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REPRESENTATIONS OF WEYL GROUPS OVER AN ARBITRARY FIELD

A.O. Morris

Introduction

The representation theory of symmetric groups over fields of characteristic zero is well developed and documented with a number of books devoted to the subject. The original approach was due to G. Frobenius and I. Schur followed independently by A. Young in a long series of difficult but highly influential papers. Later, in the 1930's, W. Specht presented an alternative approach which led in an elegant way to a full set of irreducible modules, now called Specht modules. This approach has proved to be useful not only in developing a characteristic-free approach to the subject but also because of its suitability for generalization to the construction of irreducible modules for arbitrary Weyl groups. The position in the case of characteristic p, although a great deal of important work has been done by several authors, Brauer, Nesbitt, Thrall, Littlewood, Robinson, Kerber, Peel, James and others, is far less developed.

However, in 1976, G.D. James in a very important paper [8], gave an easy and ingenious construction of all the irreducible modules of the symmetric groups over an arbitrary field which reduce to Specht modules in the case of fields of characteristic zero. The ultimate aim of this lecture is to give a possible generalisation of the results to Weyl groups.

Before proceeding to describe this work, a word about the position on the ordinary representation theory of Weyl groups may be useful. The irreducible characters of all the individual Weyl groups have been known for some time. What we are concerned with is to provide a unified approach to the representation theory. S.D. Mayer in his 1971 Warwick Ph.D. thesis [13] has presented such an attempt for the irreducible characters of Weyl groups, his work appearing in a series of papers [14,15,16]. In 1972, I.G. MacDonald [12] in the now surprisingly obvious way, generalised the Specht approach to give irreducible representations of Weyl groups, but however, without giving all of the irreducible representations in general. Since then T.A. Springer [20] has given a complete construction of all the irreducible representations of Weyl groups, the representation spaces being the ℓ-adic cohomology groups of a Borel variety of a connected reductive algebraic group. He later [21] simplified this work so that the representation spaces of the earlier work go through also over the complex field and lead to representations over the rational field. Reference should also be made to papers by T.A. Springer and R. Hotta [22], T. Shoji [9] and G. Lusztig [11], the latter paper in particular generalising I.G. MacDonald's work.

In this lecture, however, we shall attempt to obtain the irreducible representations in the more, at least to the author, simple minded and combinatorial language of the classical theory. The structure of the lecture will be as follows. The first section will give a description of G.D. James' construction mentioned earlier. Our approach will follow closely that due to James [8], his method being far more suggestive for a possible generalisation to arbitrary Weyl groups. The next section will present a generalisation of James' work to the hyperoctahedral groups, that is, Weyl groups of type B_n due to the author, Al-Aamily in his Ph.D. thesis [1] and their joint work with M.H. Peel [2]. In a final section, progress on a possible generalisation of these ideas to arbitrary Weyl groups is considered. The familiar concepts of Young diagrams, Young tableaux, standard tableaux etc. which are so crucial in the development of the representation theory of the symmetric groups, are seen to have equally familiar counterparts in the context of roots systems and Weyl groups. It should be emphasised that this section had not reached its final form with some problems still to be overcome.

1. The irreducible representations of the symmetric groups (Weyl groups of type A_n)

As noted earlier, the approach given here is the crucial one given by James [8,9,10]. Alternative approaches, one more reminiscent of that used by Specht is given in Peel [17,18] and another in Farahat and Peel [7] develops a more general setting in which the construction works.

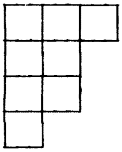
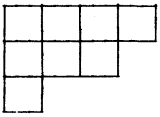
Let K be an arbitrary field and S_n be the symmetric group on n letters. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a *partition* of n, that is, $n = \sum_{i=1}^{\infty} \lambda_i$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$,

1.1 Definition The *Young diagram* $[\lambda]$ associated to λ consists of n squares arranged in consecutive rows so that the first row has λ_1 squares, the second has λ_2 squares and so on. The rows are counted from top to bottom and arranged so that they all start from the same left extremity. If $[\lambda]$ is a Young diagram, the *dual* Young diagram $[\lambda']$ is obtained by interchanging the rows and columns in $[\lambda]$, λ' is the partition conjugate to λ. If $[\lambda]$ and $[\mu]$ are Young diagrams associated to partitions λ and μ of n, we say that $[\lambda]$ *dominates* $[\mu]$ (written $[\lambda] \trianglerighteq [\mu]$) provided that

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for $j=1,2,\ldots$. This is a partial order on Young diagrams.

A λ-*Young tableau* or λ-*tableau* is one of the n! arrays of integers obtained by replacing each square in $[\lambda]$ by one of the integers $1,2,\ldots,n$, with no repetitions. We say that a λ-tableau is a *standard* λ-tableau if the numbers increase from left to right across along each row and from top to bottom down each column. For example, if $\lambda = (3,2^2,1)$,

$[\lambda] =$  , $[\lambda'] =$  $= [4,3,1]$,

$\begin{array}{lll} 5 & 3 & 8 \\ 1 & 2 & \\ 4 & 6 & \\ 7 & & \end{array}$ and $\begin{array}{lll} 1 & 3 & 7 \\ 2 & 5 & \\ 4 & 8 & \\ 6 & & \end{array}$ are $(3,2^2,1)$-tableaux, the second of which is a standard $(3,2^2,1)$-tableau.

If t is a tableau and $\sigma \in S_n$, define σt in the obvious way; for example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 3 & 4 & 2 & 7 & 8 & 6 \end{pmatrix} = (1\ 5\ 2)(6\ 7\ 8)$, then $\sigma \begin{pmatrix} 1 & 3 & 7 \\ 2 & 5 & \\ 4 & 8 & \\ 6 & & \end{pmatrix} = \begin{pmatrix} 5 & 3 & 8 \\ 1 & 2 & \\ 4 & 6 & \\ 7 & & \end{pmatrix}$. Let R_t be the subgroup of S_n fixing the rows of t and C_t be the subgroup of S_n fixing the columns of t, i.e. R_t and C_t are the *row-* and *column-stabilizer* of t respectively. Note that $R_{\sigma t} = \sigma R_t \sigma^{-1}$ and $C_{\sigma t} = \sigma C_t \sigma^{-1}$.

1.2 Definition An equivalence relation $\sim$ is defined on the set of λ-tableaux by $t_1 \sim t_2$ if there is $\sigma \in R_{t_1}$ such that $t_2 = \sigma t_1$. The equivalence class containing t will be called the *tabloid* $\{t\}$. Thus, a tabloid may be regarded as a "tableau with unordered row entries", for example, $\begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \end{array} \sim \begin{array}{lll} 2 & 3 & 1 \\ 4 & 5 & \end{array}$ and a full set of (3 2)-tabloids is given by

$$\begin{Bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{Bmatrix} \begin{Bmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{Bmatrix} \begin{Bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 5 \\ 3 & 4 & \end{Bmatrix} \begin{Bmatrix} 1 & 4 & 5 \\ 2 & 3 & \end{Bmatrix} \begin{Bmatrix} 2 & 3 & 4 \\ 1 & 5 & \end{Bmatrix} \begin{Bmatrix} 2 & 3 & 5 \\ 1 & 4 & \end{Bmatrix} \begin{Bmatrix} 2 & 4 & 5 \\ 1 & 3 & \end{Bmatrix} \begin{Bmatrix} 3 & 4 & 5 \\ 1 & 2 & \end{Bmatrix}.$$

S_n acts on the set of λ-tabloids by $\sigma\{t\} = \{\sigma t\}$; this action is easily shown to be well-defined.

1.3 Definition If $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_m)$ is a partition of n, let M^λ be the vector space over K spanned by λ-tabloids. By extending the above action linearly to KS_n we have that M^λ is a KS_n-module. It is easily verified that

1.4 Theorem M^λ is the permutation module of S_n on the Young subgroup $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_m}$. M^λ is a cyclic KS_n-module generated by any one λ-**tabloid** and $\dim_K M^\lambda = \dfrac{n!}{\lambda_1!\lambda_2!\ldots\lambda_m!}$.

1.5 Definition If t is a λ-tableau, define $\kappa_t \in KS_n$ by

$$\kappa_t = \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\sigma.$$

The λ-*polytabloid* e_t associated with the tableau t is given by

$$e_t = \kappa_t\{t\}.$$

We shall say that e_t *contains* the tabloid $\{t^*\}$ if it appears in e_t with non-zero coefficient.

The *Specht module* S^λ for the partition λ is the submodule of M^λ spanned by λ-polytabloids. Since $\sigma\kappa_t = \kappa_{\sigma t}\sigma$, we have $\sigma e_t = e_{\sigma t}$ and it follows that

1.6 Theorem S^λ is a cyclic KS_n-module generated by any one polytabloid.

The following combinatorial results are crucial in the representation theory of the symmetric group.

1.7 Lemma Let t be a λ-tableau and let t' be a μ-tableau. Suppose that a,b belong to the same row of t implies that a,b belong to different columns of t'. Then $\mu \trianglerighteq \lambda$.

1.8 Lemma Let t and t' be λ-tableaux. The following conditions are equivalent:

(i) $\{t'\}$ is contained in e_t,

(ii) there exist $\rho \in R_{t'}$, $\gamma \in C_t$ such that $\rho t' = \gamma t$,

(iii) a,b belong to the same row of t' implies a,b belong to different columns of t.

Proof By definition, $\{t'\}$ is contained in e_t if and only if $\{t'\} = \gamma\{t\}$ for some $\gamma \in C_t$; hence if and only if $\rho t' = \gamma t$ for some $\rho \in R_{t'}$, $\gamma \in C_t$, proving the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is proved by the usual combinatorial argument.

1.9 Lemma Let t' be a λ-tableau and t a μ-tableau. If there exist a,b in the same row of t' and in the same column of t then $\kappa_t\{t'\} = 0$.

Proof If t' is a λ-tableau and a,b are in the same column of t (and hence in the same row of t')

$$(1-(ab))\{t'\} = 0.$$

Select signed coset representatives $\sigma_1,\ldots,\sigma_k$ for the subgroup of C_t consisting of 1 and (ab); then $\kappa_t\{t'\} = (\sigma_1+\ldots+\sigma_k)(1-(ab))\{t'\} = 0$.

We can now prove

1.10 Lemma Let t and t' be λ-tableaux.

(i) If $\{t'\}$ is not contained in e_t then $\kappa_t\{t'\} = 0$.

(ii) If $\{t'\}$ is contained in e_t then $\kappa_t\{t'\} = \pm e_t$.

Proof (i) follows immediately from 1.8 and 1.9.

To prove (ii), if $\{t'\}$ is contained in e_t, then by 1.8, there exists $\rho \in R_{t'}$, $\gamma \in C_t$ such that $\rho t' = \gamma t$ and hence

$$\kappa_t\{t'\} = \kappa_t\{\rho t'\} = \kappa_t\gamma\{t\} = \pm\kappa_t\{t\} = \pm e_t.$$

Similarly, we can also prove

1.11 Lemma If $\mu \not\trianglerighteq \lambda$, then for any λ-tableau t' and any μ-tableau t

$$\kappa_t\{t'\} = 0.$$

A bilinear form is defined on M^λ by $\left\langle \{t\},\{t'\} \right\rangle = 1$ if $\{t\} = \{t'\}$
$= 0$ if $\{t\} \neq \{t'\}$.

This bilinear form is a symmetric, non-singular, S_n-invariant bilinear form on M^λ.

1.12 Theorem Let U be a submodule of M^λ. Then either $U \supseteq S^\lambda$ or $U \subseteq S^{\lambda\perp}$.

Proof Suppose $u \in U$ and t is a λ-tableau. Then

$$\langle u,e_t\rangle = \langle u,\kappa_t\{t\}\rangle = \sum_{\sigma\in C_t} \langle u,(\operatorname{sgn}\sigma)\sigma\{t\}\rangle = \sum_{\sigma\in C_t} \langle (\operatorname{sgn}\sigma)\sigma^{-1},\{t\}\rangle$$

$= \sum_{\sigma\in C_t} \langle (\operatorname{sgn}\sigma)\sigma u,\{t\}\rangle = \langle \kappa_t u,\{t\}\rangle$. But by 1.10, $\kappa_t u$ is a multiple of e_t, say $\kappa_t u = \beta e_t$. If $\beta \neq 0$ for some $u \in U$, then $e_t \in U$ and so $S^\lambda \subseteq U$. However, if $\beta = 0$ for all $u \in U$, then $\langle u,e_t\rangle = 0$ for all $u \in U$, that is $U \subseteq S^{\lambda\perp}$.

From this we have immediately

1.13 Theorem $S^\lambda/S^\lambda \cap S^{\lambda\perp}$ is zero or absolutely irreducible.

Proof By the above theorem, any submodule of S^λ is either S^λ or is contained in $S^\lambda \cap S^{\lambda\perp}$. Therefore $S^\lambda/S^\lambda \cap S^{\lambda\perp}$ is irreducible or zero. The absolute irreducibility can be deduced by considering the rank of the Gram matrix (see [9]).

1.14 Corollary When $K = Q$ the S^λ (λ is a partition of n) give a complete set of irreducible KS_n-modules.

We now obtain a necessary and sufficient condition for $S^{\lambda\perp}$ to be contained in S^λ. To do this, we first require a definition.

1.15 Definition A partition λ is said to be *p-regular* if no positive integer is repeated p or more times in λ, otherwise λ is said to be *p-singular*.

1.16 Lemma If the partition λ has μ_j parts equal to j, then

(i) for every pair of λ-polytabloids e_t and $e_{t'}$, $\prod_{j=1}^{\infty} \mu_j!$ divides $\langle e_t,e_{t'}\rangle$;

(ii) for every λ-polytabloid e_t there is a λ-polytabloid $e_{t'}$ such that $\langle e_t,e_{t'}\rangle = \prod_{j=1}^{\infty} (\mu_j!)^j$.

Proof For the details see James [9], but note that (i) is proved by defining an equivalence relation $\sim$ on the set of λ-tabloids by $\{t_1\} \sim \{t_2\}$ if and only if, for all i and j, i and j belong to the same row of $\{t_1\}$ when i and j belong to the same row of $\{t_2\}$; (ii) is proved by defining t' for any λ-tableau t by reversing the order of the numbers in each row of t.

<u>1.17</u> <u>Corollary</u> <u>If</u> S^λ <u>is defined over a field of characteristic</u> p <u>then</u> $S^\lambda/S^\lambda \cap S^{\lambda\perp}$ <u>is non-zero if and only if</u> λ <u>is</u> p<u>-regular.</u>

<u>Proof</u> If λ is p-regular, then $\langle e_t, e_{t'} \rangle \neq 0$ for the λ-polytabloids e_t and $e_{t'}$ by (ii) of the above lemma, thus $S^\lambda \not\subseteq S^{\lambda\perp}$. If λ is p-singular, $\langle e_t, e_{t'} \rangle = 0$ for every pair of μ-polytabloids by (i) in the above lemma, thus $S^\lambda \subseteq S^{\lambda\perp}$.

Now, if char K = p (prime on 0) and λ is p-regular, let $D^\lambda = S^\lambda/S^\lambda \cap S^{\lambda\perp}$, then James [9] proves

<u>1.18</u> <u>Theorem</u> <u>As</u> λ <u>varies over p-regular partitions of</u> n <u>the</u> D^λ <u>gives a complete set of inequivalent irreducible</u> KS_n<u>-modules.</u>

Furthermore, the dimension of each Specht module S^λ may be determined. We have

<u>1.19</u> <u>Theorem</u> $\{e_t \mid t$ is a standard λ-tableau$\}$ <u>is a K-basis for</u> S^λ. <u>The dimension of</u> S^λ <u>is the number of standard tableaux of shape</u> λ.

The proof of this theorem involves the determination of certain elements of KS_n, called Garnir elements, which annihilate a given polytabloid e_t, see Peel [18] or James [9] for details.

2. The irreducible representations of hyperoctahedral groups (Weyl groups of type B_n)

We now indicate how the construction described above can be adapted to cover hyperoctahedral groups. The details may be found in Al-Aamily [1] or Al-Aamily, Morris and Peel [2], although neither of these sources prove their results in the precise form required here.

The hyperoctahedral group O_n is the group of all permutations σ of $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for $i=1,\ldots,n$. Let (λ,μ) be a pair of partitions of n such that λ is a partition of $|\lambda|$ and μ a partition of $|\mu|$, and $|\lambda| + |\mu| = n$. Note that $|\lambda|$ or $|\mu|$ may be zero. A *double Young diagram* $[\ ,\ ]$ is defined in the obvious way. A partial order on double Young diagrams called the *dominance partial order* may be defined as follows: if $[\lambda,\mu]$ and $[\lambda',\mu']$ are double Young diagrams with $|\lambda| + |\mu| = |\lambda'| + |\mu'| = n$, then $[\lambda,\mu] \trianglerighteq [\lambda',\mu']$ if $|\lambda| > |\lambda'|$, or if $|\lambda| = |\lambda'|$, $|\mu| = |\mu'|$, $\lambda \trianglerighteq \lambda'$ and $\mu \trianglerighteq \mu'$.

A (λ,μ)-*tableau* is one of the $2^n.n!$ arrays of integers obtained by replacing each square in $[\lambda,\mu]$ by one of the integers $\pm 1, \pm 2, \ldots, \pm n$, with i and -i $(i=1,\ldots,n)$ not appearing simultaneously. A (λ,μ)-tableau t will be sometimes written (t_λ, t_μ). A (λ,μ)-tableau is called a *standard* (λ,μ)-tableau if all the integers are positive and t_λ and t_μ are both standard tableaux.

For example $(21,2^2)$ is a pair of partitions of 7, $\left\{\begin{matrix}2 & -1 \\ 7 & \end{matrix}\ ,\ \begin{matrix}-5 & 6 \\ 3 & 4\end{matrix}\right\}$ is a $(21,2^2)$-tableau, $\left\{\begin{matrix}1 & 2 \\ 7 & \end{matrix}\ ,\ \begin{matrix}3 & 5 \\ 4 & 6\end{matrix}\right\}$ is a standard $(21,2^2)$-tableau.

If t is a (λ,μ)-tableau and $\sigma \in O_n$, then σt may be defined in the obvious way, for example, if $\sigma = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -5 & 6 & -4 & 3 & 7 & -1 & 2\end{pmatrix}$ then $\sigma\left\{\begin{matrix}2 & -1 \\ 7 & \end{matrix}\ ,\ \begin{matrix}-5 & 6 \\ 3 & -4\end{matrix}\right\} = \left\{\begin{matrix}6 & 5 \\ 2 & \end{matrix}\ ,\ \begin{matrix}-7 & -1 \\ -4 & -3\end{matrix}\right\}$.

A row permutation of t is an element σ which permutes entries in each row of t and may change the sign of elements in t_μ. Let R_t be the group of row permutations of t, thus R_t is isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_\ell} \times O_{\mu_1} \times O_{\mu_2} \times \ldots \times O_{\mu_m}$.

A column permutation of t permutes the elements in each column of t and may change the sign of any entry in t_λ. Let C_t be the group of column permutations of t. It can be verified that if $\sigma \in O_n$, then $R_{\sigma t} = \sigma R_t \sigma^{-1}$ and $C_{\sigma t} = \sigma C_t \sigma^{-1}$.

2.1 Definition An equivalence relation $\sim$ is defined on the set of (λ,μ)-tableaux by $t_1 \sim t_2$ if there exists $\sigma \in R_{t_1}$ such that $t_2 = \sigma t_1$. The equivalence class containing t will be called the *tabloid* $\{t\}$. O_n acts on the set of (λ,μ)-tabloids by $\sigma\{t\} = \{\sigma t\}$ which again is easily shown to be well defined.

2.2 Definition If (λ,μ) is a pair of partitions of n, let $M^{\lambda,\mu}$ be the vector space over K spanned by (λ,μ)-tabloids. By extending the above action linearly to KO_n, we have that $M^{\lambda,\mu}$ is a KO_n-*module* and we have

2.3 Theorem $M^{\lambda,\mu}$ is the "permutation" module of O_n on the subgroup $S_{\lambda_1} \times \dots \times S_{\lambda_\ell} \times O_{\mu_1} \times \dots \times O_{\mu_m}$. $M^{\lambda,\mu}$ is a cyclic KO_n-module generated by any one (λ,μ)-tabloid and $\dim_K M^{\lambda,\mu} = 2^{n-|\lambda|} \dfrac{n!}{\lambda_1!\dots\lambda_\ell!\mu_1!\dots\mu_m!}$.

2.4 Definition If t is a (λ,μ)-tableau, define $\kappa_t \in KO_n$ by

$$\kappa_t = \sum_{\sigma\in C_t} (\mathrm{sgn}\ \sigma)\sigma.$$

The (λ,μ)-*polytabloid* e_t associated with the tableau t is given by

$$e_t = \kappa_t\{t\}.$$

The *Specht module* $S^{\lambda,\mu}$ for the pair of partitions (λ,μ) is the submodule of $M^{\lambda,\mu}$ spanned by (λ,μ)-polytabloids.

2.5 Theorem $S^{\lambda,\mu}$ is a cyclic KO_n-module generated by any one polytabloid.

The corresponding combinatorial result to 1.7 is

2.6 Lemma Let t be a (λ,μ)-tableau and t' be a (λ',μ')-tableau such that for all a, if a occurs in t_λ then $\pm a$ occurs in t'_λ . Suppose that a,b belong to the same row of t_γ implies that c,d belong to the same column of $t'_{\gamma'}$, where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ. Then $(\lambda',\mu') \trianglerighteq (\lambda,\mu)$.

2.7 Lemma Let t and t' be (λ,μ)-tableaux. The following conditions are equivalent:

(i) $\{t'\}$ is contained in e_t ,

(ii) there exist $\rho \in R_{t'}$, $\gamma \in C_t$ such that $\rho t' = \gamma t$,

(iii) a,b belong to the same row of t_γ implies that c,d belong to the same column of t'_γ , where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ.

2.8 Lemma Let t be a (λ,μ)-tableau and t' a (λ',μ')-tableau. If there exist a,b in the same row of $t'_{\gamma'}$ such that c,d are in the same column of t_γ , where $c = \pm a$, $d = \pm b$ and $\gamma = \lambda$ or μ then $\kappa_t\{t'\} = 0$.

By a similar method to that used in the case of the symmetric group we can now prove

2.9 Lemma Let t and t' be (λ,μ)-tableaux.

(i) If $\{t'\}$ is not contained in e_t then $\kappa_t\{t'\} = 0$.

(ii) If $\{t'\}$ is contained in e_t then $\kappa_t\{t'\} = \pm e_t$.

2.10 Lemma If $(\lambda',\mu') \not\trianglerighteq (\lambda,\mu)$ and $|\lambda'| \geq |\lambda|$ then for any (λ',μ')-tableau t' and any (λ,μ)-tableau t, $\kappa_t\{t'\} = 0$.

Having defined the obvious bilinear form $<\ ,\ >$ on $M^{\lambda,\mu}$, we can now prove with only a slight modification of 1.12

2.11 Theorem Let U be a submodule of $M^{\lambda,\mu}$. Then either $U \supseteq S^{\lambda,\mu}$ or $U \subseteq (S^{\lambda,\mu})^{\perp}$.

From this it follows immediately.

2.12 Theorem $S^{\lambda,\mu}/S^{\lambda,\mu} \cap (S^{\lambda,\mu})^{\perp}$ is zero or absolutely reducible.

2.13 Corollary When $K = \mathbb{Q}$, the $S^{\lambda,\mu}$ ((λ,μ) is a pair of partitions of n) give a complete set of irreducible KO_n-modules.

We now adapt the definition of p-regular to cover this case.

2.14 Definition A pair of partitions (λ,μ) is said to be *p-regular* for $p \neq 2$ when both λ and μ are p-regular. The pair (λ,μ) is said to be *2-regular* when $|\lambda| = 0$ and μ is 2-regular.

The corresponding result to Lemma 1.16 is

2.15 Lemma If the pair of partitions (λ,μ) has m_j^γ parts equal to j ($\gamma=\lambda$ or μ) then

(i) for every pair of (λ,μ)-polytabloids e_t and $e_{t'}$, $\prod_{j=1}^{\infty} m_j^\lambda!\, m_j^\mu!$ divides $\langle e_t, e_{t'}\rangle$;

(ii) for every (λ,μ)-polytabloid e_t there is a (λ,μ)-polytabloid $e_{t'}$ such that $\langle e_t, e_{t'}\rangle = 2^{|\lambda|} \prod_{j=1}^{\infty} (m_j^\lambda! m_j^\mu!)^j$.

From this follows

2.16 Corollary If $S^{\lambda,\mu}$ is defined over a field of characteristic p then $S^{\lambda,\mu}/S^{\lambda,\mu} \cap (S^{\lambda,\mu})^\perp$ is non-zero if and only if (λ,μ) is p-regular.

Now put $D^{\lambda,\mu} = S^{\lambda,\mu}/S^{\lambda,\mu} \cap (S^{\lambda,\mu})^\perp$. Then we have

2.17 Theorem As (λ,μ) varies over all p-regular pairs of partitions of n, the $D^{\lambda,\mu}$ give a complete set of inequivalent irreducible KO_n-modules.

Furthermore, in this case also, the dimension of each Specht module $S^{\lambda,\mu}$ may be determined.

2.18 Theorem $\{e_t \mid t$ is a standard (λ,μ)-tableau$\}$ is a K-basis for $S^{\lambda,\mu}$. The dimension of $S^{\lambda,\mu}$ is the number of standard tableaux of shape (λ,μ).

3. Irreducible representations of Weyl groups

We first briefly present the basic requirements about Weyl groups. As general references, we give Bourbaki [4] and Carter [5].

Let V be an ℓ-dimensional real euclidean space with positive definite inner product f.

$\Phi \subset V$ is a *root system* if

(i) Φ is a finite subset of V which spans V,

(ii) for $\alpha \in \Phi$, $\tau_\alpha \Phi = \Phi$, where τ_α is a reflection in the hyperplane $\langle\alpha\rangle^\perp$ defined by

$$\tau_\alpha(x) = x - \frac{2f(\alpha,x)}{f(\alpha,\alpha)}\alpha$$

for all $x \in V$,

(iii) if $\alpha,\beta \in \Phi$, then $\frac{2f(\alpha,\beta)}{f(\alpha,\alpha)} \in \mathbb{Z}$,

(iv) for $\alpha \in \Phi$ and $k \neq \pm 1$, $-\alpha \in \Phi$ and $k\alpha \notin \Phi$.

$W(\Phi) = \langle \tau_\alpha | \alpha \in \Phi \rangle$ is a finite group called the Weyl group of the root system Φ. For a fixed ordering in V, each root system Φ contains a *simple system* π such that

(i) π is linearly independent over $\mathbb{R}$,

(ii) if $\alpha \in \Phi$, then α is a linear combination of the elements in π in which all the coefficients are either all non-negative or non-positive.

Thus $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ contains the elements with non-negative coefficients and is called the *positive system* relative to the ordering determined by π. The following facts will be used:-

(i) $W(\Phi) = \langle \tau_\alpha | \alpha \in \pi \rangle$,

(ii) there are $|W(\Phi)|$ simple systems in Φ given by $w\pi$, $w \in W(\Phi)$, π any simple system in Φ (i.e. $W(\Phi)$ acts transitively on simple systems).

(iii) Let $\ell(w)$ be the minimal length of any expression for $w \in W(\Phi)$ as a product of the τ_α, $\alpha \in \pi$ and define $\mathrm{sgn}(w) = (-1)^{\ell(w)}$, $w \in W(\Phi)$.

(iv) To each group there corresponds a graph called the *Dynkin diagram;* the Weyl group is *irreducible* if its Coxeter graph is connected. Irreducible Weyl groups have been classified and correspond to root systems of type A_ℓ $(\ell \geq 1)$, C_ℓ $(\ell \geq 2)$, D_ℓ $(\ell \geq 4)$, E_6, E_7, E_8, F_4, G_2. (For example, $W(A_\ell) \simeq S_{\ell+1}$ $(\ell \geq 1)$ and $W(C_\ell) \simeq O_\ell$ $(\ell \geq 2)$.)

A *subsystem* Ψ of a root system Φ is a subset of Φ which is itself a root system in the space which it spans. A *Weyl subgroup* $W(\Psi)$ of $W(\Phi)$ corresponding to the subsystem Ψ is the subgroup generated by τ_α, $\alpha \in \Psi$. The graphs which are Dynkin diagrams of Weyl subgroups may be obtained by a standard algorithm, due independently to Dynkin [6] and Borel and de Siebenthal [3]. This involves the extended Dynkin diagram which is obtained from the Dynkin diagram by the addition of one further node corresponding to the negative of the highest root. The Dynkin diagrams of all possible Weyl subgroups are obtained as follows: Take the extended Dynkin diagram of Φ and remove one or more nodes in all possible ways. Take also the duals of the diagrams obtained in this way from Φ the root system obtained from Φ by interchanging long and short roots. Then repeat the process with the diagrams obtained, and continue any number of times. For example, the Dynkin diagrams of Weyl subgroups of

(1) $W(A_\ell)$ are $A_{i_1}+\ldots+A_{i_m}$, $\sum_{r=1}^{m}(i_r+1) = \ell+1$, $i_1 \geq i_2 \geq \ldots \geq i_m \geq 0$.

(2) $W(C_\ell)$ are $A_{i_1}+\ldots+A_{i_m}+C_{j_1}+\ldots+C_{j_n}$, $\sum_{r=1}^{m} i_r+1+\sum_{s=1}^{n} j_s = n$,

$$i_1 \geq i_2 \geq \ldots \geq i_m \geq 0,\ j_1 \geq j_2 \geq \ldots \geq j_n \quad 0.$$

(3) $W(G_2)$ are G_2, A_2, $\tilde{A}_2$, $A_1+\tilde{A}_1$, A_1, $\tilde{A}_1$, ϕ.

Suppose from now on that Φ be a root system represented by the Dynkin diagram Γ, and that Ψ is a subsystem of Φ with Dynkin diagram Δ. Choose an ordering on Φ (which will remain fixed from now on) and let Φ^+ and Ψ^+ be the set of positive roots in Φ and Ψ for this ordering. The subsystems of Φ with Dynkin diagram Δ which are conjugate to Ψ are given by $w\Psi$, $w \in W(\Phi)$. The positive roots in $w\Psi$ are taken to be $w\Psi^+$, i.e. $(w\Psi)^+ = w\Psi^+$. Let X_Δ be the collection of positive systems of subsystems of Φ with Dynkin diagram Δ, that is, $X_\Delta = \{w\Psi^+ | w \in W(\Phi)\}$. The elements of X_Δ are called Δ-*tableaux*. We note that if $w \in W(\Phi)$, $W(w\Psi) = wW(\Psi)w^{-1}$ since $\tau_{w\alpha} = w\tau_\alpha w^{-1}$ for all $\alpha \in \Psi$. An equivalence relation is defined on X_Δ by $\Psi_1^+ \sim \Psi_2^+$ if $\Psi_2^+ = w\Psi_1^+$ for some $w \in W(\Psi_1)$. The equivalence class $\{\Psi^+\}$ containing Ψ^+ under this equivalence relation will be called a Δ-*tabloid*. $W(\Phi)$ acts on the set of Δ-tabloids, if for $w \in W(\Phi)$ we put

$$w\{\Psi^+\} = \{w\Psi^+\}.$$

This is well defined, for if $\{\Psi_1^+\} = \{\Psi_2^+\}$, then $\Psi_2^+ = \sigma\Psi_1^+$ for some $\sigma \in W(\Psi_1)$. Then, since $w\sigma w^{-1} \in wW(\Psi_1)w^{-1} = W(w\Psi_1)$ and $w\Psi_2^+ = w\sigma\Psi_1^+ = (w\sigma w^{-1})(w\Psi_1^+)$, we have $\{w\Psi_1^+\} = \{w\Psi_2^+\}$. Let K be an arbitrary field and let M^Δ be the vector space over K whose basis elements are the various Δ-tabloids. Extend the action of $W(\Phi)$ on Δ-tabloids linearly on M^Δ, then M^Δ is a $KW(\Phi)$-module.

3.1 Lemma M^Δ is the permutation module of $W(\Psi)$ on the subgroup $W(\Psi)$. M^Δ is a cyclic $KW(\Phi)$-module generated by any one Δ-tabloid and $\dim_K M^\Delta = |W(\Phi)|/|W(\Psi)|$.

Proof $W(\Phi)$ is clearly transitive on Δ-tabloids, for if $\{\Psi_1^+\}$ and $\{\Psi_2^+\}$ are Δ-tabloids, then $\Psi_2^+ = w\Psi_1^+$ for some $w \in W(\Phi)$ and then $\{\Psi_2^+\} = \{w\Psi_1^+\} = w\{\Psi_1^+\}$ and clearly $W(\Psi)$ fixes $\{\Psi^+\}$.

3.2 Definition Ψ' is called a *dual* subsystem to Ψ if Ψ' is a maximal subsystem in Φ/Ψ. It is easily seen that if Ψ' is a maximal subsystem in Φ/Ψ so is $w\Psi'$, $w \in W(\Psi)$.

3.3 Definition A subsystem Ψ in Φ is called an *admissible* subsystem if its dual subsystem is unique up to an element of $W(\Psi)$.

3.4 Lemma If Ψ is an admissible subsystem in Φ with dual Ψ', $\alpha \in \Phi/\Psi$ then $\alpha \in w(\Psi')$ for some $w \in W(\Psi)$.

Proof If $\alpha \in \Phi/\Psi$ then $\{\pm\alpha\}$ is a root system in Φ/Ψ, thus α is contained in some maximal subsystem. As Ψ is admissible, the result follows.

Note Not all subsystems are admissible, for example, A_2 and $A_1+\tilde{A}_1$ are dual to A_1 in G_2.

3.5 Definition If Ψ is an admissible subsystem, with dual Ψ', let

$$\kappa'_\Psi = \sum_{w\in W(\Psi')} (\operatorname{sgn} w)w.$$

If Ψ'' is also a dual subsystem to Ψ, then $\Psi'' = \sigma\Psi'$ for some $\sigma \in W(\Psi)$ and

$$\kappa''_\Psi = \sum_{w\in W(\sigma\Psi')} (\operatorname{sgn} w)w = \sigma\left\{\sum_{w\in W(\Psi')} (\operatorname{sgn} w)w\right\}\sigma^{-1} = \sigma\kappa'_\Psi\sigma^{-1}.$$

Let Ψ be an admissible system with Dynkin diagram Δ and with dual Ψ'; the Δ-*polytabloid* associated with Ψ is defined by

$$e'_\Psi = \kappa'_\Psi\{\Psi^+\}.$$

We note that if Ψ'' is also a dual to Ψ, and $\Psi'' = \sigma\Psi'$, $\sigma \in W(\Psi)$ then

$$e''_\Psi = \kappa''_\Psi\{\Psi^+\} = \kappa''_\Psi\sigma\{\Psi^+\} = \sigma\kappa'\{\Psi^+\} = \sigma e'_\Psi \quad ,$$

that is, e'_Ψ is unique up to an element of $W(\Psi)$.

Now, if $\sigma \in W(\Phi)$,

$$\sigma\kappa'_{\Psi} = \sum_{w\in W(\Psi')} (\text{sgn } w)\sigma w = \left(\sum_{w\in W(\Psi')} (\text{sgn } w)\sigma w\sigma^{-1}\right)\sigma$$

$$= \left(\sum_{w\in W(\sigma\Psi')} (\text{sgn } w)w\right)\sigma = \kappa_{\sigma\Psi'}\sigma$$

and thus

$$\sigma e'_{\Psi} = \sigma\kappa'_{\Psi}\{\Psi^+\} = \kappa_{\sigma\Psi'}\sigma\{\Psi^+\} = \kappa'_{\sigma\Psi'}\{\sigma\Psi^+\} = e'_{\sigma\Psi} .$$

Let S^Δ be the submodule of M^Δ spanned by all Δ-polytabloids, then S^Δ is called a *Specht module*, and we have

3.6 Theorem S^Δ is a cyclic module generated by any one Δ-polytabloid.

Proof Follows directly from above.

3.7 Lemma Let Ψ and $\mathfrak{X}$ be admissible subsystems of Φ with Dynkin diagram Δ. The following conditions are equivalent:

(i) $\{\Psi^+\}$ appears with non-zero coefficient in $e_{\mathfrak{X}}'$;

(ii) there exist $\rho \in W(\Psi)$, $\gamma \in W(\mathfrak{X}')$ such that $\rho\Psi^+ = \gamma\mathfrak{X}^+$;

(iii) if $\alpha \in \Psi^+$ then $\alpha \notin \mathfrak{X}'^+$.

Proof Since $e_{\mathfrak{X}}' = \kappa_{\mathfrak{X}}'\{\mathfrak{X}^+\} = \sum_{w\in W(\mathfrak{X}')} (\text{sgn } w)w\{\mathfrak{X}^+\}$, it follows immediately that $\{\Psi^+\}$ appears with non-zero coefficient in $e_{\mathfrak{X}}'$ if and only if there exist $\rho \in W(\Psi)$, $\gamma \in W(\mathfrak{X}')$ such that $\rho\Psi^+ = \gamma\mathfrak{X}^+$ proving the equivalence of (i) and (ii).

(ii) $\Longrightarrow$ (iii) Without loss of generality, assume that $\alpha \in \Psi^+$. Thus if $\rho \in W(\Psi)$, $\alpha \in \rho\Psi^+ = \gamma\mathfrak{X}^+$, $\alpha \notin \gamma\, \mathfrak{X}'^+$, i.e. $\alpha \notin \mathfrak{X}'^+$ since $\gamma \in W(\mathfrak{X}')$.

(iii) $\Longrightarrow$ (ii) If $\alpha \in \mathfrak{X}'^+$, then $\alpha \notin \Psi^+$ and $\alpha \in \Phi/\Psi$. By (3.4), $\mathfrak{X}' = \rho\Psi'$ for some $\rho \in W(\Psi)$ and thus $\gamma\,\mathfrak{X}'^+ = \rho\Psi'^+$ for some $\gamma \in W(\mathfrak{X}')$. Hence $\gamma\mathfrak{X}^+ = \rho\Psi^+$ for some $\rho \in W(\Psi)$, $\gamma \in W(\mathfrak{X}')$.

3.8 Lemma Let Ψ and $\mathcal{X}$ be admissible subsystems of Φ with Dynkin diagram Δ. If there exists $\alpha \in \Psi^+$ such that $\alpha \in \mathcal{X}'^+$ then

$$\kappa_{\mathcal{X}}'\{\Psi^+\} = 0.$$

Proof If $\alpha \in \Psi^+ \cap \mathcal{X}'^+$ then $\tau_\alpha \in W(\Psi) \cap W(\mathcal{X}')$ and $(e-\tau_\alpha)\{\Psi^+\} = \{\Psi^+\}-\{\Psi^+\} = 0$. Since $\{e,\tau_\alpha\}$ is a subgroup of $W(\mathcal{X}')$, we select signed coset representatives $\sigma_1,\sigma_2,\ldots,\sigma_k$ for $\{e,\tau_\alpha\}$ in $W(\mathcal{X}')$ and obtain

$$\kappa_{\mathcal{X}}'\{\Psi^+\} = (\sigma_1+\ldots+\sigma_k)(e-\tau_\alpha)\{\Psi^+\} = 0.$$

We can now prove

3.9 Lemma Let Ψ and $\mathcal{X}$ be admissible subsystems of Φ with Dynkin diagram Δ.

(i) If $\{\Psi^+\}$ does not appear in $e_{\mathcal{X}}'$ then $\kappa_{\mathcal{X}}'\{\Psi^+\} = 0$;

(ii) If $\{\Psi^+\}$ appears with non-zero coefficient in $e_{\mathcal{X}}'$ then $\kappa_{\mathcal{X}}'\{\Psi^+\} = (\mathrm{sgn}\,\gamma)e_{\mathcal{X}}'$ for some $\gamma \in W(\mathcal{X}')$.

Proof (i) follows directly from 3.7 and 3.8.

To prove (ii), if $\{\Psi^+\}$ appears with non-zero coefficient in $e_{\mathcal{X}}'$ then by 3.7, there exist $\rho \in W(\Psi)$, $\gamma \in W(\mathcal{X}')$ such that $\rho\Psi^+ = \gamma\mathcal{X}^+$ and hence

$$\kappa_{\mathcal{X}}'\{\Psi^+\} = \kappa_{\mathcal{X}}'\{\rho\Psi^+\} = \kappa_{\mathcal{X}}'\gamma\{\mathcal{X}^+\} = (\mathrm{sgn}\,\gamma)\kappa_{\mathcal{X}}'\{\mathcal{X}^+\} = (\mathrm{sgn}\,\gamma)e_{\mathcal{X}}'.$$

The obvious bilinear form $\langle\ ,\ \rangle$ is put on M^Δ (as in sections 1 and 2). Then, using exactly the same proof as in these two sections, we can prove

3.10 Theorem Let U be a submodule of M^Δ. Then either $U \supseteq S^\Delta$ or $U \subseteq S^{\Delta\perp}$. Furthermore, $S^\Delta/S^\Delta \cap S^{\Delta\perp}$ is zero or irreducible.

There remain many problems. If $D^\Delta = S^\Delta/S^\Delta \cap S^{\Delta\perp}$, when is D^Δ non-zero? Is there a concept in this more general setting which corresponds to p-regular partitions in the case of symmetric groups? The partial order of dominance on partitions was crucial in that case in proving that the D^Δ for distinct p-regular partitions are non-isomorphic. Can we prove in our case that the non-zero D^{Δ} are distinct if the corresponding root systems are not of equal size? (cf. MacDonald [12]). What is the dimension of D^{Δ}?

3.11 Definition A Δ-tableau Ψ^{+} is called a *standard Δ-tableau* if Ψ^{+} and its dual Ψ'^{+} are subsets of Φ^{+}; that is, the positive roots of Ψ and its dual are composed of positive roots of Φ. $\{\Psi^{+}\}$ is a *standard Δ-tabloid* if there is a standard Δ-tableau in the equivalence class $\{\Psi^{+}\}$. e_{Ψ}' is a *standard Δ-polytabloid* if Ψ^{+} is a standard Δ-tableau.

3.12 Conjecture The dimension of S^{Δ} is the number of standard Δ-polytabloids.

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