PAUL ROBERTS

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Astérisque, tome 87-88 (1981), p. 353-378
<http://www.numdam.org/item?id=AST_1981__87-88__353_0>

ON THE CONSTRUCTION OF GENERIC RESOLUTIONS

OF DETERMINANTAL IDEALS

by

Paul Roberts

I. INTRODUCTION

We consider the problem of constructing a minimal free resolution of the ideal generated by the minors of given size of a generic matrix. More precisely, let Z be the ring of integers, let X_{ij}, $i = 1, \ldots, n$; $j = 1, \ldots, m$ be mn indeterminates, and let A be the polynomial ring $Z[X_{ij}]$. Let D_t denote the ideal of A generated by the $t \times t$ minors of the matrix (X_{ij}). Then we would like to find a minimal free resolution of the A-module A/D_t.

If Z is replaced by a field of characteristic zero, this problem, and in fact a more general one, has been solved by Lascoux [3]; the resolution in this case has been constructed in different ways by Nielsen [4] and Roberts [6]. All of these resolutions are built from Schur functors, and each module in the resolution is a direct sum of tensor products of Schur functors applied to certain free modules F and G. This is no longer true in the generic case. We will give a simple example to show this later, but the main reason is that the constructions all use in some form or another the fact that the category of modules over the rational group algebra of a symmetric group is semi-simple, either by defining maps through direct sum decompositions or by the direct use of the Young idempotents. The proof of exactness given by Lascoux also uses Bott's theorem on the vanishing of cohomology, which also fails in the generic case.

Before discussing some ways of getting around these problems, we should explain more exactly what is meant by a generic minimal resolution. Since D_t is a graded ideal of the graded ring $A = Z[X_{ij}]$, A/D_t will have a resolution

$$X. = \to X_i \to X_{i-1} \to \dots \to X_0 \to A/D_t \to 0$$

with each X_i a sum of modules of the form $A[k]$, i.e. A with the grading shifted k degrees. We can also require the generators of X_i to be mapped to a minimal set of homogeneous generators of $\mathrm{Ker}(X_{i-1} \to X_{i-2})$. Since Z is not a field, such a resolution is not unique (although it can be shown that the number of times each $A[k]$ occurs in each X_i is unique). By a generic minimal free resolution we mean something stronger; namely, a resolution in which each map: $X_i \to X_{i-1}$ can be defined by a matrix with entries of degree > 0. The existence of a resolution $X.$ with this property is equivalent to any of the following (see Roberts [7] for a discussion of this):

1. For any ring R, $X. \otimes_Z R$ is a minimal free resolution of $(A/D_t) \otimes_Z R$ over $R[X_{ij}]$.
2. The Betti numbers of $Z/pZ \otimes_Z (A/D_t)$ do not depend on p.
3. $\mathrm{Tor}_i^A(A/D_t, Z)$ is a free Z-module for all i.

It is at present not known whether a generic minimal free resolution exists. We note that Nielsen [5] has constructed a resolution for A/D_t (and much more), but that it is not minimal.

As noted above, the modules in a generic resolution will not be sums of Schur functors, at least not in a natural way. Nonetheless, they should clearly be functors which reduce to these when tensored with the rational numbers.

To describe this further we need more notation. Let m, n, and t be integers with $t \geqslant 1$ and $m \leqslant n$, and let (X_{ij}) be an $(m+t-1) \times (n+t-1)$ matrix of indeterminates. Let F be a free $A\ (= Z[X_{ij}])$-module of rank $n + t - 1$ and G one of rank $m + t - 1$; then (X_{ij}) defines a map $\phi\colon F \to G$. Then if Z is replaced by $\mathbb{Q}$, the modules in a resolution of A/D_t can be written as sums of modules of the form $S_I F \otimes S_J G$. where S_I, S_J denote Schur functors of certain partitions I, J and the boundary maps are induced by ϕ. Thus in the generic case one might hope to find similar functors which are extensions of functors of this type rather than sums of them. There is one further complication, however, in that there can be more than one functor over Z which gives the same Schur functor over $\mathbb{Q}$; the symmetric and divided powers are the best known example of this. We will, in fact, use both Schur functors and their duals; there exist others as well, but it is not clear whether they occur naturally in these resolutions.

The idea of what follows is based partly, of course, on the characteristic zero case and partly on some constructions of Nielsen [4] which have been extended to the generic case by Akin, Buchsbaum, and Weyman [2]. They constructed the Koszul complex (which here means the case of 1×1 minors of a matrix) by defining Schur functors of complexes and applying this to the map

$\phi: F \to G$ above. In characteristic zero Cauchy's formula (see Lascoux [3], p. 210) gives a decomposition as described above; in the generic case one has a filtration whose factors are of the desired type.

In this paper we will construct resolutions in more general cases so that they have filtrations of this sort. In the first few sections we define the basic materials and prove some combinatorial results which originate from the characteristic zero case but which we prove directly. They will be used later to show the exactness of some of the complexes we construct, and, in addition, to justify the general methods used in their construction.

II. THE MODULES IN A GENERIC MINIMAL RESOLUTION.

We discuss in this section the form that the modules in a generic resolution should have; for the most part, this means reviewing the results which hold in characteristic zero.

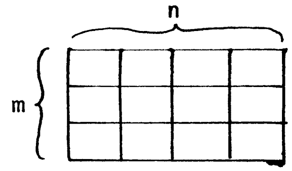
Let F have rank $n + t - 1$ and G rank $m + t - 1$ as before, and let $\phi: F \to G$ be the map defined by a generic $(m+t-1) \times (n+t-1)$ matrix. Let (mn) denote the partition $(\overbrace{n,n,\ldots,n}^{m})$. Let $I = (i_1, i_2, \ldots, i_m)$ be any partition (we use the notation in which we assume $i_1 \geq i_2 \geq \ldots \geq i_m \geq 0$; the Young diagram associated to I will have m rows of lengths i_j). If $i_j \leq n$ for all J, we will say $I \leq (mn)$; in this case let $(mn) - I$ denote the partition $(n - i_m, \ldots, n - i_1)$. For any partition J, $\tilde{J}$ will denote its adjoint; that is, the partition whose Young diagram is that of J with rows and columns reversed.

First assume $t = 1$ (the case of 1×1 minors). Let $X.$ denote a minimal free resolution; then Cauchy's formula gives

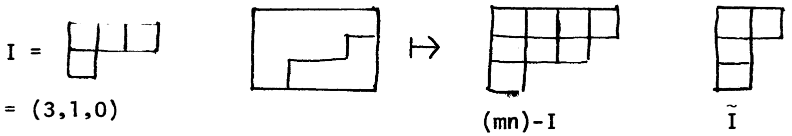
$$X_i \cong \bigoplus_{\substack{|I|=i \\ I\leq(mn)}} S_{\tilde{I}}\, F \otimes S_{(mn)-I}\, G$$

$$\text{where } |I| \text{ denotes } \sum_{j=1}^{m} i_j$$

This can be represented "pictorially" in terms of Young diagrams as follows: the partition (mn) is represented by the rectangular Young diagram



The Young diagrams in the decomposition of X_i can be found by taking all Young diagrams which fit into the rectangle, removing each one in turn from the lower right corner and taking its adjoint: e.g.,



Thus in the example given here one of the summands of X_4 is $S_{(2,1,1,0)}F \otimes S_{(4,3,1)}G$.

How suppose $t > 1$. We will describe the partitions in this case by constructing them from the case $t = 1$ for the same m and n (recall that the matrix is of size $(m+t-1) \times (n+t-1)$, not $m \times n$).

We say that a partition I has Durfee square k if its Young diagram contains a square of side k but no larger square. Equivalently, this says that $i_k \geqslant k$ and $i_{k+1} \leqslant k$. Suppose $I \subseteq (mn)$ and I has Durfee square k. Let $I_t = (i_1+t-1,\ldots,i_k+t-1,i_{k+1},\ldots,i_m)$. Thus the Young diagram of I_t is that of I with t-1 squares added to each of the first k rows, so the Young diagram of $\tilde{I}_t$ is that of $\tilde{I}$ with t-1 squares added to each of the first k columns. Similarly, let $((mn)-I)_t$ denote the partition whose Young diagram is that of (mn) - I with t-1 squares added to each of the first n-k columns. Then

$$X_i = \bigoplus_{\substack{|I|=i \\ I\subseteq(mn)}} S_{\tilde{I}_t} F \otimes S_{((mn)-I)_t} G$$

We show next that this definition agrees with that derived from Bott's Theorem (Lascoux [ ], Corollaire 5.10). The more general formula given there (changing notation to agree with that used here) is

$$X_i = \bigoplus_{|I|-n(I)=i} S_{\tilde{I}} F \otimes S_{I'} G^*$$

where the integer n(I) and partition I' are defined in non-trivial cases to be the unique ones so that

$$H^{n(I)}(S_I Q^*) \cong S_{I'}(G^*)$$

where Q is the canonical bundle of the Grassmannian of quotients of G* of rank m. When worked out, this comes to:

1. $i_k \geq k + t - 1$ so that $S_{I'}G^* \neq 0$
   $i_1 \leq n + t - 1$ so that $S_I F \neq 0$.
2. $I' = (i_1 - t + 1, \ldots, i_k - t + 1, \underbrace{k, \ldots, k}_{t-1}, i_{k+1}, \ldots, i_m)$
   where k is the size of the Durfee square of I.

The first condition says that there is a partition $J \subseteq (mn)$ with Durfee square k such that $I = J_t$; in fact, we have

$J = (i_1-t+1, \ldots, i_t-t+1, i_{k+1}, \ldots, i_m)$, so that $(mn) - J =$ $(n-i_m, \ldots, n-i_{k+1}, n-i_k+t-1, \ldots, n-i_m+t-1)$. Noting that $n-i_{k+1} \geq n-k$ and $n-i_k+t-1 \leq n-k$ (by condition 1. above), we see that adding $t-1$ squares to the first $n-k$ columns of $(mn) - J$ gives

$$((mn)-J)_t = (n-i_m, \ldots, n-i_{k+1}, \underbrace{n-k, \ldots, n-k}_{t-1}, n-i_k+t-1, \ldots).$$

Thus $((mn)-J)_t = (m+t-1)n - I'$, so, since the rank of G is $m+t-1$, we have

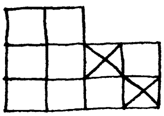
$$S_{I'}G^* \cong S_{((mn)-J)_t} G.$$

Thus the two definitions give the same sets of Schur functors.

We now introduce some notation for the bases of the modules X_i. Of course, this is done by applying the standard bases for Schur functors to the above decompositions; however, we wish to give two special forms of these bases for later use. The first will be used in the combinatorial computations of the next section, and the second will come into the actual constructions.

Let m, n, and t be as above. Let $g_1, \ldots, g_{m+t-1}$ be a basis for G and $f_1, \ldots, f_{n+t-1}$ a basis for F; denote $\{g_i\}$ by B_G and $\{f_i\}$ by B_F. Order B_G and B_F in the usual way.

To the Young diagram (mn) add $t-1$ squares to the top of each of the first $n-m$ columns. By "diagonal square" we will mean a square in (mn) (i.e., not one of the added ones) in the diagonal starting at the lower right corner, marked x in the following example $(m=2, n=4, t=2)$:



By "standard tableau" we mean a tableau in which:

1. Each non-diagonal square contains an element of B_F or B_G.
2. Each added square contains an element of B_G.
3. Each diagonal square contains either t elements of B_G arranged in a column or t elements of B_F arranged in a row.
4. If a square contains an element or elements of B_F, so do all squares to the right of and below it.
5. It follows from 4. that we can produce two Young tableaux by taking all the columns of elements of B_G and the rows of elements of B_F and arranging them so that the top squares and right hand squares respectively are in line.

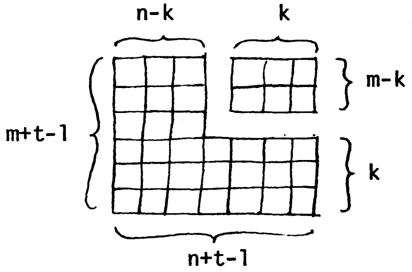
$$\left(\text{Example:}\quad \begin{matrix} g_1 g_1 & & \\ g_2 g_3 & \begin{smallmatrix} g_2 \\ g_3 \end{smallmatrix} & f_1 \\ g_3 f_5 f_4 & & f_3 f_1 \end{matrix} \quad \rightarrow \quad \begin{matrix} g_1 g_1 g_2 & \\ g_2 g_3 g_3 & f_1 \\ g_3 f_5 f_4 f_3 f_1 & \end{matrix}\right)$$

We require that the tableau of elements of B_G and the adjoint of the tableau of elements of B_F be standard.

It follows from this construction that the set of standard tableaux in which exactly i squares contain elements of B_F is a basis for X_i. The advantage of this basis is that it uses the same diagram regardless of the size of the Durfee square; its disadvantage is that it does not immediately show the Young tableaux involved. We now give a second notation which is better in this respect but depends on the size of the Durfee square.

For given k, we now construct a basis for that part of X_i of Durfee square k (this means that the partition I in the summand

$S_{\tilde{I}_t} F \otimes S_{((mn)-I)_t} G$ has Durfee square k). We take a diagram of the type:



We then order $B_G \cup B_F$ by letting $g_1 < \ldots < g_{m+t-1} < f_{n+t-1} < \ldots < f$, and consider tableaux in which each square is filled with an element of $B_G \cup B_F$. Such a tableau will be called standard if:

1. The upper left $(m+t-1-k) \times (n-k)$ rectangle contains only elements of B_G and the lower right $k \times (k+t-1)$ rectangle contains only elements of B_F.
2. Each row is non-decreasing, and strictly increasing in elements of B_F.
3. Each column is non-decreasing, and strictly increasing in elements of B_G.

Again it is easy to check that the standard tableaux with $i + k(t-1)$ boxes containing elements of B_F form a basis for the part of X_i of Durfee square k.

III. A COMBINATORIAL RESULT.

We prove a theorem in this section which will be used later to show that certain complexes are exact. We will be using the following method to show they are exact: the complexes will have length mn as sketched above, so to show they are exact it will be enough to do so after localizing at a prime ideal of height $<$ mn. Since such a prime ideal cannot contain all the $t \times t$ determinants (the height of that ideal is mn) it is enough to invert one $t \times t$ minor and show that the resulting complex is exact. We then should have the complex homotopic to zero, so we can divide by an appropriate ideal and change bases of F and G to turn (X_{ij}) into the matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where I is a $t \times t$ identity matrix. We remark that for this to work we must define the boundary maps in a natural way, so they will allow us to change bases. The theorem of this section is the combinatorial part of the proof that this complex is homotopic to zero.

Suppose $\phi: F \to G$ is defined by the matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ as above. Rechoose bases for F and G by letting $B_G = \{g_1, g_2, \ldots, g_{m-1}, k_1, \ldots, k_t\}$ and $B_F = \{f_1, \ldots, f_{n-1}, h_i, \ldots, h_t\}$, where $\phi(f_i) = 0$ and $\phi(h_i) = k_i$.

We now consider standard tableaux of the first kind defined in the last section. Recall that each square contains one element of B_F or B_G unless it is on the diagonal, when it contains t, either all in B_F or all in B_G.

We define a "corner" to be a square which either contains an element or t elements of B_G while all to the right or below contains elments of B_F or contains elements of B_F while all above or to the left contain elements of B_G. We say that a corner α "can be moved" if it satisfies the following conditions.

Suppose first that α contains a single element of B_G. Then α can be moved if

1. There is an i such that k_i is in the column above (and including) α but h_i is not in the row to the right of α.
2. If i is the largest such integer, the tableau obtained by removing k_i, shifting the elements of B_G below k_i up one position, and

adding h_i to the row to the right of (and including) α in the proper order, i.e.:

$$\begin{array}{lcl} k_{i-1} & & k_{i-1} \\ k_i & & k_{i+1} \\ k_{i+1} & \to & \vdots \\ \vdots & & k_t \\ k_t h_t h_{t-1} \cdots h_j h_\ell & & h_t h_{t-1} \cdots h_j h_i h_\ell \\ (\ell < i < j) & & \end{array}$$

is a standard tableau.

If α is a diagonal square containing elements of B_G, we say that α can be moved if

1. α contains exactly $\{k_1, \ldots, k_t\}$.
2. Replacing $\{k_1, \ldots, k_t\}$ by $\{h_1, \ldots, h_t\}$ gives a standard tableau (it suffices here that there be no h_i in the row to the right of α).

The definitions if α contains elements of B_F are the same with h_i and k_i, and "row" and "column" interchanged.

Let S denote the set of all standard tableaux, and let S_i denote the set of those with i squares filled with elements of B_F (the basis for X_i). Define a map $\tau: S \to S$ as follows: for each standard tableau T, take all corners which can be moved, and choose the first one starting from the lower left and going up and to the right up to and including the diagonal; if none of these can be moved, choose the first one starting from the upper right going down and to the left. Let α be the corner so chosen; $\tau(T)$ is then defined to be the standard tableau obtained replacing k_i by h_i or vice versa or $\{k_1, \ldots, k_t\}$ by $\{h_1, \ldots, h_t\}$ or vice versa as described above. Note that if $T \in S_i$, then $\tau(T) \in S_{i-1}$ or S_{i+1}.

THEOREM 1. τ is well-defined and τ^2 = the identity.

PROOF: To see that τ is well defined we must show that at least one corner can be moved. If α can be moved, then clearly the same corner can be moved back by interchanging the same h_i and k_i, and to show that $\tau^2 = 1$ it will suffice to show that no previous corner in the above ordering can be moved. To show this we need to examine precisely why the second conditions for a corner to be moved might fail. We will assume the corner in question is in B_G; the case for B_F is the same.

Start with the bottom right hand corner. Since this column has $m+t-1 = \text{rank}(G)$ squares, each k_i must occur. Assume some h_i does not; if they all do, we can begin again with the next corner and apply the same argument. Thus we have:

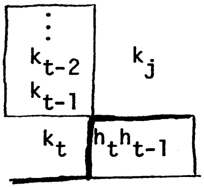
$$\begin{array}{l} g_1 \\ \vdots \\ g_{m-t+1} \\ k_1 \\ \vdots \\ \underline{k_t}\,\big|\, h_t h_{t-1} \cdots h_{i+1} h_j \qquad j < i. \end{array}$$

Only one thing can prevent this corner from being moved. The tableau obtained by inserting h_i is necessarily standard, so there must be a problem in removing k_i and a moment's reflection shows that we must have

$$\begin{array}{l|l} k_i & \\ k_{i+1} & \\ \vdots & h_t \cdots h_{i+1} \\ & \vdots \\ k_t & h_t \cdots h_{i+1} \end{array}$$

Thus at the next corner we have $\{h_{i+1}, \ldots, h_t\}$, and some k_j with $i + 1 \leqslant j \leqslant t$ must be missing. Hence we can repeat this argument with $\{k_1, \ldots, k_t\}$ replaced by $\{h_{i+1}, \ldots, h_t\}$. The only thing to check is that there will be no problem inserting k_j, but the facts that the column immediately to the left is longer (since we are at a corner) and that this column contains $\{k_{j+1}, \ldots, k_t\}$ imply that this is true.

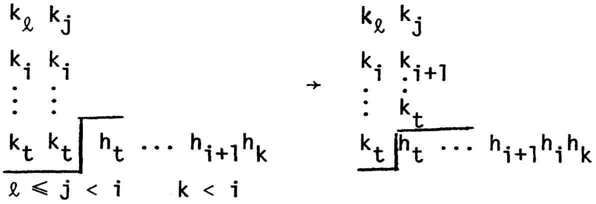
This process can be continued until the diagonal is reached. The last corner before the diagonal will look something like



In the associated Young tableau k_j will be to the right of the top entry in the diagonal square, and, since there must be t entries in this square, that entry must be a g_i. Thus the problem described above cannot occur and this corner can be moved.

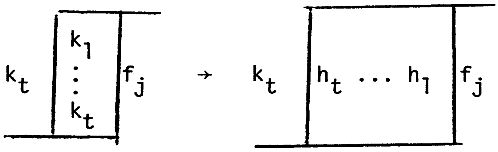
Summarizing, we see that the only way no corner before the diagonal can be moved is for each one to have the complete set of k_i's above it and the complete set of h_i's to its right. If the next corner is a diagonal square it will also contain the complete set of h_i's or k_i's, and in any case one diagonal square will contain these.

Before checking the upper right corner we show that $\tau^2(T) = T$ if a corner of T below the diagonal can be moved. From the above discussion it follows that the only situation which must be checked is:



It is clear that for the lower corner of the second diagram to be moved, we would have to remove a k_s for $s < i$; since $i > j$ this will not give a standard tableau. Thus $\tau^2(T) = T$ in this case.

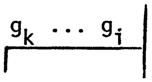
If no corner below the diagonal can be moved but a diagonal corner can, we will have



and again the lower corner cannot be moved.

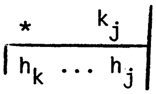
Thus the only part left to check is the upper right corner. If α is the first corner from the upper right, supposing as before that α contains an element of B_G, there will be no elements at all to the right of α, so there are two possible reasons why α could not be moved:

1. α contains a g_i rather than a k_i.

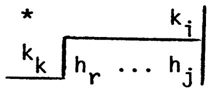


In this case the next corner will have only g_j's above it and we can start there instead.

2. We have

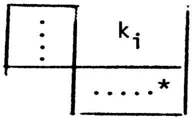


with $i < j$. In this case $*$ will be either k_ℓ for $\ell \leqslant i < k$ or a g_ℓ; in either case, the next corner can be moved. Note that after h_k is moved we have

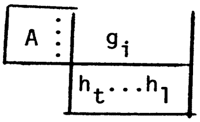


and since $r > i$ and $r < k$ the new h_r corner cannot be moved; hence we will have $\tau^2(T) = T$ in this case; the same is true if the original corner would be moved instead.

To complete the proof we have to check what happens if no corner moves before we arrive back at the diagonal. If the last corner before the diagonal contains a k_i we have



where $*$ is h_1 or an f_i; thus this corner can be moved. If the last corner before the diagonal contains a g_i we have



In fact, the diagonal box A cannot contain $\{k_1, \ldots, k_t\}$ since this would not give a standard tableau. Thus the diagonal square below g_i must contain $\{h_1, \ldots, h_t\}$ from the previous discussion, and hence it must be a corner. It is clear that this corner can be moved. It is easy to check that we have $\tau^2(T) = T$ in either of the last two cases. Thus we have shown that $\tau(T)$ is defined and $\tau^2(T) = T$ for any standard tableau T.

It follows from this theorem that if we had a complex $X_.$ defined naturally in such a way that the bases described above gave bases for X_i for each i, and whenever T was a standard tableau in S_i and $\tau(T) \in S_{i-1}$ we had $d_i(T) = \tau(T)$, (when ϕ is defined by the matrix $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$) then $X_.$ would be homotopic to zero when ϕ is $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and exact in general. However, the condition that $d_i(T) = \tau(T)$ is much too strong; it is possible in some cases, however, to put an ordering on the standard tableaux so that

$$d_i(T) = \tau(T) + \text{higher terms.}$$

This is enough to prove that the complex is homotopic to zero.

IV. SOME EXPLICIT RESOLUTIONS.

We discuss here some cases where the results of the previous section can be used to construct resolutions explicitly. We use the following terminology: we represent Schur functors and dual Schur functors by generators and relations as in Towber [8]. We recall the main facts: $S_I(F)$ is defined as the module whose generators are tableaux with shape I and entries in F and whose relations, in addition to multilinearity, are:

1. Antisymmetry in columns.
2. Suppose the tableau T has columns

$$\begin{matrix} f_1 & f'_1 \\ \vdots & \vdots \\ f_k & f'_\ell \end{matrix} \qquad (\ell \leqslant k)$$

Let r be an integer $\leqslant \ell$. For each r-tuple $1 \leqslant i_1 < i_2 < \dots < i_r \leqslant k$, let $T_{i_1 \dots i_r}$ denote T with f_{i_j} and f'_j interchanged for all j.

Then we have the relation

$$T \equiv \sum_{i_1<\dots<i_r} T_{i_1\dots i_r}.$$

The most important case of relation 2 is when $r = 1$, and it is enough to check this case when we wish to show that relation 2 is preserved by a map into a torsion-free Abelian group.

For the dual Schur functors the definitions are similar but with signs changed; we have symmetry in columns, and, because of this, use divided powers in the columns, and the case $r = 1$ of the second relation becomes

$$T \equiv -\sum_{i=1}^{k} (T \text{ with } f_i \text{ and } f_i' \text{ interchanged}).$$

We remark that the second relation implies that two columns of the same length can be interchanged (perhaps with change of sign in the case of dual Schur functors). We will denote dual Schur functors by S_I^*.

We now consider several cases of resolutions for specific m, n, or t.

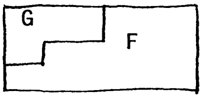
Case 1: $t = 1$. We give a brief outline of the construction of the Koszul complex following the lines of that constructed by Nielsen [4] and Akin, Buchsbaum, and Weyman [2]. Take the Young diagram (mn) and let X_i be the module whose generators are tableaux with entries in F and G; more specifically, with i entries in F and $mn - i$ in G, and whose relations are those defining the Schur functor $S_{(mn)}$ except that whenever two entries in F are interchanged the sign is changed (this is for relation 1 and the case $r = 1$ of relation 2). In addition, we use divided powers in columns for entries in F.

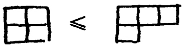
The module X_i, except for the divided powers, is a quotient of a sum of copies of $F^{\otimes i} \otimes G^{\otimes(mn-i)}$, and the convention on signs when interchanging elements in F means that if we consider this as the ith piece of the complex $(F \to G)^{\otimes mn}$, where F has degree 1, the boundary map

$$\oplus(F^{\otimes i} \otimes G^{(mn-i)}) \to \oplus(F^{\otimes(i-1)} \otimes G^{(mn-i+1)})$$

will preserve all the relations. Hence we can use this to induce a boundary map $X_i \to X_{i-1}$.

For a fixed i, we note that by rearranging each column and then putting the columns in proper order (using the relations defining $S_{(mn)}$), we can put each tableau in the form



where the entries in F form a Young diagram I with $|I| = i$ and those in G form the Young diagram $(mn) - I$. Order such diagrams by letting $I \leqslant I'$ if I' has more squares in the last column in which they differ (e.g. ).

Let X_I denote the submodule of X_i generated by tableaux of shape J for $J \leqslant I$. Then if $I' < I$ are consecutive in this ordering, the relations defining X_i restricted to X_I give the relations defining $S_{\tilde{I}}F$ and $S_{(mn)-I}G$ modulo $X_{I'}$; we give a simple example to show how this works:

$$\begin{array}{ccccccc} g_1g_2 & & g_2g_1 & & g_1g_3 & & g_1f_1 \\ g_3f_2 & = & g_3f_2 & + & g_2f_2 & + & g_3f_2 \\ f_1f_3 & & f_1f_3 & & f_1f_3 & & \underline{g_2f_3} \\ & & & & & & \text{in } X_{I'} \end{array}$$

Thus we have $X_I/X_{I'} \cong S^{*}_{\tilde{I}}F \otimes S_{(mn)-I}G$, and we have a filtration of X_I with quotients those given by Cauchy's formula.

Now consider what happens to a standard tableau under the map $X_i \to X_{i-1}$. Since $t = 1$ there is only one h and one k (the notation is that of Theorem 1), and a corner with an element of F looks like

$$\begin{array}{|l} \overline{\ \ast\ } \\ f_j \end{array}, \quad \begin{array}{|l} \overline{\ k\ } \\ h \\ \vdots \end{array}, \quad \text{or} \quad \begin{array}{|l} \overline{\ g_j\ } \\ \left.\begin{array}{l} h \\ \vdots \\ h \end{array}\right\} r \\ f_k \end{array}$$

The first two cannot be moved and contribute zero in the image; the third can be moved and contributes r times

$$\begin{array}{r|l} & g_j \\ & \overline{\ k\ } \\ r-1\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. & \begin{array}{l} h \\ \vdots \\ h \end{array} \\ & f_k \end{array}$$

Since we are using divided powers, however, the coefficient $\frac{1}{r!}$ is replaced by $\frac{1}{(r-1)!}$, and this cancels the factor of r. It follows easily from these facts that if T is a standard tableau with $\tau(T) \in S_{i-1}$, then the image of T in X_{i-1} is $\tau(T)$ + terms higher in the ordering given above. Thus this complex is exact.

We insert here a simple example of the problems which arise when one tries to use direct sums of Schur functors in the generic case. In the case of the resolution of 1×1 minors of a 2×2 matrix, the map from X_2 to X_1 would be

$$(S^*_{(1,1)}F \otimes S_{(2)}G) \oplus (S^*_{(2)}F \otimes S_{(1,1)}G) \to S^*_{(1)}F \otimes S_{(2,1)}G.$$

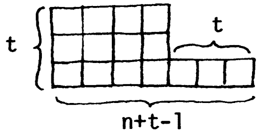
To preserve the relations, the first component of this map would have to send $\begin{array}{l} f_1 \\ f_2 \end{array} \otimes g_1 g_2$ to

$$f_1 \otimes \begin{matrix} g_1 g_2 \\ f_2 \end{matrix} + f_1 \otimes \begin{matrix} g_2 g_1 \\ f_2 \end{matrix} - f_2 \otimes \begin{matrix} g_1 g_2 \\ f_1 \end{matrix} - f_2 \otimes \begin{matrix} g_2 g_1 \\ f_1 \end{matrix}$$

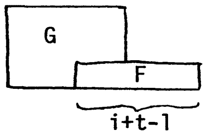
The basis element $\begin{matrix} f \\ h \end{matrix} \otimes gg$ then goes to $2f \otimes \begin{matrix} gg \\ k \end{matrix}$, and there is 2-torsion in the homology $H_1(X.)$. One could take divided powers in $S_\infty G$, but this would create 2-torsion in $H_2(X.)$ from the element $\begin{matrix} f \\ h \end{matrix} \otimes kk$. We note that in the filtration given above $S^*_\theta F \otimes S_\infty G$ is a quotient, and there is no map from this particular functor to X_1.

In the next few examples we will be mainly concerned with that part of a resolution with a given size Durfee square. The modules will have generators which are tableaux of the type described in the second form of the basis constructed in section 2, with entries in F and G in certain places but not necessarily standard. The relations will be similar to those defining Schur functors. We note that the Durfee square zero complex is just A in degree zero.

Case 2: $1 \times n$. (Eagon-Northcott). Here we only have to construct the Durfee square 1 part and the diagram looks like



X_i will be defined by tableaux as follows:



which are antisymmetric in entries in F and in entries in the same column in G; the only other relation needed, since the rank of G is t, is that we can interchange columns of elements of G of the same length (t or $t-1$). In addition, we take divided powers of equal columns of elements of G of length $t-1$ (this is really a divided power of G^*).

The map $d_i\colon X_i \to X_{i-1}$ is defined as follows: let c_j be a column of $t-1$ elements of G for $j = 1, \ldots, i-1$, and let

$$T = \boxed{\begin{array}{ll} T' \quad c_1 \ldots c_{i-1} & \\ & \boxed{f_1 \ldots f_{i+1}} \end{array}}$$

$$\text{Then} \quad d_i(T) = \sum_{j=1}^{i-1} \sum_{k=1}^{i+t-1} (-1)^{k+1} \boxed{\begin{array}{ll} T' \quad c_j \quad c_2 \ldots \hat{c}_j \ldots & \\ \quad \phi(f_k) & \boxed{f_2 \ldots \hat{f}_k \ldots f_{i+t-1}} \end{array}}$$

It is easy to verify that this makes $X.$ a complex.

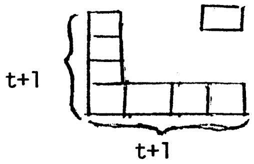
If we now order the standard tableaux letting $T < T'$ if when the first entry in F from the left at which they differ has $f_i \in T$ and $f_j \in T$, then $i < j$. It then follows that if T is a standard tableau with $\tau(T) \in S_{i-1}$, we have

$$d_i(T) = \tau(T) + \text{higher terms}$$

and this complex is exact.

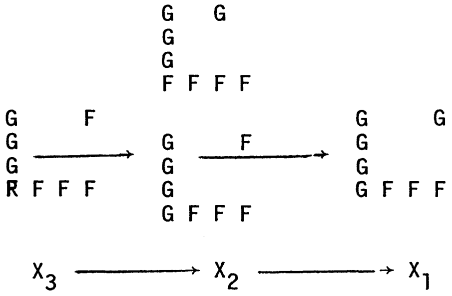
<u>Case 3: 2 × 2</u>. (Gulliksen-Negard). We do this case in some detail, as it is the first one where nontrivial problems occur, but it can be carried out more explicitly than the more general ones, which will only be outlined.

The important part is Durfee square 1, and the diagram is

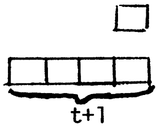


X_i will be defined by tableaux where the upper left $t \times 1$ rectangle has entries in G and the lower right $1 \times t$ rectangle has entries in F. The boundary maps will be induced by those of the tensor product.

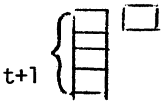
Schematically this complex looks like:



The relations are all relations which involve no entries in F in the upper left corner of the following:

1. Relations defining the Schur functor associated to 

   We recall here that signs are changed when we transpose elements in F; thus this gives in X_3 the dual Schur functor applied to F; we also use divided powers in this case.

2. Relations defining the Schur functor 

 except that relation 2 is replaced by

$$\begin{array}{ll} g_1 & g_{t+1} \\ \vdots & \\ g_t & \\ f_1\cdots f_{t+1} & \end{array} = \sum_{i=1}^{t} \begin{array}{ll} g_1 & g_i \\ \vdots & \\ g_{t+1} & \text{(ith place)} \\ \vdots & \\ g_t & \\ f_1\cdots\cdot f_{t+1} & \end{array} + \sum_{j=1}^{t}(-1)^{j+1} \begin{array}{ll} g_1 & f_j \\ \vdots & \\ g_t & \\ g_{t+1}f_1\cdot\cdot f_j\cdot\cdot f_{t+1} & \end{array}$$

(Note: this relation was modeled after those given by Akin, Buchsbaum, and Weyman [1] in their Durfee square 1 complex).

3. Any tableau with two or more entries in G in the bottom row is zero.

It is easy to see that this gives the correct Schur functors in X_1 and X_3 and the correct filtration in X_2. We check two things: first, we show that relation 2 is compatible with the other ones; it is clearly antisymmetric in $f_1, \ldots, f_t$ (this is the reason for the more complicated definition). The other thing to show is that it is compatible with the relation

$$\begin{array}{ll} g_1 & g_{t+1} \\ \vdots & \\ g_t & \\ f_1 f_2\cdots f_{t+1} & \end{array} + \begin{array}{ll} g_1 & g_{t+1} \\ \vdots & \\ g_t & \\ f_{t+1} f_2\cdots f_1 & \end{array} = \begin{array}{ll} g_1 & f_1 \\ \vdots & \\ g_t & \\ g_{t+1} f_2\cdots f_{t+1} & \end{array}$$

In fact, if we substitute the right hand side of relation 2 for each term in the left hand side of the latter equation, we get

$$\sum_{t=1}^{t} \left(\begin{array}{ll} g_1 & g_i \\ \vdots & \\ g_{t+1} & \\ \vdots & \\ g_t & \\ f_1\cdots f_{t+1} & \end{array} + \begin{array}{ll} g_1 & g_i \\ \vdots & \\ g_{t+1} & \\ \vdots & \\ g_t & \\ f_{t+1}\cdots f_1 & \end{array} \right) + \sum_{j=2}^{t}(-1)^{j+1} \left(\begin{array}{ll} g_1 & f_j \\ \vdots & \\ g_{t+1}f_1\cdots f_{t+1} & \end{array} + \begin{array}{ll} g_1 & f_j \\ \vdots & \\ g_{t+1}f_{t+1}\cdots f_1 & \end{array} \right)$$

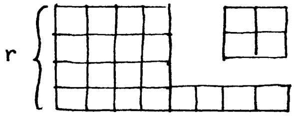
$$+ \begin{array}{ll} g_1 & f_1 \\ \vdots & \\ g_{t+1} f_2\cdots f_{t+1} & \end{array} + \begin{array}{ll} g_1 & f_{t+1} \\ \vdots & \\ g_{t+1} f_2\cdots f_1 & \end{array}$$

$$= \underbrace{[-t + (t-1) + 2]}_{=1} \quad \begin{array}{ll} g_1 & f_1 \\ \vdots & \\ g_{t+1} & f_2 \ldots f_{t+1} \end{array}$$

The other thing to notice is that it is indeed possible to use the map defined by the tesnor product, since if ϕ is applied to any entry in f, it is possible via the above relations to bring the resulting tableau to proper form.

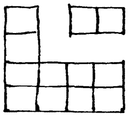
It is again easy to define an ordering on standard tableaux so that $d_i(T) = \tau(T)$ + higher terms whenever $\tau(T) \in S_{i-1}$.

The methods of the three examples can be combined and will give a complex whose diagram is



where r is the rank of G. The idea is to use the method of example 1 in the upper right corner, of example 2 in the left side, and example 3 to connect them. In particular, this gives a version of the Durfee square 1 part of the resolution of D_t for any m, n, and t. The point to notice here is that as in example 3, the relations defined are enough to move any entry of the form $\phi(f_i)$ into the proper position, so we can use the maps induced by the tensor product to define boundary maps.

We conclude with a discussion of the 3×3 case. Here the diagram for the Durfee square 2 complex (in the case $t = 2$) looks like



If we try to carry out a construction as in example 3 we run into a problem when we try to apply d_i to the tableau

$$\begin{matrix} g_1 & & g_4 & g_5 \\ g_2 & & & \\ g_3 & f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 & f_7 \end{matrix}$$

The problem is that we cannot bring $\phi(f_1)$, for example, into proper position, and to define a map which preserves all relations means introducing higher coefficients in the formulas and torsion in the homology. It could be that such torsion exists, but on the other hand this type of tableau should have a non-trivial image not only in the next lower term of the Durfee square 2 complex, but also down in the Durfee square 1 complex. This is similar to the problem which arose attempting to use sums of tensor products of Schur functors, and it could be that one has to combine the constructions of the various size Durfee square complexes rather than construct each separately.

We remark that this problem did not exist in the Durfee square one complex since the part of Durfee square zero is so simple.

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