}\begin{pmatrix} (s_{1j}|t_{1j}) & \cdots & (s_{1j}|t_{\lambda'_j j}) \\ \vdots & & \vdots \\ (s_{\lambda'_j j}|t_{1j}) & \cdots & (s_{\lambda'_j j}|t_{\lambda'_j j}) \end{pmatrix} .$$

Again there exists a generalized Laplace expansion, but some terms appear several times, so a straightening algorithm works only under suitable assumptions on R.

To be more precise let r and t be non-negative integers, $r \leq t$. Define the natural number c_{rt} by

$$c_{rt} := \Sigma_{s=o}^{r} \binom{r}{s} \cdot \binom{t}{s} .$$

<u>Theorem</u>.

If all c_{rt}-multiples of 1_R with $r+t \leq \max\{\alpha_i,\beta_j\}$ are invertible in R, then the standard bipermanents of content (α,β) form an R-basis of $R_{\alpha\beta}$. □

Corollary

If $\mathbb{Q}$ is a subring of R s.t. $1_{\mathbb{Q}} = 1_R$ then the standard bipermanents form an R-basis of the letter place algebra $R_m{}^n$. □

A proof of the above theorem and more details about bipermanents can be found in [Cl IV].

In contrast to the results for bideterminants, the straightening of bipermanents is not "characteristic-free". The same is true for the following

III. Combinations of Symmetrization and Alternation Processes.

Let $\lambda = (\lambda_1,\ldots,\lambda_h) \vdash n$.

$H(\lambda) := \{\sigma : (\lambda) \rightarrowtail\!\!\!\twoheadrightarrow (\lambda) \;/\; \forall_i \; \forall_{j \le \lambda_i} \; \exists_{j'} \; \sigma((i,j)) = (i,j')\}$

is the group of row (= horizontal) permutations, and

$V(\lambda) := \{\sigma : (\lambda) \rightarrowtail\!\!\!\twoheadrightarrow (\lambda) \;/\; \forall_j \; \forall_{i \le \lambda'_j} \; \exists_{i'} \; \sigma((i,j)) = (i',j)\}$

is the group of column (= vertical) permutations with respect to (λ).

Recall that a λ-tableau is a mapping $T : (\lambda) \longrightarrow \mathbb{N}$. Hence the composition $T\circ\sigma$, σ any permutation of (λ), is again a λ-tableau.

Two λ-tableaux S and S' of the same content are said to be column-equivalent (for short: $S \underset{c}{\sim} S'$) if there is a $\sigma \in V(\lambda)$ such that $S' = S\circ\sigma$.

Now I can define to a λ-bitableau (S,T)

(1) the L-symmetrized bideterminant

$$(\boxed{S}\,|T) := \sum_{S' \underset{c}{\sim} S} (S'|T) \quad ,$$

(2) the P-symmetrized bideterminant

$$(S|\boxed{T}) := \sum_{T' \underset{C}{\sim} T} (S|T') \quad ,$$

(3) the <u>LP-symmetrized</u> <u>bideterminant</u>

$$(\boxed{S}|\boxed{T}) := \sum_{S' \underset{C}{\sim} S} \sum_{T' \underset{C}{\sim} T} (S'|T') \quad ,$$

(4) the <u>L-alternated</u> <u>bipermanent</u>

$$(\boxed{S}|T)^{\#} := \sum_{\sigma \in H(\lambda)} \operatorname{sgn}(\sigma) \; (S\circ\sigma|T)^{\#} \quad ,$$

(4) the <u>P-alternated</u> <u>bipermanent</u>

$$(S|\boxed{T})^{\#} := \sum_{\tau \in H(\lambda)} \operatorname{sgn}(\tau) \; (S|T\circ\tau)^{\#} \quad , \text{ and}$$

(6) the <u>LP-alternated</u> <u>bipermanent</u>

$$(\boxed{S}|\boxed{T})^{\#} := \sum_{\sigma \in H(\lambda)} \sum_{\tau \in H(\lambda)} \operatorname{sgn}(\sigma\tau) \; (S\circ\sigma|T\circ\tau)^{\#} \; .$$

By a simple computation one gets the following

<u>Lemma</u>

Let (S,T) be a λ-bitableau. Let $V(\lambda)_T := \{\sigma \in V(\lambda) / T\circ\sigma = T\}$; so $V(\lambda)_T$ is the stabilizer subgroup of T in $V(\lambda)$. Then

$$\boxed{(\boxed{S}|T)^{\#} = |V(\lambda)_T| \cdot (S|\boxed{T})} \;;$$

i.e. the L-alternated bipermanent to (S,T) equals (up to the factor $|V(\lambda)_T|$) the P-symmetrized bideterminant to (S,T). □

According to [Cl I] straightening algorithms exist for all six classes of polynomials.

Now using Corollary 3.4 in [CEP] and results of section 4 in [Cl I] one easily gets the following

Theorem

If $\mathbb{Q}$ is a subring of R such that $1_{\mathbb{Q}} = 1_R$, then the elements of type (1),(2),(3),(4),(5) or (6) which correspond to the standard bitableaux form an R-basis of the polynomial ring $R_m{}^n$. □

§ 2 Applications in the Representation Theory of S_n

I. The Group Algebra of S_n.

Note that $S_n \ni \sigma \longmapsto \begin{pmatrix} \sigma(1) & 1 \\ \vdots & \vdots \\ \sigma(n) & n \end{pmatrix}$ (resp. $\sigma \longmapsto \begin{pmatrix} 1 & \sigma(1) \\ \vdots & \vdots \\ n & \sigma(n) \end{pmatrix}$)

yields an isomorphism $R[S_n] \xrightarrow{\simeq} R_{(1^n)(1^n)}$ of left (resp. right) $R[S_n]$-modules. Thus one can interpret elements of $R_{(1^n)(1^n)}$ as elements of the group algebra.

Theorem

(1) The standard bideterminants (resp. bipermanents) of content $((1^n),(1^n))$ form an R-basis of $R[S_n]$.
(No further assumptions on R are necessary!)

Now let R be a field, char R $\nmid$ n!. Then $R[S_n]$ is semisimple and the following holds:

(2) $R[S_n] = \sum^{\oplus}_{\lambda \vdash n} \sum^{\oplus}_{S \in ST^{\lambda}(1^n)} \ll (\boxed{S} \mid T) \;/\; T \in ST^{\lambda}(1^n) \gg_R$ is a direct decomposition of $R[S_n]$ into minimal right ideals $\ll (\boxed{S} \mid T) \;/\; T \in ST^{\lambda}(1^n) \gg_R$,

(3) $R[S_n] = \sum^{\oplus}_{\lambda \vdash n} \sum^{\oplus}_{T \in ST^{\lambda}(1^n)} \ll (S \mid \boxed{T}) \;/\; S \in ST^{\lambda}(1^n) \gg_R$ is a direct decomposition of $R[S_n]$ into minimal left ideals, and

(4) $R[S_n] = \sum_{\lambda \vdash n}^{\oplus} \ll (\boxed{S} \mid \boxed{T}) \; / \; S, T \in ST^{\lambda}(1^n) \gg_R$ is a direct decomposition of $R[S_n]$ into minimal two-sided ideals. □

More details can be found in [CL III, §7].

II. The Ordinary Irreducible Representations of S_n.

The map $(i|j) \longmapsto (X_j)^{i-1}$ extends to an epimorphism $F : R_m^{\;n} \twoheadrightarrow R[X_1, \ldots, X_n]$ of (right) $R[S_n]$-algebras and the right $R[S_n]$-module

$$\mathcal{S}_{\lambda'}(R) := \left(\begin{array}{c|c} 1 \; 2 \; \ldots \; \lambda_1 & 1 \; 2 \; \ldots \lambda_1 \\ 1 \; 2 \; \ldots \lambda_2 & \lambda_1{+}1 \ldots \lambda_1{+}\lambda_2 \\ \vdots & \vdots \\ 1 \; 2 \; . \; \lambda_h & \ldots . \; n \end{array} \right) \cdot R[S_n]$$

is mapped isomorphically onto the classical Specht module involving Vandermonde determinants.

Theorem

Let R be a field, char $R \nmid n!$. Then $\{\mathcal{S}_{\lambda}(R) \; / \lambda \vdash n\}$ is a full set of pairwise inequivalent irreducible $R[S_n]$--modules. □

III. The Modular Irreducible Representations of S_n.

Let R be a field of prime characteristic p, $p \mid n!$.

Theorem

If $\lambda \vdash n$ is <u>p-regular</u> (i.e.: no p of the λ_i's are equal) then $\mathcal{S}_{\lambda'}(R)$ has a unique minimal (non-zero) submodule:

$$\mathfrak{D}_{\lambda'}(R) := \left(\begin{array}{l} 1\ 2\ \ldots\ \lambda_1 \\ 1\ 2\ ..\ \lambda_2 \\ \vdots \\ 1\ 2\ .\ \lambda_h \end{array} \middle| \boxed{\begin{array}{c} 1\ 2\ \ldots\ \lambda_1 \\ \lambda_1+1 .. \lambda_1+\lambda_2 \\ \vdots \\ \ldots.\ n \end{array}} \right) \cdot R[S_n] ,$$

and $\{ \mathfrak{D}_{\lambda'}(R) \ / \ \lambda \vdash n$ p-regular$\}$ is a full set of pairwise inequivalent irreducible right $R[S_n]$-modules. □

This result [Cl I] duals the following theorem of James.

Theorem [J]

If λ is p-regular then the Specht module $\mathcal{S}_\lambda(R)$ has a unique maximal submodule $\mathcal{F}_\lambda(R)$ ($\neq \mathcal{S}_\lambda(R)$) and $\{ \mathcal{S}_\lambda(R) / \mathcal{F}_\lambda(R) \ / \ \lambda \vdash n$ p-regular$\}$ is a full set of pairwise inequivalent irreducible right $R[S_n]$-modules. □

In [Cl III] an algorithm for the computation of the matrices for the modular irreducible representations of the symmetric groups S_n has been developed. By hand I computed the matrices up to $n = 5$ for all relevant primes. At the present we are writing a computer program for this algorithm.

IV. Various Module Constructions in the Language of Letter Place Algebras

In this subsection I want to indicate how classical S_n-module constructions can be expressed very naturally in terms of letter place algebras. Let me illustrate these constructions by examples.

(i) inner tensor products of Specht modules:

Example.

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 2 & 3 \\ 1 & 2 & & | & 4 & 5 & \end{pmatrix} \cdot R[S_5] \otimes_R \begin{pmatrix} 1 & 2 & | & 1 & 2 \\ 1 & 2 & | & 3 & 4 \\ 1 & & | & 5 & \end{pmatrix} \cdot R[S_5] \cong$$

$$\ll \left(\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & \\ 4 & 5 & \\ 4 & 5 & \\ 4 & & \end{matrix} \middle| \begin{matrix} S \\ \\ \\ T \\ \\ \end{matrix} \right) / \quad S \in ST^{(3,2)}(1^5), \ T \in ST^{(2,2,1)}(1^5) \gg_R .$$

(ii) <u>some induced modules</u>:

$R \otimes_{R[S_\alpha]} R[S_n]$ is isomorphic to $R_{\alpha(1^n)}$ as well as to

$$\left(\begin{matrix} 1 & \alpha_1+1 & \cdots & . \\ 2 & \vdots & & \vdots \\ \vdots & \alpha_1+\alpha_2 & & n \\ \alpha_1 & & & \end{matrix} \middle| \begin{matrix} 1 & \alpha_1+1 & \cdots & . \\ 2 & \vdots & & \vdots \\ \vdots & \alpha_1+\alpha_2 & & n \\ \alpha_1 & & & \end{matrix} \right)^{\#} \cdot R[S_n]$$; here and in

the following example $\alpha = (\alpha_1, \alpha_2, \ldots)$ is an improper partition of n.

If AS_α denotes the sign-representation of the Young subgroup S_α then $AS_\alpha \otimes_{R[S_\alpha]} R[S_n]$ is isomorphic to $(1\ 2\ ..\ n|1\ 2\ \ldots\ n) \cdot R_{\alpha(1^n)}$ as well as to

$$\left(\begin{matrix} 1 & 2 & \cdots & \cdots & \alpha_1 \\ \alpha_1+1 & \cdots & & \cdot & \alpha_1+\alpha_2 \\ \cdots & & & & \\ \cdots & n & & & \end{matrix} \middle| \begin{matrix} 1 & 2 & \cdots & \cdots & \alpha_1 \\ \alpha_1+1 & \cdots & & \cdot & \alpha_1+\alpha_2 \\ \cdots & & & & \\ \cdots & n & & & \end{matrix} \right) \cdot R[S_n] .$$

In an extrem simple way one can define

(iii) <u>$R[S_n]$-modules to skew tableaux:</u>

Let $\lambda \vdash n_1$ and $\mu \vdash n_2$. If (λ) is a subset of (μ), $(\mu)\setminus(\lambda)$ is called a <u>skew diagram</u>, and a mapping $T : (\mu)\setminus(\lambda) \longrightarrow \mathbb{N}$ is a <u>skew</u>

tableau of shape $\mu\setminus\lambda$. The notion of bitableau (resp. bideterminant) is easily generalized to skew bitableau (resp. skew bideterminant).

To every skew tableau with n entries belongs an $R[S_n]$-module.

Example.

$$\left(\begin{array}{cccccc|cccccc} & & & & 2 & 6 & & & & & 8 & 9 \\ & & 1 & 3 & 4 & & & & 5 & 6 & 7 & \\ 1 & 2 & & & & & 1 & 2 & & & & \\ 2 & 4 & & & & & 3 & 4 & & & & \end{array}\right) \cdot R[S_9] = \left(\begin{array}{ccc|ccc} 2 & 6 & & 8 & 9 & \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & & 1 & 2 & \\ 2 & 4 & & 3 & 4 & \end{array}\right) \cdot R[S_9]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 2 & 3 \\ 1 & 2 & & 4 & 5 & \\ 2 & 4 & & 6 & 7 & \\ 2 & 6 & & 8 & 9 & \end{array}\right) \cdot R[S_9] .$$

As a special case of (iii) let me mention

(iv) Littlewood-Richardson products:

These are modules of the following type.

If $\lambda \vdash n_1$ and $\mu \vdash n_2$ then $(\mathcal{S}_\lambda(R) \# \mathcal{S}_\mu(R)) \otimes_{R[S_{n_1} \times S_{n_2}]} R[S_{n_1+n_2}]$ is the Littlewood-Richardson product with respect to the partitions λ and μ. ($\#$ denotes the outer tensor product, see [CR].)

Example.

$$\left(\begin{array}{ccccc|ccccc} & & & 4 & 5 & & & & 6 & 7 \\ & & & 4 & 5 & & & & 8 & 9 \\ & & & 4 & 5 & & & & 10 & 11 \\ 1 & 2 & 3 & & & 1 & 2 & 3 & & \\ 1 & 2 & & & & 4 & 5 & & & \end{array}\right) \cdot R[S_{11}] \cong \left(\mathcal{S}_{(2,2,1)}(R) \# \mathcal{S}_{(3,3)}(R)\right) \otimes_{R[S_5 \times S_6]} R[S_{11}]$$

V. Specht Series.

A <u>Specht series</u> of an $R[S_n]$-module M is a chain $M = M_1 > M_2 > \dots > M_{r+1} = 0$ of $R[S_n]$-submodules M_i, where each factor M_i/M_{i+1} is isomorphic to a Specht module $\mathcal{S}_{\lambda(i)}(R)$ $(i=1,\dots,r)$.

Letter place algebras are an efficient tool to construct Specht series for some classes of $R[S_n]$-modules in a very homogeneous and systematic way (see [Cl II]); examples are certain induced and subduced $R[S_n]$-modules, tensor spaces, and last but not least one gets a characteristic-free version of the classical Littlewood-Richardson rule in a module theoretical setting.

<u>Theorem</u> [Cl II]
Specht series for Littlewood Richardson products can be constructed explicitly.

□

The proof of this theorem shows a close connection between
(i) lattice permutations,
(ii) symmetrized bideterminants, and
(iii) Capelli operators to skew tableaux.

Final Remarks

Similar results hold for the general linear groups.

Extending the letter place algebra concept to letter place spaces of formal power series one can construct series of infinite-dimensional irreducible representations for the countable infinite symmetric group (see [Cl V]).

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