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INDECOMPOSABILITY OF RESTRICTED TANGENT BUNDLES

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1. Introduction

Let X be a closed subvariety of a projective space $\mathbf{P}^n = P$ which is non-singular in codimension one. J. Simonis [7] proved that if X is linearly normal in P and the restricted cotangent bundle $\Omega_P^1|X$ splits, then X is a rational curve. The purpose of this note is first to present a simplified version of Simonis' proof (see Theorem 1) and second and more importantly, to show that Simonis' theorem implies that, if the conormal bundle sequence

$$0 \to N_{X|P}^{v} \to \Omega_P^1|X \to \Omega_X^1 \to 0$$

splits, then X is a linear subspace of P. The latter result was first proved by Van de Ven [8]. Morrow and Rossi [5] gave a different geometric proof, and used it as a main ingredient in the proof of the result that the linear submanifolds of P are the only submanifolds which have a neighbourhood biholomorphic to a neighbourhood of the zero section of the normal bundle. The main purpose of Simonis' article was to generalize the result by A. Van de Ven that $\mathbf{P}^1$ is the only non-singular projective variety on which every algebraic vector bundle split into line bundles. A. Grothendieck suggested in [3] that the latter result holds for all projective varieties.

In our proof of Simonis' theorem, we use the result that if X is a non-singular curve in $\mathbf{P}^n$ not contained in a hyperplane, then a hyperplane in general position intersect X transversally in deg X points no n of which are linearly dependent. This useful result seems to go back at least to Bertini (see e.g. [1]) when the ground field is of characteristic zero. In arbitrary characteristic the result may be proved using Bertini's techniques and Lluis' theorem that the only strange projective curves are the lines and the conics in characteristic two [4]. Since there seems to be no adequate references for the result on hyperplane sections a proof of it is given below (see Proposition 8), as well as a simple coordinate free proof of Lluis' theorem.

2. Indecomposability of the restricted tangent bundle

Recall that an embedding of a variety X into a projective space $\mathbb{P}^n = P$ is called linearly normal if the map $H^0(P, \mathcal{O}_P(1)) \to H^0(X, \mathcal{O}_P(1)|X)$ is an isomorphism, or in other words if X is not contained in a hyperplane and if the hyperplanes cut out a complete linear system on X.

Theorem 1. (Simonis, [7] Theorem 3.1.1, p. 262) *Let* X *be a linearly normal closed subvariety of* $\mathbb{P}^n$ *which is non-singular in codimension one. If the restricted cotangent bundle* $\Omega_P^1|X$ *on* X *is decomposable, then* X *is a rational curve of degree* n.

Proof. Assume first that X is a non-singular curve. Then it suffices to prove that X is of degree n. Indeed, if D is a hyperplane section and K the canonical divisor, then, by the Riemann-Roch theorem, $n+1 = \ell(D) = \deg X+1-g+\ell(K-D)$. However, for a general hyperplane section we have $g \geq \ell(K-D)$ with strict inequality when $g \neq 0$. Hence $\deg X \geq n$ with equality only if X is rational.

Assume that $\Omega_P^1(1)|X$ decomposes into the direct sum of two bundles E and F of rank e and f. Write $\mathbb{P}^n = \mathbb{P}(V)$ and let $L = \mathcal{O}_P(1)|X$. Then from the exact sequence

$$0 \to \Omega_P^1(1) \to V_P \to \mathcal{O}_P(1) \to 0$$

restricted to X we obtain isomorphisms $\Lambda^{n+1}V \cong L \otimes \Lambda^n \Omega_P^1(1) \cong L \otimes \Lambda^e E \otimes \Lambda^f F$ and choosing a basis for V we obtain an isomorphism $\Lambda^e E \otimes \Lambda^f F \cong L^{-1}$.

Let A and B be quotient spaces of V of dimensions e and f and denote by u and v the composite maps $E \to V_X \to A_X$ and $F \to V_X \to B_X$.

Simonis made the following crucial observation: *The union of the sets* Z(u) *and* Z(v) *where the maps* u *and* v *drop rank coincides with a hyperplane section of* X. It holds, because Z(u) and Z(v) are the zeroes of the determinants of u and v, and consequently the union of the sets is the zeroes of the section of L, dual to the map

$$\overset{e}{\Lambda} u \otimes \overset{f}{\Lambda} v : \overset{e}{\Lambda} E \otimes \overset{f}{\Lambda} F \to \overset{e}{\Lambda} A \otimes \overset{f}{\Lambda} B.$$

We shall use Simonis' observation in the particular case when $A = \oplus_{i=1}^{e} k(p_i)$ and $B = \oplus_{i=1}^{f} k(q_i)$ where $p_1, \ldots, p_e$ and $q_1, \ldots, q_f$ are two sets of linearly independent points on X and where A and B are considered as quotients of V by the maps corresponding to the inclusions in P of the linear spaces $[p_1, \ldots, p_e]$ and $[q_1, \ldots, q_f]$ spanned by $p_1, \ldots, p_e$ and $q_1, \ldots, q_f$. In this case we clearly have $\{p_1, \ldots, p_e\} \subseteq Z(u)$ and $\{q_1, \ldots, q_f\} \subseteq Z(v)$. Conversely, by the following argument of Castelnuovo's [2] we obtain inclusions $Z(u) \subseteq [p_1,\dots,p_e]$ and $Z(v) \subseteq [q_1,\dots,q_f]$: Fix $p_1,\dots p_e$ and let $q_1,\dots,q_f$ vary over all sets of f linearly independent points of X. By the above observation of Simonis we have for each choice of $q_1,\dots,q_f$ that $Z(u)$ is contained in a hyperplane containing the points $p_1,\dots,p_e$ and $q_1,\dots,q_f$. The intersection of all such hyperplanes when $q_1,\dots,q_f$ vary is, however, clearly $[p_1,\dots,p_e]$.

Now let H be a hyperplane in general position and let $H \cap X = \{r_1,\dots,r_d\}$. Then $d \geq n$ and we can pick n distinct linearly independent points $p_1,\dots,p_e$, $q_1,\dots,q_f$ from $H \cap X$. If $d > n$, then there is one further point $p \in H \cap X$. By the above observation of Simonis, $p \in Z(u)$ or $p \in Z(v)$, where u and v are obtained from the points $p_1,\dots,p_e$ and $q_1,\dots,q_f$ as above. Then by the preceding inclusion we must have $p \in [p_1,\dots,p_e]$ or $p \in [q_1,\dots,q_f]$. This, however, contradicts the fact (see Proposition 5), that no n points on a generic hyperplane section are linearly dependent. We conclude that $d = n$ and the theorem is proved when X is a curve.

The reduction from X of arbitrary dimension to the case of a curve is standard, see e.g. Simonis [7], Proposition 1.2.3 and Remark 1.2.4.

Theorem 2. (Van de Ven [8], Theorem 2, p. 151-152 and Morrow-Rossi [5], Theorem 3.9, p. 256).

Let X *be a closed subvariety of* $\mathbb{P}^n = P$ *which is non-singular in codimension one and such that the sequence*

$$0 \to N^{v}_{X|P} \to \Omega^1_P | X \to \Omega^1_X \to 0$$

splits. Then X *is a linear subspace of* P.

Proof. Assume that X is contained in a linear subspace L of $\mathbb{P}^n$. Then we have a commutative diagram of bundles on X

$$\begin{array}{ccccccccc} 0 & \to & N^{v}_{X/P} & \to & \Omega^1_P | X & \to & \Omega^1_X & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N^{v}_{X/L} & \to & \Omega^1_L | X & \to & \Omega^1_X & \to & 0. \end{array}$$

An easy diagram chase shows that if the top sequence splits then the bottom one does. Hence we may reduce to the case that X is not contained in a hyperplane, that is when the map w: $H^0(P, \mathcal{O}_P(1)) \to H^0(X, \mathcal{O}_P(1)|X)$ is injective. We shall show that in this case X can not be a proper subvariety of $\mathbb{P}^n$.

Let $V = H^0(P, \mathcal{O}_P(1))$ and $W = H^0(X, \mathcal{O}_P(1)|X)$. The surjection $W_X \to \mathcal{O}_P(1)|X$ defines a morphism g: $X \to Q = \mathbb{P}(W)$ which is a closed immersion, because the composite map $V_X \to W_X \to \mathcal{O}_P(1)|X$ defines X as a subvariety of P. Moreover, by definition g embeds X in Q as a linearly normal subvariety.

The composite map of g with the projection of Q to P defined by the inclusion $V \to W$ is the identity on X. Hence we obtain a commutative diagram of bundles on X

$$\begin{array}{ccccccccc} 0 & \to & N^{\vee}_{X/P} & \to & \Omega^1_P|X & \to & \Omega^1_X & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N^{\vee}_{X/Q} & \to & \Omega^1_Q|X & \to & \Omega^L_X & \to & 0. \end{array}$$

Again an easy diagram chase shows that if the top sequence splits, then the bottom one does. If the bottom sequence splits, it follows from Theorem 1 that either $X = Q$ and thus $X = P$ or X is a rational curve in Q of degree $d = \dim W - 1$. The latter case is, however, excluded because then, as we shall now check, the sequence of the proposition does not split. Indeed the immersion $g\colon X \to Q$ of $X = \mathbf{P}^1$ as a curve of degree d is such that $\mathcal{O}_Q(1)|X = \mathcal{O}_X(d)$. Hence the inclusion $\Omega^1_Q \to W_Q(-1)$ gives an inclusion $\Omega^1_Q|X \to W_X(-d)$. If the map $\Omega^1_Q|X \to \Omega^1_X = \mathcal{O}_X(-2)$ had a section, we would then obtain an inclusion $\mathcal{O}_X(-2) \to W_X(-d)$ which can only happen if $d \leqq 2$, that is when X is a plane conic. On the plane conic X we have the sequence

$$0 \to \Omega^1_Q|X \to V_X(-2) \to \mathcal{O}_X \to 0$$

which after tensoring by $\mathcal{O}_X(2)$ gives an exact sequence in cohomology

$$\mathcal{O} \to H^0(X, \Omega^1_Q(2)\,|\,X) \to H^0(X, V_X) \to H^0(X, \mathcal{O}_X(2)).$$

It is easily checked that the right hand map in the latter sequence is an isomorphism and thus that $H^0(X, \Omega^1_Q(2)\,|\,X) = 0$. This is, however, incompatible with the existence of a section to the map $\Omega^1_Q\,|\,X \to \Omega^1_X = \mathcal{O}_X(-2)$.

3. Strange curves and the independence of points on hyperplane sections

Recall that a curve is called strange if all tangent lines to the curve pass through the same point.

Proposition 3. (Lluis [2] Theorem §3, p. 51. See also Samuel [6] Theorem, p. 76). *The only non-singular strange curves in $\mathbf{P}^n$ are the straight lines and the conics in characteristic two.*

Proof. Let X be a strange curve in $\mathbf{P}^n = \mathbf{P}(V) = P$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N^{\vee}_{X/P} & \to & \Omega^1_P\,|\,X & \to & \Omega^1_X & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & N^{\vee}_{X/P} & \to & V_X(-1) & \to & E(-1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X & = & \mathcal{O}_X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which defines the rank two bundle E on X. It is readily verified that at every point $p \in X$ the induced quotient $V = V_X(p) \to E(p)$ corresponds to the inclusion $\mathbf{P}(E(x)) \subseteq P$ of the tangent line to X at p into $\mathbf{P}^n$. Consequently, if all tangent lines pass through a point q of P the corresponding quotient $V = V_p(q) \to k(q)$ factors via E to a map $E \to k(q) \otimes \mathcal{O}_X = \mathcal{O}_X$. Composing with the inclusion $\Omega_X^1 \to E(-1)$ we obtain a map $\Omega_X^1(1) \to \mathcal{O}_X$. If the latter map were zero, the quotient $E \to \mathcal{O}_X$ would factor through the map $E \to \mathcal{O}_X(1)$ of the diagram which is impossible. Hence the map $\Omega_X^1(1) \to \mathcal{O}_X$ is injective and we must have $2g-2-d \leq 0$. We conclude that $g = 0$ and $d = 1$ or 2 and consequently that X is either a line or a conic. It is, however, easy to check that conics are only strange in characteristic two.

Lemma 4. (Bertini [1], §7.3, p. 166 (when $r = 2$)) *Let* X *be a curve in* $\mathbf{P}^n$ *which is not contained in a hyperplane and let* r *be an integer* $2 \leq r \leq n-1$. *Assume that, for almost all choices of* r-*points* $p_1, \ldots, p_r$ *on* X, *the linear space they span contains at least one further point of* X. *Then* X *is a strange curve.*

Proof. We may assume that r is the least integer for which the assumption of the lemma holds. Choose r points $p_1, \ldots, p_r$ on X in general position and let

p_0 be another point of $X \cap [p_1,\ldots,p_r]$. By the minimality of r the linear space $L = [p_0,p_1,\ldots,p_{r-2}]$ has dimension $r-2$. Let Y be the image of X by the projection to $\mathbf{P}^{n-r+1}$ with center on L. By the assumption of the lemma and the fact that $p_{r-1} \neq p_r$ it follows that almost all points on Y are the images of at least two distinct points on X. Let p and q be two distinct points that map to a point y on Y in general position. Then the tangents t_p and t_q map to the tangent t_y to Y at y and consequently the spaces $[L,t_p]$ and $[L,t_q]$ spanned by L and the tangents in p and q are equal. The condition that $t_q \subseteq [L,t_p]$ is closed on the set of points $p \in X$ and $q \in X \cap [L,p]$ and consequently it holds for all such pair of points. In particular, if t_i denotes the tangent to X at p_i, we have $t_{r-1} \subseteq [L,t_r]$.

Let $H = [p_1,\ldots,p_{r-1},t_r]$. Then $[L,t_r] \subseteq H$. We shall show that the inclusion $t_{r-1} \subseteq H$ implies that t_{r-1} and t_r intersect. The points $p_1,\ldots,p_r$ were chosen in general position. Hence we may assume that for $i \leqq r-2$ the point p_i is not in the space $H_{i-1} = [p_1,\ldots,p_{i-1},t_{r-1},t_r]$ whenever $\dim H_{i-1} < n$. In particular, if $\dim H_{r-2} < n$ then $\dim H_{r-2} = r-2 + \dim[t_{r-1},t_r]$. However, $H_{r-2} \subseteq H$ since $t_{r-1} \subseteq H$ and clearly $\dim H \leqq r < n$. We conclude that we have an inequality $r-2 + \dim[t_{r-1},t_r] \leqq r$ and consequently that t_{r-1} and t_r intersect. Since the condition that t_p and t_q intersect is a closed condition on the pair of points (p,q) of X^2 it follows that any two tangents of X intersect.

If for almost all choices of three points on X the tangents in these points intersect in three different points, then the curve is clearly contained in the plane spanned by any two of the tangents. Hence X is plane which is impossible because we have assumed that $n \geqq 3$ and that X is not contained in a hyperplane. We conclude that any three tangents meet in one point, that is, the curve is strange.

Proposition 5. (Bertini-Lluis) *Let* X *be a non-singular curve in* $\mathbf{P}^n$ *not contained in any hyperplane. Then any* n *different points in the intersection of* X *with a hyperplane in general position are linearly independent.*

Proof. The proposition is obvious for $n = 2$, so we may assume that $n \geqq 3$. If the proposition does not hold, there is a positive integer $2 \leqq r \leqq (n-1)$ such that on a hyperplane in general position any r different points are linearly independent but there are $(r+1)$-points that are linearly dependent. Let U be the open subset of X^r of points $(p_1,\ldots,p_r)$

such that $p_1,\ldots,p_r$ are linearly independent and let V be the closed subset of U of points $(p_1,\ldots,p_r)$ such that $[p_1,\ldots,p_r]$ contains at least one further point of X. If $V = U$, then by Lemma 4 the curve X is strange and thus by Proposition 3 we must have $n = 1$ or 2. Hence we may assume that V is a proper closed subset of U and in particular that $\dim V < r$.

Denote by Q the dual space of $\mathbf{P}^n$ and let Z in $V \times Q$ be the set of points $(p_1,\ldots,p_r,h)$ such that the points $p_1,\ldots,p_r$ are contained in h. The fiber of the map $Z \to V$ induced by the projection on the first factor is an $(n-r)$-dimensional linear subspace of Q. Consequently $\dim Z \leq \dim V+n-r < n$ and the projection of Z onto the second factor is not all of Q. Hence a hyperplane in general position does not contain r points that span a space containing one further point of X. That is, any $(r+1)$-points of the intersection of X with the hyperplane are linearly independent.

REFERENCES

[1] E. Bertini, Complementi di geometria proiettive. N. Zanichelli. Bologna 1927.

[2] C. Castelnuovo, Ricerche di geometria sulle curve algebriche. Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. XXIV(1889).

[3] A. Grothendieck, Sur la classification des fibres holomorphes sur la sphere de Riemann. Amer. J. Math. 79(1957), 121-138.

[4] E. Lluis, Variedades algebraicas con ciertas condiciones en sus tangentes. Bol. Soc. Mat. Mexicana (2) 7(1962), 47-56.

[5] J. Morrow and H. Rossi, Submanifolds of $\mathbb{P}^n$ with splitting normal bundle sequence are linear. Math. Ann. 234(1978), 253-261.

[6] P. Samuel, Lectures on old and new results on algebraic curves. Tata Institute of Fundamental Research. Bombay 1966.

[7] J. Simonis, A class of indecomposable algebraic vector bundles. Math. Ann. 192(1971), 262-278.

[8] A. Van de Ven, A property of algebraic varieties in complex projective spaces. Colloque de géométrie différentielle globale. Bruxelles (1958), 151-152.