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THE FORMAL COMPLETION OF THE SECOND CHOW GROUP,

A K-THEORETIC APPROACH

by

Jan Stienstra

1. Introduction

Let X be a smooth projective surface over a perfect field $\underline{k}$ of characteristic $p \geq 0$. In case $p = 0$ it will be assumed that $\underline{k}$ is algebraic over $\mathbb{Q}$ (because there are problems if $\Omega^1_{\underline{k}/\mathbb{Q}}$ does not vanish).

Recall that the n-th Chow group of X, denoted $CH^n(X)$, is the group of codimension n cycles on X modulo those which are rationally equivalent to zero. Bloch's formula $CH^n(X) = H^n(X, \mathcal{K}_{n,X})$ allows us to study Chow groups using algebraic K-theory. Here $\mathcal{K}_{n,X}$ is the sheaf of abelian groups on X which is associated to the pre-sheaf (open U) $\mapsto K_n(\Gamma(U, \mathcal{O}_X))$ [16].

We will discuss the structure of the formal completion of the second Chow group (at the origin). This formal completion is a covariant functor $\widehat{CH}^2_X$ from the category of augmented local artinian $\underline{k}$-algebras to the category of abelian groups. It is defined by

$$(1.1) \qquad \widehat{CH}^2_X(A) = \ker[H^2(X \times_{\underline{k}} \operatorname{Spec} A, \mathcal{K}_{2, X \times_{\underline{k}} \operatorname{Spec} A}) \xrightarrow{q} H^2(X, \mathcal{K}_{2,X})] ;$$

the map q in (1.1) is induced by the augmentation map $A \to \underline{k}$. (See also [4], where $\widehat{CH}^2_X$ is called F^2_0.)

We want to unravel the structure of $\widehat{CH}^2_X$ by studying a morphism of functors

$$(1.2) \qquad H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$$

(see §3 below). Suitable conditions on X imply the injectivity of this map and the pro-representability of its cokernel (functor). (See [17] for pro-representability of functors.) It seems likely that this cokernel is naturally isomorphic to the formal group at the origin of the Albanese variety $\mathrm{Alb}\,X$.

The present text describes the main results of the author's thesis [19]. Proofs are to be found in [19]. An appendix is added to show that there are no non-trivial morphisms from $H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2$ into a pro-representable functor. This result is not in [19]. It shows that, if (1.2) is injective and has a pro-representable cokernel, $H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2$ is the smallest subgroup (functor) of $\widehat{CH}^2_X$ for which the corresponding quotient is pro-representable. In another paper I want to relate the cokernel of (1.2) to the formal group at the origin of $\mathrm{Alb}\,X$.

I would like to thank Spencer Bloch for explaining the problem to me and the stimulating conversations we had. In particular, it was his idea to study the kernel and cokernel of (1.2). I also thank Fred Flowers for the careful typing of the manuscript.

§2. Algebraic preliminaries

(2.0) The schemes $X \times_{\underline{k}} \mathrm{Spec}\,A$ and X have the same underlying topological space. The map $\mathcal{K}_{2, X \times_{\underline{k}} \mathrm{Spec}\,A} \to \mathcal{K}_{2,X}$ between sheaves on this space splits. We denote its kernel by $\mathcal{K}_{2, X \otimes A/X}$. So we have

$$(2.0.1) \qquad \widehat{CH}^2_X(A) = H^2(X, \mathcal{K}_{2, X \otimes A/X}) .$$

Since $\mathcal{K}_{2,X\otimes A/X}$ is the sheaf associated to the presheaf

$$(\text{open } U) \mapsto \ker[K_2(\Gamma(U,\mathcal{O}_X)\otimes_{\underline{k}} A) \to K_2(\Gamma(U,\mathcal{O}_X))]$$

it seems natural to start our study of $\widehat{CH}^2_X$ with a description of $\ker[K_2(R\otimes_{\underline{k}} A) \to K_2(R)]$ for a $\underline{k}$-algebra R and an augmented local artinian $\underline{k}$-algebra A.

(2.1) For an ideal I in a ring S one can define relative K-groups $K_n(S,I)$ such that there is a long exact sequence

(2.1.1)
$$\begin{aligned} \cdots \to K_3(S) &\xrightarrow{\pi_3} K_3(S/I) \to \\ \to K_2(S,I) \to K_2(S) &\xrightarrow{\pi_2} K_2(S/I) \to \\ \to K_1(S,I) \to K_1(S) &\xrightarrow{\pi_1} K_1(S/I) \to \cdots \end{aligned}$$

(see [11],[13]).

If I is contained in the Jacobson radical of S, one has the following results.

(2.1.2) the maps π_1 and π_2 are surjective

(2.1.3) $K_1(S,I) = 1+I = \ker[S^* \to (S/I)^*]$.

We can reformulate this as follows

(2.1.3a) The group $K_1(S,I)$ is the abelian group, which has a presentation with generators $\langle a\rangle$, one for every $a\in I$, and defining relations
$$\langle a\rangle + \langle b\rangle = \langle a+b-ab\rangle \quad \text{for } a,b\in I.$$

Of course $\langle a\rangle$ corresponds to $1-a \in 1+I$.

(2.1.4) The group $K_2(S,I)$ is the abelian group, which has a presentation with generators $\langle a,b\rangle$, one for every $(a,b)\in R\times I \cup I\times R$ and defining relations

(D1) $\langle a,b\rangle = -\langle b,a\rangle$ for $a\in I$

(D2) $\langle a,b\rangle + \langle a,c\rangle = \langle a,b+c-abc\rangle$ for $a\in I$ or $b,c\in I$

(D3) $\langle a,bc\rangle = \langle ab,c\rangle + \langle ac,b\rangle$ for $a\in I$.

These facts are proven in [3], [14], [11]. It should be noticed that the notation $<a,b>$ in (2.1.4) corresponds to $<-a,b>$ in other articles.

The above results hold in particular for $S = R\otimes_{\underline{k}} A$ and $I = R\otimes_{\underline{k}} \underline{m}_A$, where $\underline{m}_A$ is the maximal ideal of A. In this case the homomorphism $S \to S/I$ splits. So π_3 is surjective and $\ker[K_2(R\otimes_{\underline{k}} A) \to K_2(R)] = K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A)$.

(2.2) Assume for a moment char $\underline{k} = 0$. Put $S = R\otimes_{\underline{k}} A$ and $I = R\otimes_{\underline{k}} \underline{m}_A$. Define $\Omega^1_{S,I} = \ker[\Omega^1_{S/\mathbb{Z}} \to \Omega^1_{R/\mathbb{Z}}]$. There is an isomorphism

(2.2.1) $$K_2(S,I) \cong \Omega^1_{S,I}/dI$$

defined by $<a,b> \mapsto \sum_{n\geq 1} \frac{1}{n} a^n b^{n-1} db \mod dI$ (see [4], [14]). Assume now that R has no zero divisors and that $\underline{k}$ is algebraic over $\mathbb{Q}$. Let $\underline{k}'$ be the algebraic closure of $\underline{k}$ in R. Then one has an exact sequence

(2.2.2) $$\underline{k}'\otimes_{\underline{k}} \underline{m}_A \xrightarrow{1\otimes d} R\otimes_{\underline{k}}\Omega^1_{A,\underline{m}_A} \longrightarrow [\Omega^1_{S,I}/dI] \longrightarrow [\Omega^1_{R/\underline{k}}/dR]\otimes_{\underline{k}}\underline{m}_A \longrightarrow 0 .$$

One can analyze the functor $\hat{C}H^2_X$ using this exact sequence, sheafified with respect to R (see § 3). Essentially this is the method Bloch uses in [4], though his groups are slightly different and his arrows go the opposite way.

For $p > 0$ there is no isomorphism (2.2.1) and the analysis of $K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A)$ has to be done with K-theoretical means. After all it will appear that this method works in characteristic zero as well as for positive characteristics.

(2.3) Following [6] we define

(2.3.1) $$\hat{C}K_q(A) = \ker[K_q(A[x]) \to K_q(A)],$$

the functor of <u>formal curves</u> on K_q evaluated at A.

(2.3.2) $$C_nK_q(R) = \ker[K_q(R[t]/(t^{n+1})) \to K_q(R)], \quad \text{for } n \geq 1.$$

Varying n we get a projective system $C_{\bullet}K_q(R)$, with structure maps induced by the canonical projections $R[t]/(t^{n+1}) \to R[t]/(t^n)$. We define

(2.3.3) $CK_q(R) = \varprojlim C_nK_q(R)$,

the functor of curves on K_q evaluated at R .

The latter group carries a topology given by the filtration with the subgroups $\ker[CK_q(R) \to C_nK_q(R)]$. The standard reference for C_nK_q and CK_q is [5] (see also [19]). Note however that in [5], $\varprojlim C_nK_q(R)$ is denoted as $\hat{C}K_q(R)$ instead of $CK_q(R)$. The above notations are taken from [6].

According to (2.1.3.a) and (2.1.4) we have generators and relations for the groups $C_nK_1(R)$ and $C_nK_2(R)$. These sets of generators are in fact too large. It suffices to take

(2.3.4) for $C_nK_1(R)$ the elements $<at^m>$ with $a \in R$, $1 \leq m \leq n$.

(2.3.5) for $C_nK_2(R)$ the elements $<at^m, b>$ with $a, b \in R$, $1 \leq m \leq n$ and the elements $<at^{m-1}, t>$ with $a \in R$, $1 \leq m \leq n+1$.

Let us write $<at^m>$, $<at^m, b>$ and $<at^{m-1}, t>$ also for the elements of $CK_1(R)$ and $CK_2(R)$ respectively whose image in every $C_nK_1(R)$ and $C_nK_2(R)$ is $<at^m>$, $<at^m, b>$ and $<at^{m-1}, t>$ respectively. The sets of these elements generate $CK_1(R)$ and $CK_2(R)$ topologically. (Incidentally, the relation with the generators used in [5] is given by $<at^m> = 1 - at^m$, $<at^m, b> = \{1 - abt^m, b\}$ provided $b \in R^*$ and $<at^{m-1}, t> = \{1 - at^m, t\}$ (cf [14])).

It is possible to get a similar result for $\hat{C}K_1(A)$ and $\hat{C}K_2(A)$. Using the fact that $K_q(\underline{k}[x]) = K_q(\underline{k})$ one shows easily that $\hat{C}K_q(A)$ is the kernel of the split surjection $K_q(A[x], \underline{m}_A[x]) \to K_q(A, \underline{m}_A)$. Thus one obtains a presentation for $\hat{C}K_1(A)$ and $\hat{C}K_2(A)$. As before it suffices to take as generators

(2.3.6) for $\hat{C}K_1(A)$ the elements $<ax^m>$ with $a \in \underline{m}_A$, $m \geq 1$

(2.3.7) for $\hat{C}K_2(A)$ the elements $<ax^m, b>$ with $a, b \in A$, a or $b \in \underline{m}_A$, $m \geq 1$ and the elements $<ax^{m-1}, x>$ with $a \in \underline{m}_A$, $m \geq 1$.

(2.4) Inside $CK_1(R)$, $CK_2(R)$, $\hat{C}K_1(A)$ and $\hat{C}K_2(A)$ one distinguishes the subgroups of p-typical (formal) curves [5], [6]; in fact these subgroups are direct summands. We denote these summands by $W(R)$, $TCK_2(R)$, $\hat{W}(A)$ and $T\hat{C}K_2(A)$ respectively. The projection operator onto the typical parts is in all four cases denoted by E. The group $W(R)$ is in fact the group of p-Witt vectors of R and $\hat{W}(A)$ is the group of "formal" p-Witt vectors of A. The letter E refers to the relation with the Artin-Hasse exponential; indeed in $CK_1(R)$ one has $E\langle t\rangle = \prod_{n\in\mathbb{N}\setminus p\mathbb{Z}} (1-t)^{\frac{\mu(n)}{n}}$, where μ is the Mobius function (see [5]).

The operator E kills all elements $\langle at^m\rangle$, $\langle at^m, b\rangle$, $\langle at^{m-1}, t\rangle$, $\langle ax^m\rangle$, $\langle ax^m, b\rangle$ and $\langle ax^{m-1}, x\rangle$ for which m is not a power of p [19]. Thus one finds the following sets of (topological) generators

(2.4.1) for $W(R)$ the elements $E\langle at^{p^r}\rangle$ with $a\in R$, $r\geq 0$

(2.4.2) for $TCK_2(R)$ the elements $E\langle at^{p^r}, b\rangle$ and $E\langle at^{p^r-1}, t\rangle$ with $a, b\in R$, $r\geq 0$

(2.4.3) for $\hat{W}(A)$ the elements $E\langle ax^{p^r}\rangle$ with $a\in \underline{m}_A$, $r\geq 0$

(2.4.4) for $T\hat{C}K_2(A)$ the elements $E\langle ax^{p^r}, b\rangle$ with $a, b\in A$, a or $b\in \underline{m}_A$, $r\geq 0$ and the elements $E\langle ax^{p^r-1}, x\rangle$ with $a\in \underline{m}_A$, $r\geq 0$.

(Convention: for $p = 0$ we only take $r = 0$ and $p^r = 1$.)

In case $p = 0$, one has the following isomorphisms

(2.4.5)
$$\begin{aligned} R &\xrightarrow{\sim} W(R) & a &\mapsto E\langle at\rangle \\ \Omega^1_{R/\mathbb{Z}} &\xrightarrow{\sim} TCK_2(R) & adb &\mapsto E\langle at, b\rangle \\ \underline{m}_A &\xrightarrow{\sim} \hat{W}(A) & a &\mapsto E\langle ax\rangle \\ \Omega^1_{A,\underline{m}_A} &\xrightarrow{\sim} T\hat{C}K_2(A) & adb &\mapsto E\langle ax, b\rangle \end{aligned}$$

(Recall: $\Omega^1_{A,\underline{m}_A} = \ker[\Omega^1_{A/\mathbb{Z}} \to \Omega^1_{\underline{k}/\mathbb{Z}}]$, by definition.)

(2.5) The groups $W(R)$, $TCK_2(R)$, $\hat{W}(A)$ and $T\hat{C}K_2(A)$ are modules over the ring $W(\underline{k})$ of p-Witt vectors over $\underline{k}$. Multiplication with the Witt vector $E<at>$, in particular, is induced by the substitution $t \mapsto at$ and $x \mapsto ax$ respectively [6].

These groups are also equipped with endomorphisms V (Verschiebung) and F (Frobenius). In the case $p > 0$ the map V comes from the substitutions $t \mapsto t^p$ and $x \mapsto x^p$ respectively. The map F is the corresponding transfer map, explicitly given by

(2.5.1)
$$\begin{aligned} FE<at^{p^r}> &= E<a^pt^{p^r}> \\ FE<at^{p^r},b> &= E<a^pb^{p-1}t^{p^r},b> \quad \text{and} \\ FE<at^{p^r-1},t> &= E<at^{p^{r-1}-1},t> \end{aligned}$$

and similar formulas with x instead of t. In the case $p = 0$ we take $V = F =$ identity map.

In fact, $W(R)$, $TCK_2(R)$, $\hat{W}(A)$ and $T\hat{C}K_2(A)$ are left modules over the Dieudonné ring $\mathfrak{D}$. Recall that $\mathfrak{D}$ is the non-commutative polynomial ring $W(\underline{k})[F,V]$ with commutation rules $FV = VF = p$ if $p > 0$ (respectively $FV = VF = 1$ if $p = 0$), $F\alpha = \alpha^\sigma F$ and $\alpha V = V\alpha^\sigma$, where α is in $W(\underline{k})$ and α^σ is its image under the Frobenius automorphism of $W(\underline{k})$.

Every left $\mathfrak{D}$-module M is in a natural way also a right $\mathfrak{D}$-module, namely with $\omega\cdot\alpha = \alpha\cdot\omega$, $\omega F = V\omega$ and $\omega V = F\omega$ for $\alpha \in W(\underline{k})$, $\omega \in M$. We will use this right action for $W(R)$ and $TCK_2(R)$.

(2.6) From the pairings which Bloch has constructed in [6] one obtains maps

(2.6.1) $$\Phi: W(R)\otimes_{\mathfrak{D}} T\hat{C}K_2(A) \to K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A)$$

(2.6.2) $$\psi: TCK_2(R)\otimes_{\mathfrak{D}} \hat{W}(A) \to K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A)$$

One has explicitly

(2.6.3) $$\Phi(E\langle at\rangle \otimes E\langle bx, c\rangle) = \sum_{n\in\mathbb{N}\setminus p\mathbb{Z}} \frac{\mu(n)}{n} \langle a^n b^n c^{n-1}, c\rangle$$

and, in case $p > 0$,

$$\Phi(E\langle at\rangle \otimes E\langle bx^{p^r-1}, x\rangle) = \sum_{n\in\mathbb{N}\setminus p\mathbb{Z}} \frac{\mu(n)}{n} \langle b^n a^{np^r-1}, a\rangle .$$

Similar formulas hold for ψ.

In case $p = 0$, Φ and ψ correspond via the isomorphisms in (2.2.1) and (2.4.5) to the obvious maps from $R\otimes_{\underline{k}} \Omega^1_{A,\underline{m}_A}$ and $\Omega^1_{R/\mathbb{Z}} \otimes_{\underline{k}} \underline{m}_A$, respectively, to $\Omega^1_{R\otimes_{\underline{k}} A,\, R\otimes_{\underline{k}} \underline{m}_A} / d(R\otimes_{\underline{k}} \underline{m}_A)$.

In any case, the images of Φ and ψ together generate the group $K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A)$. We get therefore a surjection

(2.6.4) $$\overline{\psi}: TCK_2(R)\otimes_{\mathfrak{D}} \hat{W}(A) \to \operatorname{coker} \Phi.$$

(2.7) Assume $p = 0$. One can check without difficulty that the isomorphisms in (2.2.1) and (2.4.5) induce an isomorphism

$$[\Omega^1_{R/\underline{k}} / dR]\otimes_{\underline{k}} \underline{m}_A \xrightarrow{\sim} \operatorname{coker} \Phi .$$

In this case the kernel of $\overline{\psi}$ corresponds exactly to $dR\otimes_{\underline{k}} \underline{m}_A$.

(2.8) Assume now $p > 0$. We define

(2.8.1) $$W_m(R) = \text{image of } W(R) \text{ in } K_1(R[t]/(t^{p^{m-1}+1})) \quad \text{for } m \geq 1$$

$$TCK_{2,m}(R) = \text{image of } TCK_2(R) \text{ in } K_2(R[t]/(t^{p^{m-1}+1})) \quad \text{for } m \geq 2$$

$$TCK_{2,1}(R) = \Omega^1_{R/\underline{k}}$$

The group $W_m(R)$ is the usual one, i.e. $W(R)/V^m W(R)$. The group $TCK_{2,m}(R)$ is the same as $TC_m K_2(R)$ of [5]. It is also the same as the $W_m\Omega^1_{\operatorname{Spec} R}$ of the De Rham-Witt complex [9]. Only in characteristic 2 is it necessary to define $TCK_{2,1}(R)$ separately; the other definition (that is, as the image of $TCK_2(R)$ in $K_2(R[t]/(t^2))$) would give wrong results in this case. In characteristic $\neq 2$, however, there is no difference (because of Van der Kallen's theorem [10]). There is in any characteristic a surjection from $TCK_2(R)$ onto $\Omega^1_{R/\underline{k}}$.

We are going to use the notations $E<at^{p^r}>$, $E<at^{p^r}, b>$ and $E<at^{p^r-1}, t>$ also for the images of the so-called elements of $W(R)$ and $TCK_2(R)$ in $W_m(R)$ and $TCK_{2,m}(R)$, for every m. In particular,

(2.8.2) the element $E<at, b>$ of $TCK_{2,1}(R) = \Omega^1_{R/\underline{k}}$ is in fact the 1-form adb.

The endomorphisms V and F of W(R) and $TCK_2(R)$ induce maps for every $m \geq 1$

(2.8.3)
$$V: W_m(R) \to W_{m+1}(R) \quad , \quad F: W_m(R) \to W_m(R)$$
$$V: TCK_{2,m}(R) \to TCK_{2,m+1}(R) \quad , \quad F: TCK_{2,m+1}(R) \to TCK_{2,m}(R).$$

In [5] a homomorphism (of groups)

(2.8.4)
$$d: W_m(R) \to TCK_{2,m}(R)$$

is defined. It is given explicitly by

(2.8.5)
$$dE<at^{p^r}> = E<at^{p^r-1}, t> .$$

It is the same as the map $W_m(R) \to W_m\Omega^1_{\mathrm{Spec}\, R}$ of the De Rham-Witt complex. For $m = 1$ one has $W_1(R) = R$ and $TCK_{2,1}(R) = \Omega^1_{R/\underline{k}}$ and d is the ordinary differentiation $d: R \to \Omega^1_{R/\underline{k}}$, (up to sign).

For every m, $W_m(R)$ and $TCK_{2,m}(R)$ are $W(\underline{k})$-modules and the map d is linear. One has furthermore the basic relation

(2.8.6)
$$FdV = d.$$

Next recall that the group $\underset{\rightarrow}{W}(R)$ of unipotent Witt covectors is defined as the limit of the inductive system

$$W_1(R) \xrightarrow{V} W_2(R) \xrightarrow{V} \cdots \xrightarrow{V} W_r(R) \xrightarrow{V} \cdots$$

(see [8], [15]). We define the map

(2.8.7)
$$\partial_m: \underset{\rightarrow}{W}(R) \to TCK_{2,m}(R)$$

to be the limit of the maps

$$-dV^{m-r}: W_r(R) \to TCK_{2,m}(R) \qquad \text{for } r \leq m$$

$$-F^{r-m}d: W_r(R) \to TCK_{2,m}(R) \qquad \text{for } r \geq m$$

One can identify the elements of $\underset{\to}{W}(R)$ with sequences

$\underline{a} = (\cdots, a_{-n}, a_{-n+1}, \ldots, a_{-1}, a_0)$ with $a_{-n} \in R$ for all n and $a_{-n} = 0$ for $n >> 0$.

The map ∂_m is then given by the formula

(2.8.8) $$\partial_m \underline{a} = E<t, a_0 t^{p^{m-1}-1}> + E<t, a_{-1} t^{p^{m-2}-1}> + \ldots$$
$$\ldots + E<t, a_{-m+1}> + E<a_{-m}^{p-1}t, a_{-m}> + \ldots$$
$$\ldots + E<a_{-n}^{p^{n-m+1}-1} t, a_{-n}> + \ldots$$

Note that $\partial_1 : \underset{\to}{W}(R) \to TCK_{2,1}(R) = \Omega^1_{R/\underline{k}}$ is in fact the well-known map $\underline{a} \mapsto da_0 + a_{-1}^{p-1} da_{-1} + \ldots + a_{-1}^{p^n-1} da_{-n} + \ldots$ (cf [15]).

The projection $TCK_{2,m+1}(R) \to TCK_{2,m}(R)$ maps $\partial_{m+1}\underset{\to}{W}(R)$ onto $\partial_m \underset{\to}{W}(R)$. So it induces a map

$$[TCK_{2,m+1}(R)/\partial_{m+1}\underset{\to}{W}(R)] \to [TCK_{2,m}(R)/\partial_m \underset{\to}{W}(R)].$$

We define

(2.8.9) $$TCK_2(R)/\partial \underset{\to}{W}(R) = \varprojlim [TCK_{2,m}(R)/\partial_m \underset{\to}{W}(R)]$$

and

(2.9.10) $$\partial \underset{\to}{W}(R) = \ker[TCK_2(R) \to TCK_2(R)/\partial \underset{\to}{W}(R)] .$$

The group $\partial\underset{\to}{W}(R)$ is in fact a $\mathfrak{D}$-submodule of $TCK_2(R)$. It is generated by the elements $E<t, at^{p^m-1}>$ and $E<a^{p^m-1}t, a>$ with $a \in R$ and $m \geq 0$. One has, according to (2.6.3) and its analogue for ψ, the following relations

(2.8.11) $$\psi(E<t, at^{p^m-1}> \otimes E<bx>) = \Phi(E<at> \otimes E<b, xb^{p^m-1}>)$$
$$\psi(E<a^{p^m-1}t, a> \otimes E<bx>) = \Phi(E<at> \otimes E<bx^{p^m-1}, x>)$$

So the map $\overline{\psi}$ of (2.6.4) induces a surjection

(2.8.12) $$\overline{\overline{\psi}}:[TCK_2(R)/\partial \underset{\rightarrow}{W}(R)]\otimes_{\mathfrak{D}} \hat{W}(A) \to \operatorname{coker} \Phi .$$

(2.9) Theorem. Assume $p > 0$. Let R be a regular local $\underline{k}$-algebra of (Krull) dimension ≥ 1. Let A be an augmented local artinian $\underline{k}$-algebra. Then the homomorphism $\overline{\overline{\psi}}$ of (2.8.12) is an isomorphism and the homomorphism Φ of (2.7.1) is injective. So one has an exact sequence

$$0 \to W(R)\otimes_{\mathfrak{D}} T\hat{C}K_2(A) \to K_2(R\otimes_{\underline{k}} A, R\otimes_{\underline{k}} \underline{m}_A) \to$$
$$\to [TCK_2(R)/\partial \underset{\rightarrow}{W}(R)]\otimes_{\mathfrak{D}} \hat{W}(A) \to 0$$ ▨

The proof of this theorem is based on a lengthy analysis with generators and relations for the various groups. It is given in [19].

Note that if one uses the isomorphisms (2.2.1) and (2.4.5) to translate the exact sequence (2.2.2) one gets in characteristic 0 almost the same result as (2.9) would give for $p = 0$.

§3. Applications to geometry

Recall the hypotheses: X is a smooth projective surface over a perfect field $\underline{k}$ of characteristic $p \geq 0$. In case $p = 0$ it is assumed that $\underline{k}$ is algebraic over $\mathbb{Q}$.

We will discuss the characteristic zero case first, because it is technically easier and can be formulated in the more common terms of differentials. It is also treated in [4].

(3.1) Assume $p = 0$. Then we get from (2.2.2) an exact sequence

(3.1.1) $$0 \to \underline{k}'\otimes_{\underline{k}} d\,\underline{m}_A \to \mathcal{O}_X\otimes_{\underline{k}} \Omega^1_{A,\underline{m}_A} \to \mathcal{K}_{2,X\otimes A/X} \to$$
$$\to [\Omega^1_{X/\underline{k}}/d\mathcal{O}_X] \otimes_{\underline{k}} \underline{m}_A \to 0$$

for every augmented local artinian $\underline{k}$-algebra A, depending functorially upon this A; here $\underline{k}'$ is the algebraic closure of $\underline{k}$ in the function field $\underline{k}(X)$. Note that the sheaf $\underline{k}'\otimes_{\underline{k}} \underline{m}_A$ is constant. Taking cohomology groups we get an exact sequence:

$$(3.1.2) \qquad \cdots \to H^1(X,\Omega^1_{X/\underline{k}}/d\mathcal{O}_X)\otimes_{\underline{k}} \underline{m}_A \xrightarrow{\delta} H^2(X,\mathcal{O}_X)\otimes_{\underline{k}} \Omega^1_{A,\underline{m}_A}$$
$$\to \widehat{CH}^2_X(A) \to H^2(X,\Omega^1_{X/\underline{k}}/d\mathcal{O}_X)\otimes_{\underline{k}} \underline{m}_A \to 0 \; .$$

Since X is smooth, the Hodge-De Rham spectral sequence degenerates at E_1 [7]. From this it follows that

$$H^n(X,\Omega^1_{X/\underline{k}}/d\mathcal{O}_X) = H^n(X,\Omega^1_{X/\underline{k}}) \oplus H^{n+1}(X,\mathcal{O}_X)$$

for all n. In particular, $H^2(X,\Omega^1_{X/\underline{k}}/d\mathcal{O}_X) = H^2(X,\Omega^1_{X/\underline{k}})$. It can be shown that the image of the map δ of (3.1.2) is $H^2(X,\mathcal{O}_X)\otimes_{\underline{k}} d\underline{m}_A$. Thus we find the exact sequence

$$(3.1.3) \qquad 0 \to H^2(X,\mathcal{O}_X)\otimes_{\underline{k}} [\Omega^1_{A,\underline{m}_A}/d\underline{m}_A] \to \widehat{CH}^2_X(A) \to$$
$$\to H^2(X,\Omega^1_{X/\underline{k}})\otimes_{\underline{k}} \underline{m}_A \to 0 \; .$$

It depends functorially on A. The functor

$$(3.1.4) \qquad A \mapsto H^2(X,\Omega^1_{X/\underline{k}})\otimes_{\underline{k}} \underline{m}_A$$

is the formal group over $\underline{k}$ with tangent space $H^2(X,\Omega^1_{X/\underline{k}})$. So it has the same tangent space as the formal group at the origin of $\mathrm{Alb}\, X$, which we denote $\widehat{\mathrm{Alb}}_X$. Therefore it is isomorphic to $\widehat{\mathrm{Alb}}_X$. So we have:

(3.2) <u>Theorem.</u> <u>Assume</u> $p = 0$. <u>Then there is a short exact sequence</u>

$$(3.2.1) \qquad 0 \to H^2(X,\mathcal{O}_X)\otimes_{\underline{k}} [\Omega^1_{A,\underline{m}_A}/d\underline{m}_A] \to \widehat{CH}^2_X(A) \to \widehat{\mathrm{Alb}}_X(A) \to 0$$

<u>which is functorial in</u> A. ▨

<u>From now on we assume</u> $p > 0$.

(3.3) We introduce some notations. Given a functor G from (commutative rings with 1) to (abelian groups) denote by **sheaf(G)** the sheaf on X which is associated to the pre-sheaf (open U) $\mapsto$ $G(\Gamma(U, \mathcal{O}_X))$. We now define:

$$\mathcal{W} \otimes T\hat{C}K_2(A) = \mathrm{sheaf}(W(-) \otimes_{\mathfrak{D}} T\hat{C}K_2(A))$$
$$[\mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}] \otimes_{\mathfrak{D}} \hat{W}(A) = \mathrm{sheaf}([TCK_2(-)/\partial \underrightarrow{W}(-)] \otimes_{\mathfrak{D}} \hat{W}(A))$$
$$\mathcal{W}_m = \mathrm{sheaf}(W_m(-))$$
$$\mathcal{T}\mathcal{C}\mathcal{K}_{2,m}/\partial_m \underrightarrow{\mathcal{W}} = \mathrm{sheaf}(TCK_{2,m}(-)/\partial_m \underrightarrow{W}(-))$$

Following Serre [18] we define

$$\text{(3.3.1)} \qquad H^n(X, \mathcal{W}) = \varprojlim_m H^n(X, \mathcal{W}_m)$$

and similarly

$$\text{(3.3.2)} \qquad H^n(X, \mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}) = \varprojlim_m H^n(X, \mathcal{T}\mathcal{C}\mathcal{K}_{2,m}/\partial_m \underrightarrow{\mathcal{W}}) .$$

The groups $H^n(X, \mathcal{W})$ and $H^n(X, \mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}})$ are in a natural way $\mathfrak{D}$-modules. One can show that there are natural isomorphisms

$$\text{(3.3.3)} \qquad H^2(X, \mathcal{W} \otimes_{\mathfrak{D}} T\hat{C}K_2(A)) \cong H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2(A)$$

and

$$H^2(X, [\mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}] \otimes_{\mathfrak{D}} \hat{W}(A)) \cong H^2(X, \mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}) \otimes_{\mathfrak{D}} \hat{W}(A) .$$

The proof of this result uses the fact that H^2 is right exact $(\dim X = 2)$ and does not work for H^1.

(3.4) From §2 we conclude that there is an exact sequence of sheaves on X:

$$\text{(3.4.1)} \qquad 0 \to \mathcal{W} \otimes_{\mathfrak{D}} T\hat{C}K_2(A) \to \mathcal{K}_{2, X \otimes A/X} \to [\mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}] \otimes_{\mathfrak{D}} \hat{W}(A) \to 0$$

for every augmented local artinian $\underline{k}$-algebra A, depending functorially on A. Taking cohomology and using the isomorphisms in (3.3.3) we get the following exact sequence of covariant functors from the category of augmented local artinian $\underline{k}$-algebras to the category of abelian groups:

$$\text{(3.4.2)} \qquad \cdots \to H^1(X, [\mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}] \otimes_{\mathfrak{D}} \hat{W}) \to$$
$$\to H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2 \to \widehat{CH}^2_X \to H^2(X, \mathcal{T}\mathcal{C}\mathcal{K}_2/\partial \underrightarrow{\mathcal{W}}) \otimes_{\mathfrak{D}} \hat{W} \to 0 .$$

We now are going to discuss conditions which imply that the functor $H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}}) \otimes_{\mathfrak{D}} \hat{W}$ is pro-representable and that the morphism $H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$ is injective.

(3.5) It can be shown that for all m, n the sequence of sheaves on X

$$(3.5.1) \qquad 0 \to \mathcal{J}\mathcal{C}\mathcal{K}_{2,m}/\partial_m\underset{\rightarrow}{\mathcal{W}} \xrightarrow{V^n} \mathcal{J}\mathcal{C}\mathcal{K}_{2,m+n}/\partial_{m+n}\underset{\rightarrow}{\mathcal{W}} \to$$

$$\xrightarrow{\text{proj.}} \mathcal{J}\mathcal{C}\mathcal{K}_{2,n}/\partial_n\underset{\rightarrow}{\mathcal{W}} \to 0$$

is exact. One has the corresponding exact sequences of cohomology groups

$$(3.5.2) \qquad \cdots \to H^1(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,n}/\partial_n\underset{\rightarrow}{\mathcal{W}}) \xrightarrow{\delta_{n,m}} H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,m}/\partial_m\underset{\rightarrow}{\mathcal{W}})$$

$$\xrightarrow{V^n} H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,m+n}/\partial_{m+n}\underset{\rightarrow}{\mathcal{W}}) \to H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,n}/\partial_n\underset{\rightarrow}{\mathcal{W}}) \to 0 \ .$$

This is analogous to the situation for Witt vectors (see [18]). Because of this analogy we follow Serre in calling $\delta_{n,m}$ a <u>Bockstein operation</u>.

Using the sequences (3.5.2) and the fact that $H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,1}/\partial_1\underset{\rightarrow}{\mathcal{W}})$, being a quotient of $H^2(X, \Omega^1_{X/\underline{k}})$, is finite-dimensional as a vector space over $\underline{k}$, one shows that for every m, $H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,m}/\partial_m\underset{\rightarrow}{\mathcal{W}})$ is a $W(\underline{k})$-module of finite length. Therefore, the projective system $\{H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,m}/\partial_m\underset{\rightarrow}{\mathcal{W}})\}_{m\geq 1}$ has the Mittag-Leffler property. Taking the limit of (3.5.2) for varying m and fixed n we find that the sequence

$$(3.5.3) \qquad H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}}) \xrightarrow{V^n} H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}}) \to H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_{2,n}/\partial_n\underset{\rightarrow}{\mathcal{W}}) \to 0$$

is exact.

(3.6) <u>Theorem.</u> <u>The functor</u> $H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}}) \otimes_{\mathfrak{D}} \hat{W}$ <u>is pro-representable if and only if the left</u> $\mathfrak{D}$<u>-module</u> $H^2(X, \mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}})$ <u>has no V-torsion or equivalently, if and only if all the Bockstein operations</u> $\delta_{n,m}$ <u>are zero.</u>

<u>Proof.</u> This follows from the classification of smooth formal groups by means of their covariant Dieudonné module (see [12] IV(7.12) and V(6.18)) and the results in (3.5) ▨

(3.7) <u>Theorem.</u> <u>Assume that</u> $H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}})$ <u>has no V-torsion. Assume furthermore that the Frobenius endomorphism of</u> $H^2(X,\mathcal{O}_X)$ (i.e., <u>the map induced by the p-th power map on</u> $\mathcal{O}_X$) <u>is bijective.</u> <u>Then the map</u> $H^2(X,\mathcal{W})\otimes_{\mathcal{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$ <u>is injective.</u>

For a proof of this theorem see [19].

(3.8) <u>Remarks.</u> Unfortunately we cannot compute the Bockstein operations. The only surfaces for which we could verify that the Bocksteins are zero all have $H^2(X,\Omega^1_{X/\underline{k}}) = 0$ (which implies $H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}}) = 0$). Among these examples are the rational surfaces and the K3-surfaces.

I expect however that for every smooth projective surface X over a perfect field $\underline{k}$ the covariant Dieudonné module of the formal group $\widehat{Alb}_X$ is isomorphic to $H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}})/(\text{V-torsion})$. That may give us more hold on the situation.

It appears to be difficult to find a good condition which implies the injectivity of the map $H^2(X,\mathcal{W})\otimes_{\mathcal{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$. The condition that Frobenius F should act bijectively on $H^2(X,\mathcal{O}_X)$ is probably much too strong. But on the other hand, if X is a supersingular K3-surface in the sense of [1], F is zero on $H^2(X,\mathcal{W})$ and the map $H^2(X,\mathcal{W})\otimes_{\mathcal{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$ is not injective. So some hypothesis about the action of F on $H^2(X,\mathcal{W})$ may eventually appear to be necessary.

(3.9) The hypotheses in Theorem (3.7) are so strong that they have other nice consequences besides the injectivity of the map $H^2(X,\mathcal{W})\otimes_{\mathcal{D}} T\hat{C}K_2 \to \widehat{CH}^2_X$ and the pro-representability of the functor $H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_2/\partial\underset{\rightarrow}{\mathcal{W}})\otimes_{\mathcal{D}}\hat{W}$.

Define $\mathcal{J}\mathcal{C}\mathcal{K}_{2,m} = \text{sheaf}(TCK_{2,m}(-))$

and $H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_2) = \varprojlim_m H^2(X,\mathcal{J}\mathcal{C}\mathcal{K}_{2,m})$.

Then under the hypotheses of (3.7) one has (see [19]):

$$H^2(X, \mathcal{IC}\mathcal{K}_{2,m}/\partial_m \underset{\rightarrow}{\mathcal{W}}) \xleftarrow{\sim} H^2(X, \mathcal{IC}\mathcal{K}_{2,m})$$

for every m and moreover

$$\ker[H^2(X, \mathcal{IC}\mathcal{K}_{2,m+1}) \to H^2(X, \mathcal{IC}\mathcal{K}_{2,m})] \cong H^2(X, \Omega^1_X)$$

for every m. The map $\widehat{CH}^2_X \to H^2(X, \mathcal{IC}\mathcal{K}_2) \otimes_{\mathfrak{D}} \hat{W}$ which now appears admits a section, namely the map $H^2(X, \mathcal{IC}\mathcal{K}_2) \otimes_{\mathfrak{D}} \hat{W} \to \widehat{CH}^2_X$ which always exists (cf (2.6.2)). So we find:

<u>Under the hypotheses of Theorem (3.7) there is a split short exact sequence of functors</u>

(3.9.1) $\quad 0 \to H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2 \to \widehat{CH}^2_X \to H^2(X, \mathcal{IC}\mathcal{K}_2) \otimes_{\mathfrak{D}} \hat{W} \to 0 .$

As examples we can at the moment only offer:

1) rational surfaces: in this case all terms of (3.9.1) are zero.

2) K3-surfaces whose formal Brauer group $\widehat{Br}_X$ (see [2]) is isomorphic to the multiplicative group $\mathbb{G}_m$: in this case $\widehat{CH}^2_X \cong H^2(X, \mathcal{W}) \otimes_{\mathfrak{D}} T\hat{C}K_2$

$$\cong W(\underline{k}) \otimes_{\mathfrak{D}} T\hat{C}K_2 .$$

APPENDIX

In this appendix we prove the following theorem

<u>Theorem</u>. <u>Let</u> $\underline{k}$ <u>be a perfect field of characteristic</u> $p>0$. <u>Let</u> M <u>be a right</u> $\mathfrak{D}$<u>-module and let</u> G <u>be a pro-representable functor from the category of augmented local artinian</u> $\underline{k}$<u>-algebras to the category of abelian groups</u>. <u>Then there are no non-trivial natural transformations from the functor</u> $M \otimes_{\mathfrak{D}} T\hat{C}K_2$ <u>to</u> G.

<u>Proof</u>. Let A be an augmented local artinian $\underline{k}$-algebra. According to (2.4.4), $T\hat{C}K_2(A)$ is generated by the elements $E\langle ax^{p^r-1}, x\rangle$ with $a \in \underline{m}_A$, $r \geq 0$ and the elements $E\langle bx^{p^r}, a\rangle$ with $a, b \in A$, a or $b \in \underline{m}_A$,

$r \geq 0$. Using the relations (D1)-(D3) one reduces the set of generators to the set of elements $E\langle ax^{p^r-1}, x\rangle$ as before and of those elements $E\langle bx^{p^r}, a\rangle$ with $b \in \underline{m}_A$, $a \in \underline{m}_A$, or $a \in \underline{k}$, $r \geq 0$. We show that $E\langle bx^{p^r}, a\rangle = 0$ for $b \in \underline{m}_A$ and $a \in \underline{k}$. Let N be such that $b^{p^N} = 0$. There exists $c \in \underline{k}$ such that $a = c^{p^N}$. Then we have $E\langle bx^{p^r}, a\rangle = E\langle bx^{p^r}, c^{p^N}\rangle \overset{(D3)}{=\!=} p^N E\langle c^{p^N-1}bx^{p^r}, c\rangle \overset{(D2)}{=\!=} E\langle c^{p^{2N}-1}b^{p^N}x^{p^{r+N}}, c\rangle = 0$. Thus we find that the group $M\otimes_{\mathfrak{D}} T\hat{C}K_2(A)$ is generated by the elements $m \otimes E\langle ax^{p^r-1}, x\rangle$ and $m\otimes E\langle bx, a\rangle$ with $m \in M$, $a, b \in \underline{m}_A$, $r \geq 0$.

We introduce the following notation: for every $s \geq 0$ and $\ell \geq 1$ with $p \nmid \ell$ let $I_{s,\ell} \subset \underline{k}[a,b]$ be the ideal generated by the monomials $a^n b^q$ with $np^s + q\ell \geq \ell p^s$ and we define $A_{s,\ell} = \underline{k}[a,b]/I_{s,\ell}$.

Now let $\varphi\colon M\otimes_{\mathfrak{D}} T\hat{C}K_2 \to G$ be a natural transformation. We want to show that for every A, φ kills the generators of $M\otimes_{\mathfrak{D}} T\hat{C}K_2(A)$. It suffices to show $\varphi(m\otimes E\langle ax^{p^r-1}, x\rangle) = 0$ and $\varphi(m\otimes E\langle bx, a\rangle) = 0$ when $A = A_{s,\ell}$ for all s and ℓ with $p\nmid\ell$ and $p^{s-r} > \ell$. Fix s and ℓ. Put $A = A_{s,\ell}$ and $A' = \underline{k}[u]/(u^{\ell p^s})$. The substitution $a \mapsto u^{p^s}$, $b \mapsto u^{\ell}$ defines an injective ring homomorphism $f\colon A \to A'$. Since G is pro-representable, the map $G(f)\colon G(A) \to G(A')$ is injective, too.

The little computations below show that the elements $m\otimes E\langle ax^{p^r-1}, x\rangle$ and $m\otimes E\langle bx, a\rangle$ are killed by the map $M\otimes_{\mathfrak{D}} T\hat{C}K_2(A) \to M\otimes_{\mathfrak{D}} T\hat{C}K_2(A')$ which is induced by f. So φ maps them into $\ker G(f) = 0$. We are done.

It remains to give the little computations:

$$
\begin{aligned}
E\langle u^{p^s}x^{p^r-1}, x\rangle &= p^r E\langle u^{p^{s-r}}, x\rangle && \text{by (D2) and (D1)}\\
&= p^s E\langle u, u^{p^{s-r}-1}x\rangle && \text{by (D3)}\\
&= E\langle u, u^{p^{2s-r}-1}x^{p^s}\rangle && \text{by (D2)}\\
&= 0 && \text{since } p^{2s-r} > \ell p^s
\end{aligned}
$$

$$
\begin{aligned}
E\langle u^{\ell} x, u^{p^s}\rangle &= p^s E\langle u^{\ell+p^s-1} x, u\rangle && \text{by (D3)} \\
&= E\langle u^{p^s\ell+p^{2s}-1} x^{p^s}, u\rangle && \text{by (D2), (D1)} \\
&= 0 && \blacksquare
\end{aligned}
$$

The preceding theorem has the following analogue in characteristic zero. It is left to the reader to prove this.

Theorem. <u>Let</u> $\underline{k}$ <u>be an algebraic extension of</u> $\mathbb{Q}$. <u>Let</u> M <u>be a</u> $\underline{k}$<u>-vector space and let</u> G <u>be a pro-representable functor from the category of augmented local artinian</u> $\underline{k}$<u>-algebras to the category of abelian groups. Then there are no non-trivial natural transformations from the functor</u> $M\otimes_{\underline{k}}[\Omega^1_{-/\underline{k}}/d\underline{m}_-]$ <u>to</u> G.

These two theorems show that under the hypotheses of (3.7) and (3.2) the subgroups $H^2(X,\mathcal{W})\otimes_{\mathcal{D}} T\hat{C}K_2$ and $H^2(X,\mathcal{O}_X)\otimes_{\underline{k}}[\Omega^1_{-/\underline{k}}/d\underline{m}_-]$, respectively, of $\widehat{CH}^2_X$ are the smallest subgroups for which the corresponding quotient is prorepresentable.

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