Astérisque

ARTHUR J. KRENER

Boundary value linear systems

Astérisque, tome 75-76 (1980), p. 149-165

<http://www.numdam.org/item?id=AST_1980__75-76__149_0>

BOUNDARY VALUE LINEAR SYSTEMS

by

Arthur J. KRENER

-:-:-:-

ABSTRACT.- The standard linear time varying system is modified by replacing the initial conditions on the state by boundary conditions, the result is an noncausal system. The concepts of input/output map, dual system, weighing pattern, controllability, observability, minimal realization and linear quadratic regulation for such systems are discussed.

I.- INTRODUCTION.-

As a model for many causal input-output relations, an initial value linear system (1.1), (1.2) and (1.3) is appropriate.

$$\dot{x} = Ax + Bu \tag{1.1}$$

$$y = Cx \tag{1.2}$$

$$x(0) = x^{o} \quad . \tag{1.3}$$

The input u is $\ell \times 1$, the output y is $m \times 1$ and the state x is $n \times 1$. The matrices A, B and C are of the appropriate dimensions and possibly time varying.

Such a system is causal, i.e., future inputs $u(s)$, $s > t$ do not affect past outputs, $y(s)$, $s < t$. Past inputs $u(s)$, $s < t$ affect future outputs through the present state $x(t)$. The current state $x(t)$ is the memory of the system. This causality is reflected mathematically by the ordinary differential equation with initial value data. Given $x(t)$ and $u(s)$ for $s \geq t$ then $x(s)$ and $y(s)$ for $s \geq t$ are uniquely determined.

We would like to discuss a generalization of (1.1), (1.2) and (1.3) which does not assume causality. This generalization we call a boundary value linear system.

Instead of the initial conditions (1.3), we assume $x(t)$ satisfies the boundary conditions

$$V^0 x(0) + V^T x(T) = v \tag{1.4}$$

where V^0 and V^T are $n \times n$ and v is $n \times 1$. If $V^0 = I$, $V^T = 0$ and $v = x^0$ then (1.4) is equivalent to (1.3). Before we study the new system (1.1), (1.2) and (1.4) in some detail, let us briefly consider some situations where it might arise.

Consider a distributed parameter linear system, for example,

$$\frac{\partial}{\partial t} u(t,x) = \mathbf{L}u(t,x) + f(t,x) \tag{1.5}$$

where $t \geq 0$ is time as before but $0 \leq x \leq L$ is now an independent scalar variable representing a spatial coordinate. The state is $u(t,x)$ a scalar quantity which evolves according to the partial differential (1.5) where L is a linear differential operator in the space domain and $f(t,x)$ is a spatially and temporally distributed forcing term. A typical L would be the Laplacian in which case (1.5) is the standard heat equation and $f(t,x)$ represents heat sources or sinks. Of course, appropriate initial data for $u(0,x)$ and boundary data $u(t,0)$, $u(t,L)$ are needed to specify a solution. If we look for steady state solutions of (1.5), i.e., $u(t,x) = u(x)$, $f(t,x) = f(x)$ then

$$Lu(x) = -f(x)$$

with boundary conditions on $u(0)$ and $u(L)$. After appropriate change of notation we obtain (1.1), (1.2) and (1.4).

Another example is in delay differential systems,

$$\dot{x} = A_1 x(t) + A_2 x(t-T) + Bu(t)\,.$$

If one seeks solutions of period T, $x(t) = x(t-T)$ then this becomes

$$\dot{x} = (A_1 + A_2)\, x(t) + Bu(t)$$

$$x(0) - x(T) = 0\,.$$

As a last example consider the problem of smoothing data, suppose signals are generated by (1.1) where $u(t)$ a white Gaussian driving noise. Suppose all such signals begin and end in a similar fashion (1.4). The boundary data v might be known exactly or it might be assumed to be a Gaussian variable with known mean and covariance. The smoothing problem is given the observations $z(t)$, $0 \leq t \leq T$

$$z(t) = y(t) + w(t)$$

corrupted by a white Gaussian observation noise $w(t)$, estimate the signal $x(t)$, $0 \leq t \leq T$.

II. - THE INPUT/OUTPUT MAP AND DUAL SYSTEMS.

From a system theoretic point of view, (1.1), (1.2) and (1.3) defines a map from the input $u(t)$ to the output $y(t)$. More precisely the map is defined from pairs of initial state and input $(x^0, u(t))$ to pairs of final state and output $(x^T, y(t))$. We restricted our alteration to the system on a fixed interval $[0, T]$. This map is given by the well-known variation of constants formula,

$$y(t) = C(t)\,\Phi(t, 0)\,x^0 + \int_0^t C(t)\,\Phi(t, s)B(s)u(s)\,ds$$

$$x^T = x(T) = \Phi(T, 0)x^0 + \int_0^T \Phi(T, s)B(s)u(s)\,ds$$

where $\Phi(t, s)$ is $n \times n$ matrix satisfying

$$\frac{\partial}{\partial t}\,\Phi(t, s) = A(t)\,\Phi(t, s)$$

$$\Phi(t, t) = I \;.$$

If we replace the initial condition (1.3) by the boundary condition (1.4) then the variation of constants technique is generalized to techniques involving Green's function. At this point we must assume the homogeneous problem is well-posed, namely, that there exists no nontrivial solution of

$$\dot{x} = Ax \tag{2.1}$$

$$V^0 x(0) + V^T x(T) = 0 \;. \tag{2.2}$$

This is always satisfied for initial value problems ; for boundary value problems it will hold iff the matrix

$$F = V^0 + V^T \Phi(T, 0)$$

is invertible. This is a necessary and sufficient condition to guarantee the existence of a unique solution to (1.1) and (1.4) for any boundary data v and sufficient regular input $u(t)$, for example, if each component of $u(t)$ is square integrable on $[0, T]$. For a fuller discussion of this see [4].

The Green's function $G(t, s)$ for (1.1) and (1.4) is the $n \times n$ matrix valued function defined by

$$G(t, s) = \begin{cases} \Phi(t, 0) F^{-1} V^0 \Phi(0, s), & s < t, \\ -\Phi(t, 0) F^{-1} V^T \Phi(T, s), & s > t. \end{cases}$$

It follows [4] that $G(t, s)$ satisfies

$$\frac{\partial}{\partial t} G(t, s) = A(t) G(t, s) \qquad t \neq s \tag{2.3}$$

$$G(s^+, s) - G(s^-, s) = I \tag{2.4}$$

$$V^0 G(0, s) + V^T G(T, s) = 0 . \tag{2.5}$$

The solution of (1.1) and (1.4) is given by

$$x(t) = \Phi(t, 0) F^{-1} v + \int_0^T G(t, s) B(s) u(s)\, ds , \tag{2.6}$$

and the output is given by

$$y(t) = C(t) \Phi(t, 0) F^{-1} v + \int_0^T C(t) G(t, s) B(s) u(s)\, ds . \tag{2.7}$$

This is a direct generalization of the initial value situation because for these the Green's function is given by

$$G(t, s) = \begin{cases} \Phi(t, s) & s \leq t \\ 0 & s > t . \end{cases}$$

For boundary value systems the other part of the output is not the final state but rather the boundary values left unspecified by (1.4). Let W^0, W^T be $n \times n$ matrices such that the matrix

$$\begin{bmatrix} V^0 & V^T \\ W^0 & W^T \end{bmatrix} \tag{2.8}$$

is of full rank $2n$. The system (1.1), (1.2) and (1.4) describes a map from $(v, u(t))$ to $(w, y(t))$ where

$$w = W^0 x(0) + W^T x(T) \quad . \tag{2.9}$$

We denote this map by Σ,

$$\Sigma : (v, u(t)) \to (w, y(t)) \; .$$

If we assume that $u(t) \in L_2^{\ell}[0, T]$, the space of $\ell \times 1$ vector valued functions, each component of which is L_2 measurable on $[0, T]$, then

$$\Sigma : \mathbb{R}^{n \times 1} \times L_2^{\ell \times 1}[0, T] \to \mathbb{R}^{n \times 1} \times L_2^{m \times 1}[0, T] \; .$$

There is an adjoint or dual map Σ',

$$\Sigma' : \mathbb{R}^{1 \times m} \times L_2^{1 \times m}[0, T] \to \mathbb{R}^{1 \times n} \times L_2^{1 \times \ell}[0, T] \; .$$

Let $\zeta, \xi \in \mathbb{R}^{1 \times n}$, $\mu \in L_2^{1 \times m}[0, T]$ and $\nu \in L_2^{1 \times \ell}[0, T]$, then

$$\Sigma' : (\zeta, \mu(t)) \to (\xi, \nu(t))$$

is characterized by the relation

$$\xi v + \int_0^T \nu(t)\, u(t)\, dt = \zeta w + \int_0^T \mu(t)\, y(t)\, dt \; .$$

This implies that

$$\begin{aligned} \xi = \zeta\, & (W^0 + W^T \Phi(T, 0)) F^{-1} \\ & + \int_0^T \mu(t)\, C(t)\, \Phi(t, 0) F^{-1} dt \end{aligned}$$

and

$$\begin{aligned} \nu(s) = \zeta & (W^0 G(0, s) + W^T G(T, s)) B(s) \\ & + \int_0^T \mu(t)\, C(t)\, G(t, s) B(s)\, dt \; . \end{aligned}$$

The input/output map Σ is realized by the boundary value linear system (1.1), (1.2), (1.4) and (2.9). We can ask if there is a similar realization of Σ'. The answer is yes, but before discussing it we review some properties of $G(t, s)$ viewed as a function of s, see [4] for details. $G(t, s)$ satisfies the following

$$(2.10)\qquad \frac{\partial}{\partial s} G(t,s) = -G(t,s)A(s)$$

$$(2.11)\qquad G(t,t^+) - G(t,t^-) = -I$$

and for some invertible $n\times n$ matrix $L(t) = -\Phi(t,0)F^{-1}$

$$(2.12)\qquad G(t,0) = -L(t)V^0, \quad G(t,T) = L(t)V^T.$$

Suppose we define an $1\times n$ vector $\lambda(s)$ by

$$\lambda(s) = \zeta(W^0G(0,s) + W^TG(T,s)) + \int_0^T \mu(t)C(t)G(t,s)\,dt$$

and $n\times n$ matrices M^0, M^T, N^0, N^T by

$$\begin{bmatrix} V^0 & V^T \\ W^0 & W^T \end{bmatrix} \begin{bmatrix} -M^0 & -N^0 \\ M^T & N^T \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

then it is a straightforward verification using (2.10), (2.11) and (2.12) to show that

$$(2.13)\qquad \dot{\lambda} = -\lambda A - \mu C$$

$$(2.14)\qquad \nu = \lambda B$$

$$(2.15)\qquad \lambda(0)M^0 + \lambda(T)M^T = \zeta$$

$$(2.16)\qquad \lambda(0)N^0 + \lambda(T)N^T = \xi .$$

This is a boundary value linear system with input μ, output ν, state λ the boundary conditions (2.15) and unspecified boundary values (2.16).

For the initial value problem this reduces to the familiar system

$$\begin{aligned} \dot{\lambda} &= -\lambda A - \mu C \\ \nu &= \lambda B \\ \lambda(T) &= \zeta \quad \text{(specified)} \\ \lambda(0) &= \xi \quad \text{(free)} . \end{aligned}$$

III. - <u>CONTROLLABILITY AND OBSERVABILITY</u>.

As we saw in the last section, a boundary value linear system defines a mapping for input and specified boundary values to output and unspecified boundary values as given by (2.6), (2.7) and (2.9). The output (2.7) is a sum of a term depending on v and a term depending on $u(t)$. We concentrate our attention on the homogeneous boundary situation $(v=0)$ for we can obtain the nonhomogeneous output by merely adding

$$\Phi(t,0)\,F^{-1}v\ .$$

Therefore, throughout this section we shall be interested in the input/output map defined by (1.1), (1.2) and (1.4) for $v = 0$. Abusing notation, we denote this map also by Σ

$$\Sigma : u(t) \to y(t) = \int_0^T C(t)G(t,s)B(s)u(s)\,ds\ . \tag{3.1}$$

We refer to the kernel of this map

$$\Sigma(t,s) = C(t)\,G(t,s)\,B(s) \tag{3.2}$$

as the <u>weighing pattern</u> or <u>impulse response</u> of the system, and refer to (1.1), (1.2) and (1.4) as a <u>realization</u> of Σ.

Needless to say, many boundary value linear systems can realize the same input/output map and weighing pattern. Minimality refers to the question of whether (1.1), (1.2) and (1.4) are of the minimal state dimension necessary to realize $\Sigma(t,s)$. This question can be broken up into parts ; one concerned with input-state minimality and the other with state-output minimality. The former is the question of controllability and the latter that of observability.

The system (1.1), (1.2), (1.4) and (2.9) is said to be controllable if for any v, $w \in \mathbb{R}^n$ there exists a control $u(.)$ and solution $x(.)$ of (1.1) satisfying (1.4) and (2.8). This definition is a straightforward generalization of that for initial value linear systems in which case $v = x(0)$ and $w = x(T)$. Notice that controllability is a property of (1.1) and is certainly independent of the output (1.2) but it is also independent of the boundary conditions as long as (1.4) is well-posed and (2.8) is of full rank. Therefore, a boundary value linear system is controllable iff the corresponding initial value linear system obtained by replacing (1.4) by (1.3) is controllable on $[0, T]$. See [3] for a definition of this.

Define the <u>controllability Gramian</u>

$$M(t, s) = \int_s^t \Phi(s, \tau) B(\tau) B'(\tau) \Phi'(s, \tau)\, d\tau$$

where prime denotes transpose. A necessary and sufficient condition for controllability on $[0, T]$ is that $M(T, 0)$ be of full rank n. See [2] or [3] for details. For autonomous systems this is equivalent to the condition that the matrix

$$(B \quad AB \; . \; . \; . \; A^{n-1} \quad B)$$

be of full rank n.

A boundary value linear system is <u>observable</u> if knowledge of the input $u(t)$ and output $y(t)$ on $[0, T]$ is sufficient to uniquely determine the boundary condition v. This is a direct generalization of observability for initial value systems. Moreover, this definition does not depend on the particular boundary conditions (1.4) merely that they be well-posed. Therefore, a boundary value system is observable iff the corresponding initial value system is.

Define the <u>observability Gramian</u>

$$N(t, s) = \int_s^t \Phi'(\tau, s) C'(\tau) C(\tau) \Phi(\tau, s)\, d\tau \; .$$

A necessary and sufficient condition for observability is that $N(T, 0)$ be of full rank [3]. For autonomous systems this is equivalent to the condition that the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

be of full rank n.

Controllability and observability are dual concepts ; the controllability of a system is equivalent to the observability of its dual. Again we refer the reader to [2] or [3] for details.

Suppose the boundary value linear system fails to be both controllable and observable ; can we reduce it to a system of smaller state dimension which is and which has the same weighing pattern ?

If the system is not controllable then under certain conditions (see [3]) there is a time varying change of coordination in the state space which converts (1.1), (1.2) and (1.4) to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} \\ 0 \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \tag{3.3}$$

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.4}$$

$$\begin{pmatrix} V_1^0 & V_2^0 & V_1^T & V_2^T \end{pmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_1(T) \\ x_2(T) \end{bmatrix} = v \tag{3.5}$$

where x_1 and x_2 are vectors of dimension $k \times 1$ and $(n-k) \times 1$ and the matrices are of the appropriate dimensions. The first of these x_1 represents the controllable modes of the system and the second x_2 the uncontrollable modes.

This induces a similar decomposition

$$\Phi(t,s) = \begin{bmatrix} \Phi_{11}(t,s) & \Phi_{12}(t,s) \\ 0 & \Phi_{22}(t,s) \end{bmatrix} .$$

The form (3.3) of A , the differential equation and initial condition for Φ imply that

$$\Phi_{21}(t,s) = 0$$

$$\frac{\partial}{\partial t}\,\Phi_{11}(t,s) = A_{11}\,\Phi_{11}(t,s) \qquad\qquad \Phi_{11}(t,t) = I$$

and

$$\frac{\partial}{\partial t}\,\Phi_{22}(t,s) = A_{22}\,\Phi(t,s) \qquad\qquad \Phi_{22}(t,t) = I\ .$$

We also have similar decompositions

$$V^0 = (V_1^0 \quad V_2^0)$$

$$V^T = (V_1^T \quad V_2^T)$$

$$F = (F_1 \quad F_2) = \Big((V_1^0 + V_1^T\,\Phi_{11}(T,0) \qquad V_1^0 + V_1^T\,\Phi_{12}(T,0) + V_{22}^T\,\Phi_{22}(T,0) \Big)\ .$$

There exists an invertible $n \times n$ matrix P decomposed into $k \times n$ and $(n-k) \times n$ submatrices

$$P = \begin{bmatrix} P_1 \\ \\ P_2 \end{bmatrix}$$

such that

$$PF = \begin{bmatrix} P_1 F_1 & P_1 F_2 \\ & \\ 0 & P_2 F_2 \end{bmatrix}\ .$$

Since PF is invertible so are $P_1 F_1$ and $P_2 F_2$.

The boundary value problem

(3.6) $$\dot{x}_1 = A_{11}\,x_1$$

(3.7) $$P_1 V_1^0 x_1(0) + P_1 V_1^T x_1(T) = 0$$

is well-posed because

$$P_1 V_1^0 + P_1 V_1^T\,\Phi_{11}(T,0) = P_1 F_1$$

is invertible.

Define a boundary value linear system

(3.8) $$\dot{x}_1 = A_{11}\, x_1 + B_1\, u$$

(3.9) $$y = C_1\, x_1$$

(3.10) $$P_1 V_1^0 x_1(0) + P_1 V_1^T x_1(T) = v_1$$

where v_1 is $k \times 1$ vector. To see that this system has the same weighing pattern as the original system (3.3), (3.4) and (3.5), let $x_1(t)$ be the unique solution of (3.8) and (3.10) for $v_1 = 0$ and some $u(t)$.

Let $x(t)$ be defined by

$$x(t) = \begin{bmatrix} x_1(t) \\ \\ 0 \end{bmatrix}$$

then this is the necessarily unique solution of the original system for $v = 0$ and the same input $u(t)$. It follows that the two outputs are the same, hence the weighing patterns agree.

For future reference we shall show that

(3.11) $$\dot{x}_2 = A_{22}\, x_2$$

(3.12) $$P_2 V_2^0\, x_2(0) + P_2 V_2^T\, x_2(T) = 0$$

is also well-posed. Suppose $x_2(t)$ is a nontrivial solution, since (3.6) and (3.7) is well-posed there exists a unique solution $x_1(t)$ of

$$\dot{x}_1 = A_{11}\, x_1 + A_{12}\, x_2$$

$$P_1 V_1^0\, x_1(0) + P_1 V_1^T\, x_1(T) = -P_1 V_2^0 x_2(0) - P_1 V_2^T x_2(T)\ .$$

If we define

$$x(t) = \begin{bmatrix} x_1(t) \\ \\ x_2(t) \end{bmatrix}$$

then it is a nontrivial solution of (3.3) and (3.5) which was assumed to be well-posed.

In a dual fashion, a system that is not observable can be reduced to one that is. Under certain conditions [3], there exists a time varying change of coordinates taking (1.1), (1.2) and (1.4) to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{3.13}$$

$$y = C_2 x_2 \tag{3.14}$$

$$(V_1^0 \;\; V_2^0 \;\; V_1^T \;\; V_2^T) \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_1(T) \\ x_2(T) \end{bmatrix} = v$$

where x_1 and x_2 are vectors of dimension $k \times 1$ and $(n-k) \times 1$ and the matrices are of compatible dimensions. The first of these vectors x_1 represents the unobservable modes of the system and the second x_2 the observable modes. We proceed as before and define a boundary value linear system

$$\dot{x}_2 = A_{22} x_2 + B_2 u \tag{3.16}$$

$$y = C_2 x_2 \tag{3.17}$$

$$P_2 V_2^0 x_2(0) + P_2 V_2^T x_2(T) = V_2 \tag{3.18}$$

where v_2 is a $(n-k) \times 1$ vector. To see that this new system has the same weighing pattern as the original (3.13), (3.14) and (3.15), let $x_2(t)$ be the unique solution of (3.16) and (3.18) for $v_2 = 0$ and some $u(t)$. Let $x_1(t)$ be the solution of

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u$$

$$P_1 V_1^0 x_1(0) + P_1 V_1^T x_1(0) = -P_1 V_2^0 x_2(0) - P_1 V_2^T x_2(T)$$

which exists since (3.6) and (3.7) is well-posed. Then

$$x(t) = \begin{bmatrix} x_1(t) \\ \\ x_2(t) \end{bmatrix}$$

is the solution of the original system and clearly from (3.14) the output is the same.

From the above, the reader can see that the minimality of a boundary value linear system is equivalent to the minimality of the corresponding initial value linear system. For time-varying systems, this can be complicated and we refer the reader to [2] and [3] for full details.

IV.- THE BOUNDARY VALUE REGULATION.

In this section we discuss the solution of the linear quadratic regulation where boundary data rather than initial data is specified. Consider the problem of minimizing, by choice of control $u(t)$,

$$\int_0^T x'Qx+u'Ru\,dt+x(0)'P_0x(0)+x'(T)P_Tx(T) \tag{4.1}$$

where v is specified ans $x(t)$ the solution of

$$\dot{x} = Ax + Bu \tag{4.2}$$

$$V^0x(0)+V^Tx(T)=v\ , \tag{4.3}$$

where $Q(t)$, P_0 and P_T are positive semidefinite and $R(t)$ is positive definite.

We shall not consider the most general problem of this type but instead we shall assume that (4.3) splits into k constraints involving only $x(0)$ and $(n-k)$ constraints involving only $x(T)$. In this case by making a time-varying change coordinate, the boundary constants (4.3) can always be put in the form

$$x_i(0)=v_i\ ,\ i=1,\dots,k\ ;\ x_i(T)=v_i\ , i=k+1,\dots,n\ . \tag{4.4}$$

For any pair v and $u(t)$, define $J(v,u(t))$ to be the value of (4.1) subject to (4.2) and (4.4), and define $J^*(v)$ to be the minimum of $J(v,u(t))$ over all $u(t)$. We are assuming such a minimizing control denoted by $u^*(v)$ exists for each v_i. This can be shown by standard techniques, see Berkovitz [1].

Lemma.- $J^*(v)$ is a quadratic form in v and $u^*(v)$ is a linear function of v.

Proof : To show J^* is a quadratic form we must show it is homogeneous of degree 2 and satisfies the parallelogram identity. J is clearly a quadratic form in $(v, u(t))$ and hence has these properties.

Let $\lambda \neq 0$ be a scalar item

$$\begin{aligned} J^*(\lambda v) &\leqslant J(\lambda v, \lambda u^*(v)) \\ &= \lambda^2 J(v, u^*(v)) \\ &= \lambda^2 J(v, \lambda^{-1} u^*(\lambda v)) \\ &= J(\lambda v, u^*(\lambda v)) \\ &= J^*(\lambda v) \end{aligned}$$

so $J^*(\lambda v) = \lambda^2 J^*(v)$ and $u^*(\lambda v) = \lambda u^*(v)$.

Let v^1, v^2 be different boundary conditions

$$\begin{aligned} J^*(v^1)+ J^*(v^2) &= \frac{1}{4} J^*(2v^1)+ \frac{1}{4} J^*(2v^2) \\ &\leqslant \frac{1}{4} J^*(2v^1, u^*(v^1+ v^2)+ u^*(v^1- v^2)) \\ &\quad + \frac{1}{4} J^*(2v^2, u^*(v^1+ v^2)- u^*(v^1- v^2)) \\ &= \frac{1}{4} J((v^1+ v^2)+ (v^1- v^2), u^*(v^1+ v^2)+ u^*(v^1- v^2)) \\ &\quad + \frac{1}{4} J((v^1+ v^2)-(v^1- v^2), u^*(v^1+ v^2)-u^*(v^1-v^2)) \ . \end{aligned}$$

By using the parallelogram identity for J we obtain

$$J^*(v^1)+ J^*(v^2) \leqslant \frac{1}{2} J^*(v^1+v^2)+\frac{1}{2} J^*(v^1-v^2) \ .$$

If we re write this with v^1 replaced by $v^1+ v^2$ and v^2 replaced by $v^1- v^2$ we obtain

$$J^*(v^1+ v^2)+ J^*(v^1- v^2) \leqslant \frac{1}{2} J^*(2v^1)+2J^*(v^2)= 2J^*(v^1)+2J^*(v^2) \ .$$

From this it follows that J^* satisfies the parallelogram identity and

$$u^*(v^1+v^2) = u^*(v^1)+u^*(v^2) \quad .$$

Q.E.D.

From this lemma we see that to solve the boundary value regulator for any v we need only consider it for n linearly independent values of v. Assume $k \geqslant n-k$ and suppose we ignore the constraints on $x(T)$, then the problem can be solved by standard techniques [2]. We fine an $n \times n$ matrix $K(t)$ satisfying the matrix Riccati equation.

$$\dot{K} = -KA - A'K - Q + KBR^{-1}B'K \tag{4.5}$$

and terminal constraint

$$K(T) = P_T \quad . \tag{4.6}$$

If we add the zero quantity

$$\int_0^T \frac{d}{dt}[(x'Kx)dt - x'Kx]_0^T$$

to (4.1) we obtain

$$\int_0^T \| R^{-\frac{1}{2}'} B'Kx + R^{\frac{1}{2}} u \| \, dt + x'(0)\,(K(0)+P_0)\,x(0) \tag{4.7}$$

where $\|.\|$ is the ℓ_2 norm and $R = R^{\frac{1}{2}'} R^{\frac{1}{2}}$. Clearly the optimal control is given by

$$u(t) = -R^{-1}B'Kx(t) \tag{4.8}$$

and we seek to minimize

$$x'(0)(K(0)+P_0)\,x(0) \tag{4.9}$$

subject to

$$x_i(0) = v_i \qquad i=1,\ldots,k \; . \tag{4.10}$$

For $v_1 = 1$, $v_2 = \ldots = v_n = 0$ we solve for the optimal $x(0)$. Using the optimal control given by (4.8) we compute $x(T)$. Let $v_i = x_i(T)$, $i = k+1, \ldots, T$, we have solved the boundary value regulator for that v, since the solution which we have found minimizes (4.1) without the constraints on $v_{k+1}, \ldots, v_n$. Repeat the process for $v_1 = 0$, $v_2 = 1$, $v_3 = \ldots = v_n = 0$ and so on ; in this fashion we have solved the boundary value regulator for k linearly independent boundary values denoted by $v^1, \ldots, v^k$. Let $u^i(t)$ denote the corresponding optimal controls. Since $k \geq n/2$ we are at least half done.

To find additional linearly independent solutions notice we can add to P_T any positive semidefinite matrix of the form

$$\pi = \begin{bmatrix} 0 & 0 \\ & \\ 0 & * \end{bmatrix}$$

While this will change the value of (4.1), it will not change the optimal controls since $x_i(T)$ is constrained for $i = k+1, \ldots, n$. We choose π so that $v^{j'} \pi v^j$ is large for $j = 1, \ldots, k$ and we repeat the above process. If π is chosen sufficiently large, at least one of the new v^i's so generated must be linearly independent of the previous ones. We repeat until the n linearly independent boundary conditions are obtained.

We note in closing that if $n \times n$ matrices L and X are defined by

$$\dot{L} = -A'L + KBR^{-1}B'L$$

$$\dot{X} = AX - BR^{-1}B'(KX+L)$$

$$L(T) = \pi$$

$$X(T) = I$$

then $K + LX^{-1}$ is a solution of the Riccati differential equation (4.5) satisfying

$$K(T) + L(T)X^{-1}(T) = K(T) + \pi$$

so new solutions of the Riccati equation can be computed from old.

-:-:-:-

REFERENCES

[1] L.D. BERKOVITZ.- Optimal Control Theory, Springer-Verlag, New-York, 1974.

[2] R.W. BROCKETT.- Finite Dimensional Linear Systems. Wiley, New-York, 1970.

[3] H. D'ANGELO.- Linear Time-Varying Systems. Allyn and Bacon, Boston, Mass., 1970.

[4] R.H. COLE.- Theory or Ordinary Differential Equations. Appleton-Century-Crofts, New-York, 1968.

-:-:-:-

Arthur J. KRENER
Department of Mathematics
University of California
Davis, CALIFORNIA 95616 (U.S.A.)