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A ratio limit theorem

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A RATIO LIMIT THEOREM

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I.- Introduction.-

Let G be a (topological) group, μ a probability measure on G and denote by μ^n the n-th convolution power of μ. We are interested in results of the type

$$\lim_{n\to\infty} \frac{\mu^n(f)}{\mu^n(g)} \quad \text{exists} = \frac{\sigma(f)}{\sigma(g)}$$

for certain functions f,g and a certain measure σ on G. In the case of abelian groups very complete results have been obtained in [7] , in the case of discrete groups a general theorem is proved in [1] (for μ symmetric). There are only a few other results of this type for nondiscrete and nonabelian groups ([8]). We will prove a ratio limit theorem for certain positive-definite and for certain symmetric probability measures on (certain) amenable groups ; theorem 2 contains as a special case an extension of the result in [1] . It will also be shown that these results can only hold in unimodular, amenable groups.

2.- Preliminaries.-

Let G be always a locally compact group.

For a function f on G we write

$${}_xf(y) = f(xy),\ f_x(y) = f(yx)\ ,\ \tilde{f}(x) = f(x^{-1}) \quad .$$

By K we denote the linear space of real-valued continuous functions on G vanishing outside compact sets, by K^+ the positive elements in K.

For a measure μ on G and a complex-valued function f on G we write

$$\mu(f) = \int f(x)\, d\mu(x) \quad .$$

All properties of convolutions of functions and measures we use can be found in [5]

The convolutionpowers of a probability measure μ on G will be denoted by μ^n .

If G is a unimodular group the Haar measure will be denoted by λ and its differential by dx ; we have then for $y \in G$:

$$\lambda(f) = \int f(x)dx = \int f(xy)dx = \int f(yx)dx = \int \tilde{f}(x)dx \quad .$$

A measure μ on the group G is called positive definite if

$$\mu(f*\tilde{f}) \geq 0 \quad \text{for all} \quad f \in K \quad (\text{or} \quad f \in L^2(G))$$

(see [3] for properties of positive-definite measures and functions).

If μ is a positive-definite probability measure on G then all the convolution powers μ^n are positive-definite probability measures on G.

Lemma 1 : if μ is a positive-definite probability measure on the unimodular group G, $x \in G$ and $g \in L^2(G)$ then

$$|\mu^n(g_x * \tilde{g})| \leq \mu^n(g * \tilde{g}) \quad . \quad (n = 1, 2, \ldots) .$$

Proof : Since $g_x * \tilde{g}_x = g * \tilde{g}$ for every $x \in G$ we have by the Cauchy-Schwarz inequality

$$|\mu^n(g_x * \tilde{g})| \leq (\mu^n(g * \tilde{g}))^{1/2} (\mu^n(g_x * \tilde{g}_x))^{1/2} = \mu^n(g * \tilde{g}) \quad .$$

Lemma 2 : If μ is a positive-definite probability measure on G then

$$\lim_{n \to \infty} \frac{\mu^{n+1}(g * \tilde{g})}{\mu^n(g * \tilde{g})} = \|\mu\|_2 \quad \text{for all } g \in K^+, \ g \neq 0$$

($\|\mu\|_2$ is the norm of the operator $T : L^2(G) \to L^2(G)$ given by $Tf = \mu * f$ for $f \in L^2(G)$

Proof : Since by assumption T is a positive-definite operator on th Hilbertspace $L^2(G)$ there exists a unique positive-definite operator S on $L^2(G)$ such that $S^2 = T$ ([5] , C.35). Therefore we have for any positive integer n and $g \in K^+$ ($<,>$ denotes the inner product in $L^2(G)$) :

$$\begin{aligned} \mu^n(g * \tilde{g}) &= \langle T^n g, g \rangle = \langle S^{2n} g, g \rangle = \langle S^{n-1} g, S^{n+1} g \rangle \\ &\leq \langle S^{n-1} g, S^{n-1} g \rangle^{1/2} \langle S^{n+1} g, S^{n+1} g \rangle^{1/2} \\ &= \langle T^{n-1} g, g \rangle^{1/2} \langle T^{n+1} g, S^{n+1} g \rangle^{1/2} \\ &= (\mu^{n-1}(g * \tilde{g}) \, \mu^{n+1}(g * \tilde{g}))^{1/2} . \end{aligned}$$

In [2] it is shown that under the hypotheses of lemma 2 we have

$$\|\mu\|_2 = \lim_{n \to \infty} (\mu^n(g * \tilde{g}))^{1/n} \text{ for all } g \in K^+ , \ g \neq 0$$

and that the sequence $(\mu^n(g * \tilde{g}))^{1/n}$ is increasing. Therefore $\|\mu\|_2 > 0$ and to every $g \in K^+$, $g \neq 0$ there exists an integer $n_o(g)$ such that

$$\mu^n(g * g) > 0 \quad \text{for all } n \geq n_o(g) \quad .$$

From the calculation above we get then

$$\frac{\mu^n(g * \tilde{g})}{\mu^{n-1}(g * \tilde{g})} \leq \frac{\mu^{n+1}(g * \tilde{g})}{\mu^n(g * \tilde{g})}$$

for n sufficiently large ; so the sequence $\frac{\mu^{n+1}(g * \tilde{g})}{\mu^n(g * \tilde{g})}$ is increasing, therefore its limit exists and is equal to $\|\mu\|_2$.

3.- Positive-definite probability measures.

If μ is a probability measure on G then we write

$$\mu' = \sum_{n=1}^{\infty} 2^{-n} \mu^n$$

which is again a probability measure on G.

A function f on G is called central if $f(xy) = f(yx)$ for all $x, y \in G$.

For definition and properties of amenable groups see [4] .

We will prove here the following

Theorem 1 : Assume that

a) G is a locally compact, amenable, unimodular group with Haar-measure λ

b) μ is a positive-definite probability measure on G such that λ is absolutely continuous with respect to μ' (i.e. $\mu'(A) = 0 \Rightarrow \lambda(A) = 0$)

c) there exists a nonzero function $g \in K^+$ such that $g*\mu = \mu*g$.

Then for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ we have

$$\lim_{n\to\infty} \frac{\mu^n(g*f_1*\tilde g)}{\mu^n(g*f_2*\tilde g)} = \frac{\int g*f_1*\tilde g(x)\,dx}{\int g*f_2*\tilde g(x)\,dx} = \frac{\int f_1(x)\,dx}{\int f_2(x)\,dx} .$$

Proof : By lemma 1 we have

$$\tilde g*\mu^n*g(x) = \mu^n({}_x(g*\tilde g)) \leq \mu^n(g*\tilde g) = \tilde g*\mu^n*g(e) .$$

Since G is amenable $\|\mu\|_2 = 1$ ([4], Th. 3.2.2.). Now (using b))

$$0 < \int \Big(1 - \frac{\tilde g*\mu^n*g(x)}{\tilde g*\mu^n*g(e)}\Big)d\mu'(x) = \int \sum_{s=1}^{\infty} 2^{-s}\Big(1 - \frac{g*\mu^n*g(x)}{g*\mu^n*g(e)}\Big)\,d\mu^s(x) =$$

$$= \sum_{s=1}^{L} 2^{-s}\int\Big(1-\frac{\tilde g*\mu^n*g(x)}{\tilde g*\mu^n*g(e)}\Big)d\mu^s(x) + \int \sum_{s=L+1}^{\infty} 2^{-s}\Big(1-\frac{\tilde g*\mu^n*g(x)}{\tilde g*\mu^n*g(e)}\Big)d\mu^s(x)$$

$$\leq \sum_{s=1}^{L} 2^{-s}\Big(1-\frac{\mu^{n+s}(g*\tilde g)}{\mu^n(g*\tilde g)}\Big) + 2^{-L} \leq \sum_{s=1}^{L} 2^{-s}\Big(1-\frac{\mu^{n+L}(g*\tilde g)}{\mu^n(g*\tilde g)}\Big) + 2^{-L}$$

$$\leq \Big(1-\frac{{}^{n+L}(g*\tilde g)}{{}^{n}(g*\tilde g)}\Big) + 2^{-L} \to 2^{-L} \text{ for } n\to\infty \text{ (lemma 2)}$$

which implies that

$$\int\Big(1-\frac{\tilde g*\mu^n*g(x)}{\tilde g*\mu^n*g(e)}\Big)d\mu'(x) \to 0 \text{ for } n\to\infty .$$

Therefore if (n') is any subsequence of the positive integers there exists a sub-subsequence $(n'') \subset (n')$ such that ([5] , (11.27))

$$\lim \frac{\tilde g*\mu^{n''}*g(x)}{\tilde g*\mu^{n'}*g(e)} = 1 \quad \mu' \text{ almost everywhere}$$

and so by assumption b) λ almost everywhere. Lebesgue" dominated convergence theorem then implies

$$\lim \int \frac{\tilde g*\mu^{n''}*g(x)}{\tilde g*\mu^{n'}*g(e)}\, f(x)dx = \int f(x)dx = \lambda(f)$$

for all $f \in L^1(G)$. Since $\tilde g*\mu^n*g$ is positive-definite

$$\int \tilde g*\mu^n*g(x)\, f(x)dx = \mu^n(g*f*\tilde g)$$

and we obtain : every subsequence (n') of the positive integers contains a sub-sub-sequence $(n'') \subset (n')$ such that

$$\lim_{n''} \frac{\mu^{n''}(g*f_1*\tilde{g})}{\mu^{n''}(g*f_2*\tilde{g})} = \frac{\lambda(f_1)}{\lambda(f_2)} \quad ;$$

since the limit is independent of (n'), (n") this implies that

$$\lim_{n} \frac{\mu^{n}(g*f_1*\tilde{g})}{\mu^{n}(g*f_2*\tilde{g})} \text{ exists } = \frac{\lambda(f_1)}{\lambda(f_2)} \quad ;$$

The function g is by assumption positive and so we get from the invariance of the Haarmeasure

$$\int g*f*\tilde{g}(x)dx = \|g\|_1^2 \int f(x)dx$$

which proves theorem 1 .

4.- Symmetric probability measures.

A probability measure μ is called symmetric if $\mu(f) = \mu(\tilde{f})$.
If μ is symmetric then μ^{2n}(n=1, 2, ...) is positive-definite. We will write

$$\mu'' = \sum_{n=1}^{\infty} 2^{-n} \mu^{2n} \quad .$$

Theorem 2 : Assume that

a) G is a locally compact, unimodular, amenable group with Haarmeasure λ

b) μ is a symmetric probability measure on G such that λ is absolutely continuous with respect to μ''

c) there exists a nonzero central function $g \in K^+$.

Then for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ we have

$$\lim_{n\to\infty} \frac{\mu^{n}(g*f_1*g)}{\mu^{n}(g*f_2*g)} = \frac{\int g*f_1*\tilde{g}(x)dx}{\int g*f_2*\tilde{g}(x)dx} = \frac{\int f_1(x)dx}{\int f_2(x)dx} \quad .$$

Proof : Since μ^2 is positive-definite we get by theorem 1

$$\lim_{n\to\infty} \frac{\mu^{2n}(g*f_1*\tilde{g})}{\mu^{2n}(g*f_2*\tilde{g})} = \frac{\int f_1(x)dx}{\int f_2(x)dx}$$

for all $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$. We have to show that this relation also hods if 2n is replaced by 2n+1. Now

$$\mu^{2n+1}(g*f*g) = \int \mu^{2n}({}_x(g*f*g))d\,\mu(x) = \int \mu^{2n}(g*{}_xf*\tilde{g})d\,\mu(x) \quad .$$

Therefore we get for $f_1, f_2 \in L^1(G)$ with $\lambda(f_2) \neq 0$ by theorem 1 and Lebesque's dominated convergence theorem

$$\lim_{n\to\infty} \frac{\mu^{2n+1}(g*f_1*\tilde{g})}{\mu^{2n}(g*f_2*\tilde{g})} = \lim_n \int \frac{\mu^{2n}(g*_x f_1*\tilde{g})}{\mu^{2n}(g*f_2*\tilde{g})}\, d\,\mu(x) =$$
$$= \int \frac{\int_x f_1(y)dy}{\int f_2(y)dy}\, d\,\mu(x) = \frac{\int f_1(y)dy}{\int f_2(y)dy} .$$

But this implies that

$$\lim_{n\to\infty} \frac{\mu^{2n+1}(g*f_1*\tilde{g})}{\mu^{2n+1}(g*f_2*\tilde{g})} = \frac{\lim \frac{\mu^{2n+1}(g*f_1*\tilde{g})}{\mu^{2n}(g*f_2*\tilde{g})}}{\lim \frac{\mu^{2n+1}(g*f_2*\tilde{g})}{\mu^{2n}(g*f_2*\tilde{g})}} = \frac{\int f_1(x)dx}{\int f_2(x)dx}$$

ant theorem 2 follows.

5.- Some remarks.

Remark 1 : We will write Supp μ for the support of μ . Theorem 2 specialized to discrete groups gives the ratio limit theorem in [1] namely :

If G is a discrete amenable group, μ a symmetric probability measure on G whose support generates G ($G = \bigcup_{n=1}^{\infty} \text{Supp } \mu^n$) and if there exists an integer k such that $\mu^{2k+1}(e) > 0$ then

$$\lim_{n\to\infty} \frac{\mu^n(f_1)}{\mu^n(f_2)} = \frac{\sum_{x\in G} f_1(x)}{\sum_{x\in C} f_2(x)}$$

for all $f_1, f_2 \in L^1(G)$ such that $\sum_{x\in G} f_2(x) \neq 0$

Proof : We can take $g = \delta_e$ as the nonzero central function in K^+ .

Now since e Supp μ^{2k+1} we have for n = 1, 2, ...

$$\text{Supp } \mu^{2n+1} \subset (\text{Supp } \mu^{2n+1})\ (\text{Supp } \mu^{2k+1}) = \text{Supp } \mu^{2n+k+1})$$

and therefore

$$G = \bigcup_{n=1}^{\infty} \text{Supp } \mu^n \subset \bigcup_{n=1}^{\infty} \text{Supp } \mu^{2n} \subset G .$$

Since G is deiscrete the Haarmeasure λ is the counting measure and the last relation means that λ is absolutely continuous with respect to μ''. Therefore theorem 2 implies the result.

Remark 2 : The results of theorem 1 and theorem 2 can only hold for unimodular groups, namely :

If μ is a symmetric probability measure on the locally compact group G with left Haarmeasure λ and if there exists a function g on G such that

$$\lim_{n\to\infty} \frac{\mu^n(g*f*g)}{\mu^n(g*g)} = \lambda(f) \quad \text{for all} \quad f \in K ,$$

then G is unimodular.

Proof : Since μ is symmetric

$$\mu^n(g*f*\tilde{g}) = \mu^n((g*f*\tilde{g})^{\sim}) = \mu^n(g*\tilde{f}*\tilde{g})$$

and so we get $\lambda(f) = \lambda(\tilde{f})$, i.e. λ is inversion invariant.
Therefore G is unimodula ([5], (15.16)).

Remark 3 : Theorem 1 and theorem 2 can only hold for amenable groups namely :

If μ is a symmetric probability measure on the locally compact unimodular group G such taht for same $g \geq 0$

$$\lim_{n\to\infty} \frac{\mu^n(g*f*g)}{\mu^n(g*g)} = \lambda(f)$$

for all f which are characteristic functions of compact sets, then G is amenable.

Proof : Since

$$\mu^n(g*f*\tilde{g}) = \int \tilde{g}* {}^n*g(t)f(t)dt$$

the assumption is equivalent to

$$\lambda(f) - \frac{\mu^n(g*f*\tilde{g})}{\mu^n(g*\tilde{g})} = \int (1 - \frac{\tilde{g}* {}^n*g(t)}{\mu^n(g*\tilde{g})}) f(t)dt \to 0 \quad \text{for} \quad n \to \infty$$

for all f which are characteristic functions of compact sets. Now write

$$s_n(x) = \frac{\mu^n*g(x)}{(\mu^{2n}(g*\tilde{g}))^{1/2}} \quad .$$

Then

$$\tilde{s}_n*s_n(x) = \frac{\tilde{g}*\mu^{2n}*g(x)}{\mu^{2n}(g*\tilde{g})}$$

and

$$||s_n||_2^2 = \frac{1}{\mu^{2n}(g*\tilde{g})} \int \mu^n*g(x)\ \mu^n*g(x)dx =$$

$$= \frac{1}{\mu^{2n}(g*\tilde{g})} \int\int\int g(y^{-1}x)\tilde{g}(x^{-1}z)dx\ d\ \mu^n(y)d\ \mu^n(z)$$

$$= \frac{1}{\mu^{2n}(g*\tilde{g})} \int\int g*\tilde{g}(y^{-1}z)d\ \mu^n(y^{-1})d\ \mu^n(z)$$

$$= \frac{\mu^n*\mu^n(g*\tilde{g})}{\mu^{2n}(g*\tilde{g})} = 1 \quad .$$

Since $\mu^{2n}(n=1, 2, \ldots)$ is a positive-definite probability measure on G we have by lemma 1

$$0 \leq \tilde{s}_n * s_n(x) \leq 1 \quad \text{for all} \quad x \in G \quad .$$

Now let K be a compact set (of positive Haar-measure), let $\varepsilon > 0$, $\delta > 0$ be given and let f-characteristic function of K, then there exists an integer n_o such that

$$\int_K (1-\tilde{s}_n * s_n(x))dx < \varepsilon\delta \quad \text{for all } n \geq n_o \quad .$$

Therefore if $K_n(\varepsilon) = \{x \in K \mid 1-\tilde{s}_n * s_n(x) > \varepsilon\}$ the Haar-measure of $K_n(\varepsilon)$

$$\lambda(K_n(\varepsilon)) < \delta \quad .$$

This implies that to every compact set K, every $\varepsilon > 0$, $\delta > 0$ there exists an integer n_o and there exist nonnegative functions $s_n \in L^2(G)$ and subsets $K_n(\varepsilon) \subset K$ with the properties :

$$|1-\tilde{s}_n * s_n(x)| < \varepsilon \text{ for all } n \geq n_o, \ x \in K \setminus K_n(\varepsilon), \ \lambda(K_n(\varepsilon)) < \delta$$

This means that (in the terminology of [6]) the sequence $\tilde{s}_n * s_n$ converges almost uniformly to 1 for $n \to \infty$. Therefore prop. 0.1 and prop. 6.1 (see the proof there) of [6] imply that G is amenable.

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Résumé :

Dans cet article on démontre les résultats suivants :
1) Soit G un groupe localement compact, moyennable, de unimodulaire de mesure de Haar λ , μ une mesure de probabilité sur G telle que λ soit absolument continue par rapport à $\mu' = \sum_{n=1}^{\infty} 2^{-n} \mu^n$, supposons qu'il existe une fonction non nulle $g \in K^+$ (ensemble des éléments positifs de K où K est l'espace des fonctions réelles ontinues tendant vers zéro en dehors des compacts) telle que $g * \mu = \mu * g$
Alors pour tout $f_1, f_2 \in L^1(G)$ telle que $\lambda(f2) \neq 0$ on a :

$$\lim_{n \to +\infty} \frac{\mu^n(g*f_1*\tilde{g})}{\mu^n(g*f_2*\tilde{g})} = \frac{\int g*f_1*\tilde{g}(x)dx}{\int g*f_2*\tilde{g}(x)dx} = \frac{\int f_1(x)dx}{\int f_2(x)dx} \qquad (I)$$

2) Soit G un groupe localement unimodulaire, moyennable de mesure de Haar λ μ une mesure de probabilité symétrique sur G telle que λ soit absolument continue par rapport à $\mu'' = \sum_{n=1}^{\infty} 2^{-n} \mu^{2n}$, supposons qu'il existe une fonction centrale non nulle $g \in K^+$
Alors pour tout $f1, f2 \in L^1(G)$ telle que $\lambda(f2) \neq 0$ on a (I) .

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