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Astérisque, tome 87-88 (1981), p. 335-352
<http://www.numdam.org/item?id=AST_1981__87-88__335_0>

YOUNG TABLEAUX AND P.I. ALGEBRAS

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INTRODUCTION

This is a report about the relations between the theory of algebras satisfying polynomial identities (P.I. algebras) and the representation theory of the symmetric groups (S_n-rep.). The relations between S_n-reps and the Procesi-Razimyslov theory of trace identities is not discussed here. A review of these results can be found in [3].

The sequence $\{c_n(A)\}$ of a P.I. algebra A was introduced in [15] in order to prove that $A \otimes B$ is P.I. It was later understood that these dimensions are the degrees of certain S_n-characters, called the cocharacters of the P.I. algebra [18]. These sequences enable one to apply S_n-reps to study many questions about P.I. algebras. For example, Amitsur's $s_\ell^k[x]$ theorem was known with an upper bound for ℓ but not for k. Applying S_n-characters we can re-prove it, together with such bounds for both k and ℓ. This as well as other applications are discussed here. One application of P.I. theory to S_n-characters is also described in §5.

We summarize here most of the results which are known to us and which are relevant to that relation between the two theories. Most of the results which are due to Amitsur are unpublished yet, although some of them can be found in [3]. Detailed proofs are avoided here, but we do give some proofs when they are both short and illuminating. For more results on T-ideals see [23], [24].

To simplify the presentation we assume here the characteristic of the base field to be zero.

§1. Some P.I. Algebras and Identities

Let S_n be the symmetric group on $1,\dots,n$. The following are two important non-commutative (associative) polynomials:

$s_n[x_1,\dots,x_n] = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$ is the n-th standard

polynomial (of degree n), and

$$d_n[x_1,\dots,x_n;y_1,\dots,y_{n-1}] = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{n-1} x_{\sigma(n)}$$

is the n-th Capelli polynomial (of degree $2n-1$).

The definition of a P.I. algebra is well known. The most important P.I. algebra is F_k, the algebra of $k \times k$ matrices over F. Since the Capelli polynomial $D_n[x_1,\dots,x_n;y_1,\dots,y_{n-1}] = d_n[x;y]$ is alternating in $x_1,\dots,x_n$ and is multilinear, a "determinant" type argument shows that $d_n[x;y]$ is an identity for any algebra A of dimension $\dim_F A < n$. In particular, F_k satisfies $d_{k^2+1}[x;y]$. It is shown in [2] that F_k does not satisfy $d_{k^2}[x;y]$. This completely answers which Capelli identities are satisfied by F_k: since $d_{n+1}[x;y]$ is a combination of n-th Capelli polynomials, if A satisfies $d_n[x;y]$ then it also satisfies $d_m[x;y]$ for all $m \geqslant n$.

Next we note that $d_n[x_1,\dots,x_n;1,\dots,1] = s_n[x_1,\dots,x_n]$, so if A satisfies $d_n[x;y]$ then it satisfies also $s_n[x_1,\dots,x_n]$ (the converse is discussed below). The Amitsur-Levitski theorem says that F_k satisfies

$s_{2k}[x_1,\ldots,x_{2k}]$ (as a minimal identity). Roset [22] used the Grassmann algebra to give a very short proof of that theorem. Kemer [8], used the Grassmann algebra in a different way to prove that if an algebra satisfies $s_\ell[x]$ then it satisfies some $d_n[x;y]$, $n = n(\ell)$. Thus (if Char $F = 0$) , an algebra satisfies a standard identity if and only if it satisfies a Capelli identity. It is interesting to mention that the Grassmann algebra was one of the earliest examples, given by Cohn, of an algebra that does not satisfy any standard identity.

We close this section with a classical theorem of Amitsur [1], to be revisited in §6.

Theorem 1. If A satisfies an identity of degree d , then A satisfies

$$(s_{2\ell}[x_1,\ldots,x_{2\ell}])^k \quad \text{where} \quad \ell \leq [\tfrac{d}{2}] .$$

§2. P.I. Algebras and S_n-representations

The identities $I(A)$ of a P.I. algebra A are elements of $F\langle x\rangle$, the free algebra in infinitely many variables $\{x\}$. Also, $I(A) = Q$ is a two-sided ideal in $F\langle x\rangle$, closed under substitutions (a T-ideal).

A basic P.I. result says that every identity can be multilinearized and, since Char $F = 0$, the multilinear identities determine the others. We therefore restrict our attention to multilinear identities. Those of degree n are completely determined by such identities in n fixed variables. We are led to the following construction:

Fix $x_1,x_2,\ldots \in \{x\}$. For each n , let V_n be the vector space of all the multilinear polynomials in $x_1,\dots,x_n$:

$$V_n = \left\{ \sum_{\sigma \in S_n} \alpha_\sigma \, x_{\sigma(1)} \cdots x_{\sigma(n)} \,\middle|\, \alpha_\sigma \in F \right\} .$$

Clearly, $\displaystyle\sum_{\sigma \in S_n} \alpha_\sigma \sigma \leftrightarrow \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ (*)

is an isomorphism between the group algebra FS_n and V_n, as vector spaces over F.

If $Q = I(A)$ are the identities of A, then $Q_n = Q \cap V_n$ are the multilinear identities of degree n in $x_1,\dots,x_n$ and $\{Q_n\}_{n=1}^{\infty}$ determines Q (char $F = 0$).

Since FS_n is an algebra, the above isomorphism (*) induces an algebra structure on V_n. It is convenient to use (*) to identify V_n with FS_n :

$\sigma \equiv x_\sigma \overset{\text{def}}{=} x_{\sigma(1)} \cdots x_{\sigma(n)}$. The algebra structure of V_n is determined by rule $x_\sigma x_\eta \equiv \sigma\eta \equiv x_{\sigma\eta}$ for $\sigma,\eta \in S_n$, [16]. Elements of FS_n are now realized as polynomials in V_n, and we proceed to describe the polynomials realizing the idempotents corresponding to some Young Tableau T_λ, [3], [16], [19]. Here $\lambda \in \mathrm{Par}(n)$ is a partition of n, D_λ the corresponding Young diagram and T_λ a chosen Young Tableau.

<u>Example 1.</u> $\lambda = (n)$, $D_\lambda = \boxed{}\boxed{\quad\cdots\quad}\boxed{}$,

$T_\lambda = \boxed{1}\boxed{2}\boxed{\quad\cdots\quad}\boxed{n}$. The corresponding polynomial is

$$e_T(x_1,\dots,x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} .$$

This is a multilinearization of x^n, hence an identity for all rings satisfying $x^n = 0$.

<u>Example 2.</u> $\lambda = (1^n)$, $T_\lambda = \begin{array}{|c|} \hline 1 \\ \hline \\ \\ \hline n \\ \hline \end{array}$

$$e_T = \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) x_{\sigma(1)} \cdots x_{\sigma(n)} = \frac{1}{n!} s_n[x_1,\ldots,x_n] \ .$$

<u>Example 3.</u> $\lambda \in \mathrm{Par}(n)$ arbitrary with conjugate $\lambda' = (b_1,b_2,\ldots)$. Choose

$$T_\lambda = \begin{array}{|c|c|c|} \hline 1 & b_1+1 & \quad\cdots\quad \\ \hline 2 & b_1+2 & \cdots \\ \hline \vdots & \vdots & \\ & b_1+b_2 & \\ \cline{1-2} b_1 & & \\ \cline{1-1} \end{array}$$

After evaluating $e_T(x_1,\ldots,x_n)$, identify

$$x_1 = x_{b_1+1} = x_{b_1+b_2+1} = \cdots$$

$$x_2 = x_{b_1+2} = x_{b_1+b_2+2} = \cdots$$

$$\vdots$$

$e_T(x)$ becomes a scalar multiple of

$$s_{b_1}[x_1,\ldots,x_{b_1}] \cdot s_{b_2}[x_1,\ldots,x_{b_2}] \cdots \ .$$

If D_λ is an $\ell \times k$ rectangle, we obtain in this way the polynomial $(s_\ell[x_1,\ldots,x_\ell])^k$.

<u>Example 4.</u> Start with an arbitrary tableau

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline \alpha_{11} & \alpha_{12} & \cdots & \\ \hline \alpha_{21} & \alpha_{22} & \cdots \\ \cline{1-3} \vdots & \vdots \\ \cline{1-2} & \\ \cline{1-1} \end{array}$$

Distinguish the variables corresponding to the first column by $z_1,\dots,z_h$, then $e_T(x_1,\dots,x_n)$ is a combination of Capelli polynomials $q_0 d_h[z_1,\dots,z_h;q_1,\dots,q_{h-1}]q_h$ where $q_0,\dots,q_h$ are polynomials in the other variables.

We now turn back to $\{Q_n\}_1^\infty$. If $f(x_1,\dots,x_n) \in V_n$ then $\sigma f(x_1,\dots,x_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)})$, which implies that Q_n is a left ideal in $V_n \equiv FS_n$. It is almost never two-sided. Thus Q_n determines an S_n-representation so an S_n-character $\chi(Q_n)$, which can be determined by complements: FS_n is semi-simple, (char $F = 0$), so $FS_n = Q_n \oplus J_n$ for some (not necessarily unique) complementary left ideal J_n which determines a unique S_n-character $\chi(J_n)$, and $\chi(Q_n) = \chi(FS_n)-\chi(J_n)$.

<u>Definition.</u> Let $Q = I(A) \subseteq F\langle x\rangle$ be the identities of the algebra A, $Q_n = V_n \cap Q$ and $V_n \equiv FS_n = Q_n \oplus J_n$ as above, then $\chi(J_n)$ is the n-th cocharacter of A (or of Q), denoted by $\chi(J_n) = \chi_n(A)$. We call $\{\chi_n(A)\}_{n=1}^\infty$ the cocharacter sequence (c.c.s.) of A. Also $c_n(A) = \deg \chi_n(A) = \dim V_n/Q_n$ is the n-th codimension of A, and $\{c_n(A)\}_1^\infty$ is the sequence of codimensions (c.d.s.) of A.

These sequences are tools for obtaining information about the identities of a P.I. algebra. Although $\{Q_n\}_1^\infty$ determines Q, its computation has so far been next to impossible. Since characters are much easier to handle than their representations, $\{\chi_n(A)\}_1^\infty$ does look like the right invariant to begin with. Several examples will be discussed later.

§3. Codimensions

The sequence of codimensions $\{c_n(A)\}$ is a significant invariant of A, which is also useful in determining $\{\chi_n(A)\}$. Codimensions were introduced to show that if A and B are P.I. then so is $A \otimes B$, [15]. The main tool there is the exponential estimate $c_n(A) \leqslant \alpha^n$ [15, Th.4.7], the proof of which was considerably simplified by Latyshev (see [16]). We describe now his proof, which is further simplified by the Robinson-Schensted correspondence.

Definition. $\sigma \in S_n$ is "d-bad" $(d \leqslant n)$ if there exist $1 \leqslant i_1 < \dots < i_d \leqslant n$ such that $\sigma(i_1) > \dots > \sigma(i_d)$. Otherwise σ is "d-good".

Lemma. If A satisfies an identity of degree d, then V_n is spanned, modulo Q_n, by the d-good permutations.

Latyshev then bounds the number of d-good permutations by a direct combinatorial argument, to conclude that

$c_n(A) \leqslant$ number of d good permutations $\leqslant (d-1)^{2n}$.

A detailed proof of the above appears in [16, §1].

Using the Robinson-Schensted correspondence we can actually count that number $g_d(n)$ of d-good permutations.

Let $\lambda = (a_1,\dots,a_r) \in \mathrm{Par}(n)$ be a partition of n : $a_1 + \dots + a_r = n$, $a_1 \geq \dots \geq a_r \underset{\neq}{>} 0$. Clearly, $h(\lambda) = r$ is the height of the Young diagram D_λ. Let χ_λ be the corresponding S_n-irreducible character, d_λ its degree (given by the hook formula) and $I_\lambda \subset FS_n$ the corresponding minimal two-sided ideal: $\dim I_\lambda = d_\lambda^2$. Let U be an ℓ-dimensional vector space. Construct $U^{\otimes n} \overset{\mathrm{def}}{=} \underbrace{U \otimes \dots \otimes U}_{n}$, map $\varphi\colon S_n \to \mathrm{End}(U^{\otimes n})$ by $\varphi(\sigma) = \hat{\sigma}$, $\hat{\sigma}(u_1 \otimes \dots \otimes u_n) = u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(n)}$ and extend φ from FS_n onto the algebra $B(\ell,n)$ spanned by the $n!$ elements $\hat{\sigma}$: $\varphi\colon FS_n \to B(\ell,n) \subseteq \mathrm{End}(U^{\otimes n})$. A basic result in this construction of Schur is

Theorem 2. (H. Weyl): $B(\ell,n) \cong \sum\limits_{\substack{\lambda\in \mathrm{Par}(n)\\ h(\lambda)\leq \ell}} \oplus\, I_\lambda$.

Corollary. $\dim B(\ell,n) = \sum\limits_{\substack{\lambda\in \mathrm{Par}(n)\\ h(\lambda)\leq \ell}} d_\lambda^2 \overset{\mathrm{def}}{=} S_\ell^{(2)}(n)$, and since $B(\ell,n) \subseteq \mathrm{End}(U^{\otimes n})$, $S_\ell^{(2)}(n) \leq \ell^{2n}$.

The Robinson-Schensted correspondence, [9], maps each $\sigma \in S_n$ to a pair (P,Q) of standard Young tableaux having the same shape. It is one-to-one, onto, and has the following property (among others): the height $h(P) = h(Q) = d$ is the length of a maximal chain $1 \leq i_1 < \dots < i_d \leq n$ such that $\sigma(i_1) > \dots > \sigma(i_d)$. In other words, $\sigma \leftrightarrow (P,Q)$ is "d-good" if and only if $h(P) \leq d-1$. It follows that $S_{d-1}^{(2)}(n)$ is the number of d-good permutations in S_n. By the above and by Latyshev's lemma we thus have

Theorem 3. If A satisfies an identity of degree d then $c_n(A) \leq$ No. of d good permutations $= S_{d-1}^{(2)}(n) \leq (d-1)^{2n}$, which proves the exponential bound for codimensions.

Kemer [7] characterized the algebras A such that $c_n(A) = O(n^k)$ as those satisfying some very specific identities. This indicates that codimensions almost always have exponential rate of growth.

In the very few examples that have been done so far (to be discussed later), $c_n(A)$ is exponentially smaller than $(d-1)^{2n}$. However, unless the estimate "$c_n(A) \leq$ No. of d-good permutations" is improved, one cannot significantly lower the bound $c_n(A) \leq (d-1)^{2n}$: it is shown, [21], that as $n \to \infty$,

$$S_\ell^{(2)}(n) \underset{n\to\infty}{\simeq} c\cdot[\tfrac{1}{n}]^e\cdot\ell^{2n}$$

where c is some (interesting) constant and $e = \frac{1}{2}(\ell^2-1)$.

We note also that the constant appearing in the asymptotic formula for

$$S_\ell^{(\beta)}(n) \overset{\text{def}}{=} \sum_{\substack{\lambda\in\text{Par}(n)\\ h(\lambda)\leq\ell}} d_\lambda^\beta, \quad n \to \infty,$$

relates the theory of Young tableaux to a very interesting conjecture of I.G. Macdonald on the invariants of finite reflection groups (see [12], [21]).

§4. Cocharacters

Any S_n character χ_n can be written as $\chi_n = \sum_{\lambda\in\text{Par}(n)} m_\lambda\chi_\lambda$ where m_λ is the multiplicity of the irreducible character χ_λ (since char $F = 0$). This in particular applies to the cocharacter $\chi_n(A)$, and we are looking for information about its m_λ 's.

Example. [10], [14]: The infinite dimensional Grassmann algebra E satisfies $[[x_1,x_2],x_3] = 0$ $([a,b] = ab-ba)$. The polynomial $f(x) \in V_d$ is of type J_d if

$$f(x_1,\dots,x_d) = x_1 \dots x_d + \sum_{\sigma(1)\neq 1} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(d)} .$$

By [10], if A satisfies a J_d-identity, then $c_n(A) \leq (d-1)^n$, so $c_n(E) \leq 2^{n-1}$. The reverse inequality follows from studying the cocharacters $\chi_n(E)$. The Partition $\lambda = (k,1^{n-k}) \in \mathrm{Par}(n)$ defines a Γ-shaped Young diagram. For such λ, it is easy to choose a polynomial in I_λ which is not an E-identity. Hence $\chi_n(E) = \sum_{\lambda\in \mathrm{Par}(n)} m_\lambda \chi_\lambda$ and for each $\lambda = (k,1^{n-k})$, $1 \leq k \leq m$, $m_\lambda \geq 1$. Thus

$$c_n(E) = \deg \chi_n(E) \geq \sum_{k=1}^{n} d_{(k,1^{n-k})} = \sum_{k=1}^{n} \binom{n-1}{k} = 2^{n-1} .$$

It clearly follows that $c_n(E) = 2^{n-1}$ and that $\chi_n(E) = \sum_{k=1}^{n} \chi_{(k,1^{n-k})}$.

The only other cocharacters which have been determined are those of $Q = T(s_3[x_1,x_2,x_3])$, the T ideal generated by $s_3[x]$, [6], [17]. Because of the importance of F_k, the main goal in this direction should be to estimate $c_n(F_k)$ and the multiplicities in $\chi_n(F_k)$. So far only partial information had been obtained when $k \geq 3$. The results for $\chi_n(F_2)$ are quite satisfactory and appear in [19]. We here summarize the main results:

<u>Theorem 4.</u> A satisfies the Capelli polynomial $d_{\ell+1}[x;y]$ if and only if for all n,

$$\chi_n(A) = \sum_{\substack{\lambda\in \mathrm{Par}(n)\\ h(\lambda)\leq \ell}} m_\lambda \chi_\lambda .$$

<u>Corollary.</u> $\chi_n(F_k) = \sum_{\lambda\in \mathrm{Par}(n),\, h(\lambda)\leq k^2} m_\lambda \chi_\lambda$. In particular,

$$\chi_n(F_2) = \sum_{\lambda\in \mathrm{Par}(n),\, h(\lambda)\leq 4} m_\lambda \chi_\lambda .$$

Let now $\lambda \in \mathrm{Par}(n)$ with $h(\lambda) \leq 4$ and write $\lambda = (\omega_1+\omega_2+\omega_3+\omega_4, \omega_2+\omega_3+\omega_4, \omega_3+\omega_4, \omega_4)$. Let m_λ be its multiplicity in $\chi_n(F_2)$. It is shown that m_λ is very close to $\omega_1 \cdot \omega_2 \cdot \omega_3$. It then follows [19, Cor. 5.5] that $c_n(F_2)$ is asymptotically $(n \to \infty)$ sandwiched as follows:

$$\frac{\sqrt{2}}{\pi}\left[\frac{4}{\sqrt{\pi}}\ \frac{1}{n\sqrt{n}}\ 4^n\right] \lesssim c_n(F_2) \lesssim \frac{4}{\sqrt{\pi}} \cdot \frac{1}{n\sqrt{n}} \cdot 4^n$$

There are indications, [19, Rem. 5.6] that the general results for $\chi_n(F_k)$ and $c_n(F_k)$ are similar.

§5. Applications of Codimensions and Cocharacters

These sequences can be defined for any algebra A , and A is P.I. iff $c_n(A) \lneq n!$ for some n . The theorem that $A \otimes B$ is P.I. if A and B are, clearly follows from the exponential bounds $(c_n(A) \leq \alpha^n, \ldots)$ and from the inequality $c_n(A \otimes B) \leq c_n(A) \cdot c_n(B)$, because $n!$ exceeds any $(\alpha \cdot \beta)^n$.

This codimension's inequality has an interesting cocharacter interpretation: Define $\sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi_\lambda \leq \sum_{\lambda \in \mathrm{Par}(n)} m'_\lambda \chi_\lambda$ if $m_\lambda \leq m'_\lambda$ for all λ . Given two S_n-characters χ_n, ψ_n, let $\chi_n \otimes \psi_n$ denote their Kronecker (inner) product. In [20] we proved

<u>Theorem 5.</u> $\chi_n(A \otimes B) \leq \chi_n(A) \otimes \chi_n(B)$ ($A \otimes B$ is the usual tensor product of the algebras A and B .) Apply "degree" to both sides to deduce the previous codimensions inequality. Theorem 5 strongly relates P.I. algebras to Kronecker products of S_n-character. In what follows we bring applications in both directions.

Let $\chi_n = \sum_{\lambda \in Par(n)} m_\lambda \chi_\lambda$ and define $h(\chi_n) = \max\{h(\lambda) \mid m_\lambda \neq 0 \text{ in } \chi_n\}$. By applying (Weyl's) Theorem 2 one can prove (see [20])

Theorem 6. $h(\chi_n \otimes \psi_n) \leq h(\chi_n)\cdot h(\psi_n)$ for any two S_n-characters χ_n, ψ_n .

We first apply S_n-characters to P.I. theory:

Theorem 7. If A satisfies $d_{k+1}[x;y]$ and B satisfies $d_{\ell+1}[x;y]$ then $A \otimes B$ satisfies $d_{k\ell+1}[x;y]$.

Proof. By Theorem 4, $h(\chi_n(A)) \leq k$, $h(\chi_n(B)) \leq \ell$, hence $h(\chi_n(A \otimes B)) \underset{5}{\leq} h(\chi_n(A) \otimes \chi_n(B)) \underset{6}{\leq} h(\chi_n(A)\cdot h(\chi_n(B)) \leq k\cdot\ell$, so by 4, $A \otimes B$ satisfies $d_{k\ell+1}[x;y]$.

By Kemer's theorem (§1), existence of standard and Capelli identities is equivalent, so we obtain another proof of a result of E. Berman [4].

Corollary. (Berman): If A and B satisfies standard identities, then so does $A \otimes B$.

Examine now the inequality $h(\chi \otimes \psi) \leq h(\chi)h(\psi)$ of Theorem 6. Tables in [13] show that in many cases, $h(\chi \otimes \psi) \underset{\neq}{<} h(\chi)h(\psi)$. The proper question should be

Question H: Given two heights h_1, h_2 is there $N = N(h_1,h_2)$ such that for any $n \geq N$ there are two S_n-characters χ_n , ψ_n satisfying:

$h(\chi_n) = h_1$, $h(\psi_n) = h_2$ and $h(\chi_n \otimes \psi_n) = h_1h_2 = h(\chi_n)\cdot h(\psi_n)$?!

Note that for outer products $\chi \otimes' \psi$ we do have $h(\chi \otimes' \psi) = h(\chi)+h(\psi)$ as a consequence of the Littlewood-Richardson rule. Missing yet a rule for inner products, this we think, makes H rather intriguing. We do conjecture "yes" to H ! At the moment we can prove it when $h_1 = k^2$, $h_2 = \ell^2$ are squares, by applying P.I. theory! We sketch the proof: First, if $n \geqslant 2k^2-1$ then $h(\chi_n(F_k))=k^2$. This follows from the fact that F_k satisfies d_{k+1} , but not d_{k^2} , by an argument of Amitsur's which is also applied to prove (half of) Theorem 4. Given $h_1 = k^2$, $h_2 = \ell^2$, let $N = 2k^2\ell^2-1$, so $h(\chi_n(F_{k\ell})) = k^2\ell^2$ if $n \geqslant N$: there exists $\lambda \in \mathrm{Par}(n)$, $h(\lambda) = k^2\ell^2$ and χ_λ has a non-zero multiplicity in $\chi_n(F_{k\ell})$. Since $\chi_n(F_{k\ell}) = \chi_n(F_k \otimes F_\ell) \underset{5}{\leqslant}$ $\chi_n(F_k) \otimes \chi_n(F_\ell)$, there must be χ_{λ_1} in $\chi_n(F_k)$, χ_{λ_2} in $\chi_n(F_\ell)$, both with non-zero multiplicity, such that χ_λ appears in $\chi_{\lambda_1} \otimes \chi_{\lambda_2}$. But $h(\lambda_1) \leqslant k^2$, $h(\lambda_2) \leqslant \ell^2$ and $h(\lambda) = k^2\ell^2$, so necessarily $h(\lambda_1) = k^2$ and $h(\lambda_2) = \ell^2$, as was to be shown.

§6. Explicit Identities

We begin with the following "Structure" argument: For each ℓ assume $f_\ell(x_1,\ldots,x_{n(\ell)})$ is an identity for F_ℓ . Let A be an arbitrary P.I. algebra and mode out its nil radical N . Since A/N is semi-simple, there exists ℓ such that it satisfies all the identities of F_ℓ , hence in particular $f_\ell(x_1,\ldots,x_{n(\ell)})$. By using "generic elements" (or other methods) one can lift $f_\ell(x)$ back to A : there exists a power k such that A satisfies $(f_\ell(x_1,\ldots,x_{n(\ell)}))^k$. For example, assume A satisfies an identity of degree d ; Choose $f_\ell(x) = s_{2\ell}[x_1,\ldots,x_{2\ell}]$ to conclude that A satisfies $(s_{2\ell}[x])^k$, $\ell \leqslant [\frac{d}{2}]$; Choose $f_\ell(x) = d_{\ell^2+1}[x;y]$ to conclude that A satisfies $(d_{\ell^2+1}[x;y])^k$, $\ell \leqslant d$.

This method, due to Amitsur [1], usually yeilds a bound for ℓ but not for k . We now apply Young tableaux, to re-prove these results with explicit bounds for both indices k and ℓ .

It is well known that the minimal two-sided ideal $I_\lambda \subset FS_n$ $(\lambda \in \mathrm{Par}(n))$ is a direct sum of (d_λ) minimal left ideals J_λ and $(\dim J_\lambda)^2 = d_\lambda^2 = \dim I_\lambda$. As before, $Q = I(A)$, and our basic tool is

Lemma 8. If $d_\lambda \underset{\neq}{>} c_n(A)$ then $Q_n \supseteq I_\lambda$.

Proof. $I_\lambda = \oplus J_\lambda$, J_λ minimal left ideals. If some $J_\lambda \not\subseteq Q_n$ then $Q_n \cap J_\lambda = 0$, so $c_n(A) = \dim V_n/Q_n \geq \dim J_\lambda = d_\lambda$, a contradiction.

Q.E.D.

As in Theorem 3, let $c_n(A) \leq (d-1)^{2n}$. We are therefore looking for $n = n(d)$ and $\lambda \in \mathrm{Par}(n)$ such that $d_\lambda > (d-1)^{2n}$; all the elements of I_λ are then A-identities. If that λ is "rectangular" $\lambda = (k^\ell)$, we deduce from §2, Expl. 3 that A satisfies $s_\ell^k[x]$.

Such λ is found in [16] by analytic methods. Amitsur [3], gave a very short and simple method for finding such λ, which we now describe.

By the hook formula, $d_\lambda = n!/\prod_{ij} h_{ij}$ where $\{h_{ij}\}$ are the n hook numbers. Replace $d_\lambda > (d-1)^{2n}$ by the equivalent inequality $(n!/\pi\, h_{ij})^{1/n} > (d-1)^2$. It is well known that $(n!)^{1/n} > n/e$, and since the geometric mean is smaller than the arithmetic mean, $(\pi\, h_{ij})^{1/n} \leq \frac{1}{n}\sum h_{ij}$. It is therefore enough to find $\lambda \in \mathrm{Par}(n)$ such that $n^2/\sum h_{ij} \geq e(d-1)^2$.

Let λ be "rectangular": $n = k\cdot\ell$, $\lambda = (k\ell)$. The hook numbers in D_λ are

$k+\ell-1$	$\cdots$	$\ell+1$	ℓ
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k+1$	$\cdots$	3	2
k	$\cdots$	2	1

so $\sum h_{ij} = k\ell \frac{k+\ell}{2} = n \frac{k+\ell}{2}$. If $\frac{k\ell}{k+\ell} > \frac{e}{2}(d-1)^2$ we then conclude that all the polynomials in I_λ , $\lambda = (k^\ell)$, are identities, hence also $s_\ell^k[x]$ (Expl. 3). There are many such k and ℓ : Denote $\alpha = \frac{e}{2}(d-1)^2$. Since $k,\ell > \frac{k\ell}{k+\ell}$, one must have $k,\ell > \alpha$, so choose $\ell > \alpha$. If $k > \frac{\alpha\ell}{\ell-\alpha}$ then clearly $\frac{k\ell}{k+\ell} > \alpha$.

<u>We summarize:</u> Assume A satisfies an identity of degree d , and let $\alpha = \frac{e}{2}(d-1)^2$. If $\ell > \alpha$ and $k > \frac{\alpha\ell}{\ell-\alpha}$ then A satisfies the identity $(s_\ell[x_1,\ldots,x_\ell])^k$. Notice the gap: "Structure" yields the same result, with $\ell \leqslant d$ (but no bound for k).

Using the codimensions inequality $c_n(A \otimes B) \leqslant c_n(A)\cdot c_n(B)$ one can construct similar such identities for $A \otimes B$ by replacing $\frac{e}{2}(d-1)^2$ by $\frac{e}{2}(d_1-1)^2(d_2-1)^2$.

Next, consider the analog of Amitsur's $s_\ell^k[x]$ theorem, but with Capelli instead of standard polynomials. The following construction is essentially due to Amitsur (unpublished) and is based on the branching theorem.

Construct the Capelli polynomial $d_\ell[x;y]$ in two steps: first write $s_\ell[x_1,\ldots,x_\ell]x_{\ell+1} \ldots x_{2\ell-1}$ and denote $x_{\ell+1} = y_1,\ldots,x_{2\ell-1} = y_{\ell-1}$. Now, there exists $\sigma \in S_{2\ell-1}$ such that $(s_\ell[x_1,\ldots,x_\ell]x_{\ell+1} \ldots x_{2\ell-1})\sigma = d_\ell[x;y]$, [3], [16]. The construction of a product of Capelli polynomials is done similarly.

Let $\lambda \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(n+k)$ such that D_μ extends D_λ , i.e.: D_μ is obtained from D_λ by adding k boxes. A trivial consequence of the branching theorem implies that $d_\mu \geqslant d_\lambda$.

Start with a P.I. algebra A , $Q = I(A)$, $c_n(A) \leqslant (d-1)^{2n}$. Next, find $n = k\ell$ such that $(\lambda = (k^\ell))$, $d_\lambda > (d-1)^{4n}$. Let $\mu \in \mathrm{Par}(2n)$ such that D_μ extends D_λ , then $d_\mu \geqslant d_\lambda > (d-1)^{4n} \geqslant c_{2n}(A)$, hence $I_\mu \subseteq Q_{2n}$. Since this is true for any such μ , the branching theorem implies that $FS_{2n} \cdot I_\lambda \cdot FS_{2n} \subseteq Q_{2n}$. Choose

$$T_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & \ell+1 & \cdots & \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \ell & 2\ell & \cdots & k\ell \\ \hline \end{array} \qquad (k = n)$$

so, in FS_{2n} ,

$$e_T(x_1,\dots,x_{2n}) = \sum_{p\in P} p\, s_\ell[x_1,\dots,x_\ell]s_\ell[x_{\ell+1},\dots,x_{2\ell}] \cdots s_\ell[\dots,x_{k\ell}]x_{n+1}\cdots x_{2n} .$$

Denote $x_{n+1} = y_1,\dots,x_{2n} = y_n$. There exists $\sigma \in S_{2n}$ such that

$$(s_\ell[x_1,\dots,x_\ell] \cdots s_\ell[\dots,x_{k\ell}]\cdot y_1 \cdots y_n)\sigma =$$

$$= d_\ell[x_1,\dots,x_\ell;y_1,\dots,y_{\ell-1}]y_\ell \cdot d_\ell[x_1,\dots,x_\ell;y_{\ell+1},\dots,y_{2\ell-1}]\cdot y_{2\ell} \cdots$$

Now equate $x_1 = x_{\ell+1} = x_{2\ell+1} = \cdots$, $x_2 = x_{\ell+2} = x_{2\ell+2} = \cdots$, $\dots$

$y_1 = y_{\ell+1} = y_{2\ell+1} = \cdots$, $y_2 = y_{\ell+2} = y_{2\ell+2} = \cdots$, $\dots$

and $y_\ell = y_{2\ell} = \cdots = y_{k\ell} = 1$

to conclude that A satisfies $(d_\ell(x;y))^k$ (it is easy to deduce stronger results from these same arguments. See [3]). Recall that $\lambda = (k^\ell) \in Par(n)$ should only satisfy $d_\lambda > (d-1)^{4n}$. As before, let $\beta = \frac{e}{2}\cdot(d-1)^4$ (instead of $\alpha = \frac{e}{2}(d-1)^2)$, then choose $\ell > \beta$ and $k > \frac{\beta\ell}{\ell-\beta}$ to obtain explicit ℓ and k .

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