CHRISTOPHER I. BYRNES

On certain problems of arithmetic arising in the realization of linear systems with symmetries

Astérisque, tome 75-76 (1980), p. 57-65

<http://www.numdam.org/item?id=AST_1980__75-76__57_0>

ON CERTAIN PROBLEMS OF ARITHMETIC ARISING IN THE REALIZATION OF LINEAR SYSTEMS WITH SYMMETRIES*

by

Christopher I. BYRNES

-:-:-:-

INTRODUCTION.-

Now, one way of organizing, if not attacking, the standard problems of linear, finite-dimensional system theory is by taking into account the group of symmetries of the problem at hand. Thus, the realization question revolves around the general linear group $GL(n,\underline{k})$ and its action on state-space representations, while the problems of state-space feedback or of linear quadratic optimal control involve groups which are extensions of $GL(n)$. Moreover, for over a decade it has been known that transfer functions, defined over $\underline{k}$, which possess certain symmetries often possess realizations which reflect these symmetries, e.g., the driving-point impedances of linear networks or the frequency response of a Hamiltonian system. If $k = \mathbb{R}$, then it is well-known that the group of state-space transformations preserving such a realization is a classical subgroup, e.g. $O(p,q)$ or $Sp(n,\mathbb{R})$, of $GL(n,\mathbb{R})$.

All of these interconnections become far more interesting when we consider linear systems depending on a parameter or linear systems whose coefficients lie in some commutative ring. Indeed, as we shall review in the next section, even standard realization theory in these more general settings makes contact with the topology and geometry of the parameter space, or with the algebra of the ring. The problems here are, however, fairly tractible. The truly exciting, arithmetic side of the topology and algebra will come to the fore when we consider the less trivial actions of the classical groups in the second section. In particular, we make significant contact with the Hasse-Minkowski theory of quadratic forms over $\mathbb{Z}$ and, motivated by earlier investigations of Youla [14] and of Koga [12], with the quadratic analogue of the Serre conjecture. In the final section, we announce

* Partially supported by the NASA under Grant NSG-2265 and the ONR under JSEP Contract N 00014-15-C-0648.

some of the joint work done on this problem with T.E. Duncan.

I.- SOME TOPOLOGICAL AND GEOMETRIC ASPECTS OF LINEAR SYSTEM THEORY.

Some of the recent infusion of topology and geometry is accounted for by the interconnection between 3 viewpoints :

a) A system defined over a ring R of functions, say $\mathbb{R}[x_1,\ldots,x_N]$, ℓ^1, or $C(S^2)$.

b) A family of linear systems $(A(\lambda), B(\lambda), C(\lambda))$, $\lambda \in X$, say $X = \mathbb{R}^N$, S^1, or S^2.

c) A map, $X \to \mathcal{H}$, into the space of Hankel matrices (of a fixed number m of inputs, p of outputs, and n = McMillan degree).

Note that c) is almost the same as b) ; to understand the distinction it's perhaps easiest to understand realizations in this context. As has been shown ([8], [11]), the space Σ of all minimal triples (A, B, C) modulo $GL(n)$, with m, p, and n fixed as above is a smooth, algebraic manifold. On the other hand, M. Clark [9] has proved that the space $\mathcal{H}$ is also a smooth, algebraic manifold and it is not too hard to see that the natural map

$$\eta : \Sigma \to \mathcal{H}$$

which assigns to each class $[(A, B, C)]$ the associated Hankel matrix $(CA^{j-1}B) \in \mathcal{H}$ is an algebraic map. Thus, a continuous family of system, as in b), with constant McMillan degree, gives rise, after composing with η, to a map as in c). Of course, by evaluation, the data in a) gives rise to a family b), and conversely. In order to complete the circle, we would have to recover a family of realizations $(A(\lambda), B(\lambda), C(\lambda))$ from a family of Hankels $H(\lambda)$. Here we ask, e.g., that the family be C^k w.r.t. λ if $H(\lambda)$ is C^k w.r.t. λ and that the family $(A(\lambda), B(\lambda), C(\lambda))$ be minimal for every value λ. This, of course, is not possible, due to the non-existence of continuous canonical forms. If $k=\mathbb{R}$, M. Hazewinkel [10] has produced a lovely example over $X = S^1$, involving essentially the Mobius strip with X as its equator. For the complexes, one still does not have continuous canonical forms (see the survey [7]). The issue is quite simple : if one chooses the canonical observable realization of $H(\lambda)$, then the only remaining obstruction to finding a minimal realization is that of choosing, continuously in λ, a basis for the reachable subspace. Thus, as in the problem of combing the hair on a tennis ball or of choosing an orientation on "the surface which all the world knows", this obstruction lies in the topology of Σ. Over $\mathbb{R}$, Hazewinkel computed enough to show that $\pi_1(\Sigma) \neq (0)$. Over $\mathbb{C}$, the author and N.E. Hurt have shown $\pi_2(\Sigma) \neq (0)$.

However, something actually can be said, if we are willing to speak geometrically. That is, starting with c) we can, provided $\operatorname{rk} H(\lambda) = \text{constant}$, represent $H(\lambda)$ by a generalized family. Thus, we can realize $H(\lambda)$ by, in lieu of a matrix-valued function $A(\lambda)$, an endomorphism A of a vector bundle Q (the state bundle), sections b_i of Q, and dual sections c_j. By standard techniques, this generalized form of a) corresponds to a system over R whose state-module is projective but not necessarily free. Thus, slightly more general forms of a) and b) correspond precisely to the data in c).

Example 1.1 ([5]) - Suppose $H(\lambda)$ is polynomial in $\lambda \in \mathbb{R}^N$ and of constant rank, say n. Thus, H induces a map

$$H : \mathbb{R}^N \to \mathcal{H} \overset{\eta^{-1}}{\to} \Sigma$$

and, since η^{-1} is easily shown to be algebraic, a map to Σ. Therefore (by pulling back the universal family on Σ), H induces on $\mathbb{R}^N$ a vector bundle Q of rank n, a map $A : Q \to Q$, sections b_i of Q, and co-sections c_i. By the validity of the Serre conjecture, Q is trivial ; thus, H can actually be realized by matrix-valued polynomials $(A(\lambda), B(\lambda), C(\lambda))$, minimal at each point.

Example 1.2. ([4], [7]) - Here the ring is not Noetherian, consider $R = \ell^1$, the complex-valued L^1-functions on $\mathbb{Z}$. We suppose given a Hankel matrix (h_{i+j-1}) of ℓ^1 functions and ask for an ℓ^1-realization, minimal over ℓ^1. First of all, consider the Fourier transform,

$$\hat{H} : S^1 \to \mathcal{H} \overset{\eta^{-1}}{\to} \Sigma \quad .$$

The condition that $\hat{H}$ have its range in $\mathcal{H}$ is natural both from the point of view of system theory and of harmonic analysis. System theoretically, it is just the condition that the dual of any minimal realization of H is also minimal (Kalman Duality), while in terms of harmonic analysis it is one of the equivalent conditions in Wiener's Tauberian Theorem. Assuming this, $\hat{H}$ therefore induces a family on S^1 and, since η^{-1} is algebraic, the coefficients of this family are in the range of the Fourier transform. On the other hand, the state bundle $\hat{Q}$ on S^1 is trivial since the complex Grassmanns are simply connected, but what remains to be checked is that Q can be trivialized "within the range of the Fourier transform". This, of course, is possible, by the Docquier-Grauert Theorem. In summary, any Hankel over ℓ^1, such that $\hat{H}$ has constant rank, can be realized by a minimal triple (A, B, C) over ℓ^1.

However, for even dimensional spheres the corresponding statements are false. Indeed, by combining Bott Periodicity with some of the crude computations made in [3], one can produce non-trivial families on any S^{2n}. This shows how the topology of the map $X \to \Sigma$ induced by H, can actually affect some of the system theory involved. Of course, in the scalar input-output case, the existence of the rational canonical form eliminates such complications. This makes the non-trivial topology of Σ all the more fascinating. That is, R. Brockett has shown ([1]) that Σ splits into $n+1$ components, each separated by the Cauchy index of the transfer function. Therefore, if the McMillan degree is 2, there are 3 components and it is known [1] that 2 of these are diffeomorphic to $\mathbb{R}^4$, while the component corresponding to Cauchy index 0 is diffeomorphic to $\mathbb{R}^3 \times S^1$. One generator for $\pi_1(\mathbb{R}^3 \times S^1) = \mathbb{Z}$ is the family,

(1.1) $$g_\theta(s) = \frac{(\cos\theta)\, s^2 + \sin\theta}{s^2 + 1} .$$

To see that g_θ is not null-homotopic, compute its Hankel :

(1.2) $$H : S^1 \to \Sigma \to 0(2)\ , \quad \text{where}$$

$$H(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

is the component of $0(2)$ consisting of the reflections. The relationship between g_θ and the group $0(1,1) \subset GL(2,\mathbb{R})$ will be seen presently.

II.- SOME ARITHMETIC ASPECTS OF LINEAR SYSTEM THEORY.

Motivated by questions of network synthesis [14] and by stability questions [2] we consider for $R = \mathbb{R}[x_1, \ldots, x_N]$, or for R a ring of realvalued functions on a connected space, or for $R = \mathbb{Z}$:

Question : Given a symmetric transfer function $g(s)$ defined over R, does there exist an internally symmetric, minimal realization of g over R ?

By an internally symmetric, minimal realization is meant a minimal triple (A, B, C) satisfying

$$I_{p,q} A = {}^t A I_{p,q} \quad , \quad I_{p,q} B = {}^t C \quad , \tag{2.1}$$

where $I_{p,q}$ is the standard quadratic form with signature $p-q$. Note that $p-q$ is necessarily the signature of the associated Hankel and is thus an external invariant. According to Youla and Tissi, these always exist, by Sylverter's Theorem, over $\mathbb{R}$ and the subgroup which transforms one such realization to all others is $0(p,q) \subset GL(n,\mathbb{R})$.

In particular, for delay-time systems with commensurate delays, a necessary and sufficient condition for the existence of such realizations can be given [6]. That is, following E. Kamen, one can represent such a system as a symmetric Hankel matrix whose entries are polynomials in a single variable x. By the remark above, a necessary condition is a rather familiar one, viz. rank $H(x) = n$ is constant. This being assumed, the techniques of section 2 reduce the question to the validity of Sylvester's Theorem over $R = \mathbb{R}[x]$, which has been recently proved by Harder ! This quadratic analogue of Serre's conjecture is false over $\mathbb{R}[x_1, x_2]$, but (to my knowledge) the question for delay-time systems with non-commensurate delays is still open.

As always, a sharper condition can be given in the scalar case. Indeed, one can show quite readily :

Theorem 2.1.- For scalar $g(s)$, a necessary and sufficient condition for g to admit an internally symmetric, minimal realization is that the Hankel be conjugate to $I_{p,q}$.

Now, over $\mathbb{Z}$, this leads immediately to counterexamples in the indefinite case. That is, one knows that for indefinite non-degenerate quadratic forms over $\mathbb{Z}$, the complete invariants are given by the rank (= McMillan degree), the signature (= the Cauchy index, by the Hermite-Hurwitz Theorem), and the type. For example, the standard forms $I_{p,q}$ are all of type I, since they admit vectors of odd norm. However, the Hankel matrix (see (1.3)) of $g_{\pi/2}(s) = \frac{1}{s^2}$ is of type II ; i.e., it admits no vector of odd norm. More generally, we have

Corollary 2.2.- No (Hamiltonian) transfer function $g(s)$, satisfying $g(s) = g(-s)$ admits an internally symmetric realization.

The definition case is far more subtle and will be presented elsewhere. In the next and final section, we will present more applications of Theorem 2.1 to families of linear systems.

Turning to the Hamiltonian case, suppose one has, over $\mathbb{R}$, $g(s) = {}^t g(-s)$. Then it is well known that there exists an (infinitesimally) Hamiltonian realization, i.e. a minimal realization (A, B, C) satisfying

$$JA = -{}^tAJ \ , \quad JB = -{}^tC$$

where J is the standard symplectic form, and any 2 such realizations are related by $Sp(n,\mathbb{R}) \subset GL(n,\mathbb{R})$. Here, the obstructions are easy to obtain

Theorem 2.3.- If R is projective-free and if $\operatorname{rank} H(x)$ is constant, then any Hamiltonian transfer function has an infinitesimally Hamiltonian realization.

Here, too, we have the obvious converse. If R is a ring of functions, then $\operatorname{rank} H(x)$ constant is first used to obtain a generalized realization, while projective-free is used to obtain an honest realization of the form b) (section 1). Next, the rank condition on the Hankel is used to deduce the statement usually implied by Kalman duality, reducing the proof of the theorem to a classification of symplectic forms over R . Finally, one obtains a "Darboux Theorem" by using projective free.

For $R = \mathbb{Z}$, we replace $H(x)$ by $H \bmod(p)$ for primes p and use Sontag's criterion for "split" realizations [13] to implement the duality argument.

Q.E.D.

III.- THE HANKEL BUNDLES.

Let $\Sigma = \mathrm{Rat}(n)$ denote the (moduli) space of scalar input-output systems of McMillan degree n, over $\underline{k} = \mathbb{R}$, and let $R(p,q)$ denote the component of $\mathrm{Rat}(n)$ of transfer functions of Cauchy index $p-q$. By the Hermite-Hurwitz Theorem, for each $g \in R(p,q)$ the signature of the associated Hankel is $p-q$. Thus, in contrast to the situation over $\mathbb{Z}$, the indefinite case is far more interesting. That is, $R(n,0) \simeq R(0,n) \simeq \mathbb{R}^{2n}$ while the topology of the intermediate $R(p,q)$ is still not completely known. Motivated by Theorem 2.1, we form

Definition 3.1.- The positive (resp. negative) Hankel bundle H_+ on $R(p,q)$ is the vector bundle whose fiber over $g \in R(p,q)$ is the positive (resp. negative) eigenspace of the Hankel H_g.

For example, for the ring $C(R(p,q))$ and the universal system on $R(p,q)$, Theorem 2.1 implies that there exists a network symmetric realization only if the positive and the negative Hankel bundles are trivial. To put this another way, continuous, network symmetric canonical forms exist only if the Hankel bundles are trivial. That is, for $p-q$ fixed, set $\tilde{R}(0,q)$ to be the (affine) variety of triples (A,b,c) satisfying (2.1). Then, the moduli, or quotient space, $\tilde{R}(0,q)/0(p,q)$ is simply $R(p,q)$ and the existence of canonical forms is precisely the existence of a cross-section of the map

$$\tilde{R}(p,q) \to R(p,q) \quad . \tag{3.1}$$

Thus, if $p = n$ or $q = n$, a cross-section exists, since $0(n)$ is a retract of $GL(n,\mathbb{R})$. Such cross-sections have been known in the litterature for almost half a century, viz. the Cauer and Foster canonical forms for network synthesis - which exist only for the RL or RC case. In the indefinite (or RLC) case, T.E. Duncan and the author have shown that no continuous canonical forms exist, even in the scalar case. The family g_θ of (1.1) provides an easy counterexample.

Example 3.1.- The positive eigenspace of H_θ as a function of θ is (by the trigonometric addition formulae) the line in $\mathbb{R}^2$ which makes an angle $\theta/2$ with the x-axis. In particular, one cannot choose a continuous basis for H_+ over S^1. To put this another way, the induced map

$$S^1 \to \mathbb{R}\mathbb{P}^1 \simeq S^1$$

has degree 1 and, therefore, the line bundle H_+ is non-trivial.

This can also be seen by analyzing (1.2), we have a map

(1.2)' $$S^1 \to 0(2) \to 0(2)/\ \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathbb{R}\mathbb{P}^1 .$$

This is a special case of the map

(3.2) $$R(p, q) \to Gass(p, n) \simeq GL(n,\mathbb{R})/0(p, q)$$

induced by the family H_+ of p-dimensional subspaces of (the state-space) $\mathbb{R}$, which reflects the fact the Hankel bundles are associated to the $0(p, q)$ - bundle (3.1) and thus (again) the fact that non-triviality of H_+ and H_- are obstructions to the existence of continuous canonical forms.

<u>Theorem 3.2</u>.- In the indefinite case, the Hankel bundles are always non-trivial. Thus, network symmetric forms exist only in the definite cases.

This result, together with extensions to the multivariable case appear in [6]. Some of the global properties of the space of scalar Hamiltonian systems, such as the number of connected components and existence of canonical forms, have appeared implicitly in the work of R.W. Brockett and are also known to the author. The multivariable Hamiltonian case, however, appears to be novel.

-:-:-:-

REFERENCES

[1] R.W. BROCKETT.- Some Geometric Questions in the Theory of Linear Systems. <u>IEEE Trans. Automatic Control</u>, AC-21, 1976.

[2] R.W. BROCKETT and J.C. WILLEMS.- Average Value Stability Criteria for Symmetric Systems, <u>Richerche di Aut</u>., Vol.4, 1973.

[3] C.I. BYRNES.- On the Moduli Space for Linear Systems, in "The 1976 Ames Research Center (NASA) Conference on Geometric Control Theory", <u>Math. Sci. Press</u>, Brookline, MA, 1977.

[4] C.I. BYRNES.- On Classification Space Techniques in Realization Theory. AMS Notices, 1978.

[5] C.I. BYRNES.- On the Control of Certain, Deterministic, Infinite Dimensional Systems by Algebro-Geometric Techniques. To appear in Amer. J. Math.

[6] C.I. BYRNES and T.E. DUNCAN.- On Certain Topological and Geometric Invariants Arising in System Theory, To appear.

[7] C.I. BYRNES and P.L. FALB.- Applications of Algebraic Geometry in System Theory. To appear in Amer. J. of Math., April, 1979.

[8] C.I. BYRNES and N.E. HURT.- On the Moduli of Linear Dynamical Systems. Adv. in Math., Studies in Analysis, Vol.4, 1978 and in Mathematical Methods in System Theory, MIR, Moscow 1979 (Russian).

[9] J.M.C. CLARK.- The Consistent Selection of Local Coordinates in Linear System Identification. Proc. JACC, Purdue, 1976.

[10] M. HAZEWINKEL.- Moduli and Canonical Forms for Linear, Dynamical Systems, II : The Topological Case, Math. System Theory.

[11] M. HAZEWINKEL and R.E. KALMAN.- On Invariants, Canonical Forms, and Moduli for Linear, Constant, Finite-Dimensional Systems. Lecture Notes in Econ. -Math. System Theory, Vol. 131, Springer-Verlag, 1976.

[12] T. KOGA.- Synthesis of Finite Passive n-parts with Prescribed Two-Variable Reactance Matrices. IEEE Trans. Cir. Theory, CT-13, 1966.

[13] E. SONTAG.- On Split Realizations of Response Maps over Rings. Inf. and Control, 37 (1978).

[14] D.C. YOULA.- The Synthesis of Networks Containing Lumped and Distributed Elements, Networks and Switching Theory, 11, 1968.

-:-:-:-

Christopher I. BYRNES
Department of Mathematics and
Division of Applied Sciences
Harvard University
CAMBRIDGE, MASSACHUSSETS 02138