PETER SCHENZEL
WOLFGANG VOGEL

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ON LIAISON AND ARITHMETICAL BUCHSBAUM CURVES IN P^3

Peter Schenzel and Wolfgang Vogel

It seems that A.Cayley[6],p.152 was the first who posed the problem to describe the liaison class of a complete intersection in 1847.Nowadays it is well-known that a curve in P^3 is in the liaison class of a complete intersection if and only if the curve is arithmetically Cohen-Macaulay (see,e.g.[1],[2],[7],[10], or[3]).Furthermore,L.Gruson and C.Peskine[8] have given a complete classification of arithmetical Cohen-Macaulay curves in P^3. Discussing some results from[4] we will begin with this talk to investigate the next simple case,that is,the liaison classes which are characterized by finite-dimensional vector spaces of dimension ≥ 1.Using the theory of (local) Buchsbaum rings that means we will study liaison among arithmetical Buchsbaum curves in P^3.

First,we review the definition and basic results of liaison,see [10]and[11].Let $\underline{a},\underline{b}$ be two ideals of the local Gorenstein ring R.

<u>Definition:</u> The ideals $\underline{a},\underline{b}$ are algebraically linked by a complete intersection $\underline{x} = \{x_1,\ldots,x_g\} \subseteq \underline{a} \cap \underline{b}$ if

(i) $\underline{a}$ and $\underline{b}$ are ideals of pure height g, and

(ii) $\underline{a}/\underline{x}\cdot R = \mathrm{Hom}_R(R/\underline{b},R/\underline{x}R)$ and

$\underline{b}/\underline{x}\cdot R = \mathrm{Hom}_R(R/\underline{a},R/\underline{x}\cdot R)$.

Furthermore,$\underline{a}$ and $\underline{b}$ are linked (geometrically) by a complete intersection $\underline{x} = \{x_1,\ldots,x_g\}$ if $\underline{a}$ and $\underline{b}$ have no components in common and $\underline{a}\cap\underline{b} = \underline{x}\cdot R$.

Two projective varieties $X,Y \subset P^n_K$ over an algebraically closed field K are linked if the ideal sheaves of X and Y are linked in a corresponding manner.The equivalence relation generated by linkage is called liaison.

Let $C \subset P_K^3$ be a curve, that is, C is an 1-dimensional subscheme of P^3, equidimensional, locally Cohen-Macaulay and generic complete intersection. A.P.Rao [11] studied the following invariant, due to R.Hartshorne:

$$M(C) := \bigoplus_v H^1(P^3, \mathcal{J}_C(v)).$$

M(C) is a graded $S := K[x_o,x_1,x_2,x_3]$-module of finite length. M(C) is invariant up to duals and shifts in gradings, under liaison. It follows from [11] that each graded S-module of finite length, there is a liaison equivalence class, and that the module determines that equivalence class.

The equivalence class of curves C corresponding to M(C) = o, that is, C is arithmetically Cohen-Macaulay, is well-studied in the above-mentioned papers. Now, let us consider equivalence classes corresponding to a finite-dimensional vector space of dimension ≥ 1. Since $M(C) \cong H^1_{\underline{m}}(S/I(C))$, where I(C) is the defining ideal of the curve C, we have to apply the theory of local Buchsbaum rings. Therefore we will recall some basic facts on Buchsbaum rings. Denote by $e_o(\underline{x},A)$ the multiplicity of a local noetherian ring A with respect to a system of parameters $\underline{x} = \{x_1,\dots,x_d\}$, d:=dim(A), of A.

<u>Definition:</u> A local noetherian ring A with maximal ideal $\underline{m}$ is said to be a Buchsbaum ring if the difference

$$l_A(A/\underline{x}\cdot A) - e_o(\underline{x},A)$$

is an invariant i(A) of A not depending on the system of parameters $\underline{x}$ of A. This is equivalent to the condition that every system of parameters $\underline{x} = \{x_1,\dots,x_d\}$ of A is a weak A-sequence, i.e.,

$$(x_1,\dots,x_{i-1}) : x_i \subseteq (x_1,\dots,x_{i-1}) : \underline{m}$$

for every i=1,...,d.

The concept of the theory of Buchsbaum rings was given in [16], [17] and has its root in an answer in [19] to a problem of D.A.Buchsbaum [5]. For more specific information on Buchsbaum rings see, e.g. [18], [14] or the forthcoming monograph [15]. In particular, we have for a Buchsbaum ring A of dimension $d \geq 1$:

$\dim_{A/\underline{m}} H^i_{\underline{m}}(A) < \infty$ for all $i \neq d$, and

$$i(A) = \sum_{i=o}^{d-1} \binom{d-1}{i} \dim_{A/\underline{m}} H^i_{\underline{m}}(A)$$

where $H^i_{\underline{m}}$ denotes the i-th derived local cohomology functor with support $\{\underline{m}\}$.

Note, a local ring A is Cohen-Macaulay if and only if A is a Buchsbaum ring with $i(A)=o$. From [10] we know that liaison respects the Cohen-Macaulay property. In connection with our considerations we need the following more general result:

<u>Theorem 1:</u> Let R be a local Gorenstein ring of dimension $d \geq 1$ and with maximal ideal $\underline{m}$. Suppose that the ideals $\underline{a},\underline{b} \subset R$ are linked. Then we have:

(a) $R/\underline{a}$ is a Buchsbaum ring if and only if $R/\underline{b}$ is a Buchsbaum ring.

(b) for the local cohomology modules

$$H^i_{\underline{m}}(R/\underline{a}) = \mathrm{Hom}(H^{d-i}_{\underline{m}}(R/\underline{b}),E)$$

for $i=1,\ldots,\dim(R/a)$, where E is the injective hull of the residue field of R.

<u>Sketch of the proof:</u> To this we need a new invariant under liaison extending M(C) of a curve C. From the standpoint of local algebra this invariant is defined for an arbitrary ideal $\underline{a}$ of a local Gorenstein ring R. Let $E^{\bullet}_R$ be the minimal injective resolution of R over itself. Then we define

$$I^{\bullet}_{\underline{a}} = \mathrm{Hom}_R(R/\underline{a},E^{\bullet}_R) \ ,$$

that is, the complex $I^{\bullet}_{\underline{a}}$ is the dualizing complex of $R/\underline{a}$ in the sense of R.Hartshorne. Note that in the derived category the complex $I^{\bullet}_{\underline{a}}$

is isomorphic to $\underline{R}\mathrm{Hom}_R(R/\underline{a},R)$.If we abbreviate $g:=\dim(R)-\dim(R/\underline{a})$, then we call

$$K_{\underline{a}} := H^g(I^\bullet_{\underline{a}}) = \mathrm{Ext}^g_R(R/\underline{a},R) \neq 0$$

the canonical or dualizing module of $R/\underline{a}$.Factoring out the first non-vanishing cohomology module $K_{\underline{a}}$ of $I^\bullet_{\underline{a}}$ we get a short exact sequence

$$0 \longrightarrow K_{\underline{a}}[-g] \longrightarrow I^\bullet_{\underline{a}} \longrightarrow J^\bullet_{\underline{a}} \longrightarrow 0$$

where $J^\bullet_{\underline{a}}$ is up to a shift in grading the truncated dualizing complex of $R/\underline{a}$.Then we can prove that there exists a canonical isomorphism

$$J^\bullet_{\underline{b}} = \underline{R}\mathrm{Hom}_R(J^\bullet_{\underline{a}},R)[-g]$$

in the derived category,that means,$J^\bullet_{\underline{a}}$ is invariant (up to duality and shifts in gradings) under liaison.Now,from the main result of [14] we know that $R/\underline{a}$ is a Buchsbaum ring if and only if $J^\bullet_{\underline{a}}$ is quasi-isomorphic to a complex of $R/\underline{m}$-vector spaces in the derived category.Therefore we get the statement (a) of theorem 1 since $J^\bullet_{\underline{a}}$ is invariant under liaison.The statement (b) follows immediately from the local duality theorem , q.e.d.

Remarks: (i) Using the fact that $R/\underline{a}$ is a Cohen-Macaulay ring if and only if $J^\bullet_{\underline{a}} = 0$ we obtain another proof that liaison respects the Cohen-Macaulay property.

(ii) Let C be a curve in P^3.Denote by A the local ring of the vertex of the (affine) cone over C.The curve C is called arithmetically Buchsbaum if and only if the local ring A is a Buchsbaum ring.In this special case the curve C is arithmetically Buchsbaum if and only if $H^1_{\underline{m}}(A) \cong M(C)$ is a K-vector space.Furthermore,we get:

$$M(C) \cong H^1_{\underline{m}}(S/I(C)) = \underline{R}\mathrm{Hom}_S(J^\bullet_{I(C)},S)[-2]$$

by applying the local duality theorem.Since $J^\bullet_{I(C)}$ is invariant under liaison we recover the invariance of $M(C)$.We set $i(C):=i(A)$.

Examples: The simplest curve in P^3 which belongs to the liaison class corresponding to a vector space of dimension 1 is the union of 2 skew lines in P^3.Having this curve the specific data in the papers from the late 19th century yield examples of arithmetical Buchsbaum curves with $i(C)=1$.For instance:

C_4^0, C_6^3, C_7^4, C_8^8 from the paper of M.Noether [9] or

C_{10}^{11}, C_{13}^{21}, C_{16}^{34}, C_{19}^{50} from the paper of K.Rohn [12],where C_d^g is a irreducible curve of degree d and of genus g.The curve C_4^0 is the well-known (non-Cohen-Macaulay) twisted quartic curve.Let us consider the curve C_6^3.It follows from M.Noether [9],p.87,(a_3) and (a_3') and our considerations that either

C_6^3 is arithmetically Cohen-Macaulay , or

C_6^3 is arithmetically Buchsbaum.

The resolution of the curve C_6^3 is known if C_6^3 is arithmetically Cohen-Macaulay:

$$0 \longrightarrow S^3(-4) \longrightarrow S^4(-3) \longrightarrow S \longrightarrow S/I(C_6^3) \longrightarrow 0.$$

If C_6^3 is arithmetically Buchsbaum then we obtaine,in addition,the resolution of C_6^3:

$$0 \longrightarrow S(-6) \longrightarrow S^4(-5) \longrightarrow S(-2) \oplus S^3(-4) \longrightarrow S \longrightarrow S/I(C_6^3) \longrightarrow 0.$$

In order to construct this resolution we have the following more general result.

Lemma: Let C be a curve in P^3 which is linked to two skew lines in P^3 by two hypersurfaces of degree f and g,resp.Then we get the following free resolution of $S/I(C)$:

$$0 \longrightarrow S(-f-g) \longrightarrow S^4(-f-g+1) \longrightarrow S(-f) \oplus S(-g) \oplus S^2(-f-g+2)$$

$$\longrightarrow S \longrightarrow S/I(C) \longrightarrow 0.$$

Proving this lemma we must use for a curve C the property of being ideally the intersection of d hypersurfaces, that is, there exists a surjection $\bigoplus_{i=1}^{d} \mathcal{O}_{P^3}(-a_i) \dashrightarrow \mathcal{I}_C \dashrightarrow 0$, for some integers a_i. This definition is equivalent to saying that there are homogeneous elements $f_1,\ldots,f_d$ in I(C) such that

$$C = \mathrm{Proj}(S/(f_1,\ldots,f_d)).$$

Using liaison the assumptions of our lemma imply that C is ideally the intersection of 3 hypersurfaces and I(C) is generated by precisely 4 elements. Since C is arithmetically Buchsbaum with i(C)=1 we get the statement of the lemma.

Another consequence of the property of being ideally the intersection of three surfaces is that the homogeneous ideal I(C) of a curve C is generated by precisely 3 elements if and only if C is ideally the intersection of 3 surfaces and C is arithmetically Cohen-Macaulay (non-complete intersection). If one continues in this vein, one discovers that curves in P^3 such that their homogeneous ideals are generated by precisely 4 elements are much more complicated. We have the following theorem.

Theorem 2: Let C P^3 be any curve. The following conditions are equivalent:

(i) C is arithmetically Buchsbaum (non-Cohen-Macaulay) and C is ideally the intersection of three hypersurfaces, say $f_1=f_2=f_3=o$.

(ii) There are homogeneous elements f_1,f_2,f_3,f_4 which provide a minimal base for I(C), and $x_i \cdot f_4 \in (f_1,f_2,f_3)$ for i=o,1,2,3.

Our lemma and theorem 2 show that the property of being ideally the intersection of three surfaces is useful in studying curves in P^3. The analogous result of Abhyankar in P^3, namely that every curve in P^3 is ideally the intersection of three hypersurfaces, is not true in general. By[10] there exists such a curve. Furthermore, Rao[11] proved that every liaison class contains curves that are not ideally the intersection of three surfaces, and that there is a liaison class that does not contain a curve arising from a section of a rank two bundle. In this connection we get the following

Theorem 3: Every liaison equivalence class corresponding to a finite-dimensional vector space of dimension >1 does not contain any curve that is ideally the intersection of three surfaces, hence contains no curves coming from sections of rank two bundles.

Sketch of proof of theorem 2 and 3: Let $I(C)$ be the defining ideal of the curve C. $\mu(I(C))$ denotes the number of elements in a minimal basis of $I(C)$. If C is an arithmetical Buchsbaum curve with invariant $i(C) \geqslant 1$ then we obtain:

$$\mu(I(C)) \geqslant 3 \cdot i(C) + 1.$$

If in addition C is ideally the intersection of three surfaces then we get:

$$\mu(I(C)) = 4 \quad \text{and} \quad i(C) = 1.$$

Now, theorem 2 and 3 follow from these facts and the isomorphism

$$M(C) \cong \mathrm{Hom}_K((I(C)/(f_1,f_2,f_3))(f+g+h-4),K),$$

where C is ideally the intersection of the three surfaces $f_1=f_2=f_3=o$ of degrees f,g,h, resp.

We conclude by studying some examples.

1.) Take the curve C given parametrically by (s^6, s^4t^2, st^5, t^6).

We get the following properties:

$I(C) = (x_0^2x_3-x_1^3,\ x_0x_2^2-x_1^2x_3,\ x_0x_3^2-x_1x_2^2,\ x_1x_3^3-x_2^4)$, that is, $I(C)$ is generated by precisely four elements. C is not arithmetically Buchsbaum and also not ideally the intersection of three surfaces. This example shows that we can not drop the last condition in (ii) of theorem 2.

2.) Take the curve C given parametrically by (s^7,s^5t^2,st^6,t^7).
Then we obtain:
$I(C) = (f_1,f_2,f_3,f_4)$ where

$f_1=x_0^2x_2-x_1^3$, $f_2=x_0x_3^2-x_1x_2^2$, $f_3=x_0x_3^3-x_1^2x_3^2$, and $f_4=x_1x_3^4-x_2^5$,
i.e., $\mu(I(C)) = 4$.

C is not arithmetically Buchsbaum, but C is ideally the intersection of the three surfaces $f_1=f_2=f_4=o$. This example shows that curves C in P^3 with $\mu(I(C))=4$ and the property of being ideally the intersection of three surfaces are not, in general, arithmetically Buchsbaum.

3.) There exist irreducible curves C_{42}^{145} in P^3 of degree 42 and genus 145 which belong to the liaison class corresponding to a vector space of dimension 3.

In order to prove this statement we want to mention that K.Rohn [13] studied the residual intersection of special classes of space curves lying on any surface of degree 4 in the year 1897. From these specific data we get that the rational twisted cubic curve C_3^o, counted with multiplicity 6, is linked to irreducible curves C_{42}^{145} by two hypersurfaces of degree 4 and 15. Therefore our claim follows if we show that the local ring

$$A := K[x_o,x_1,x_2,x_3]_{(x_o,x_1,x_2,x_3)}/\underline{p}^3$$

is a Buchsbaum ring with invariant $i(A)=3$, where $\underline{p}$ is the defining prime ideal of C_3^o. Some nasty calculations yield our assertion.

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Martin-Luther-University,Dep.of Mathematics, 401 Halle,
German Democratic Republic.