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PISOT SEQUENCES, PISOT NUMBERS AND SALEM NUMBERS

by

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1. The sets S and H: The well known set S of Pisot (or Pisot-Vijayaraghavan) numbers is the set of algebraic integers $\theta > 1$ all of whose other conjugates lie strictly within the unit circle. The initial interest in S stems from the fact that if $\lambda \in \mathbb{Z}(\theta)$, then $\|\lambda\theta^n\| = \text{dist}(\lambda\theta^n, \mathbb{Z}) \to 0$ as $n \to \infty$. Let us denote by H the set of real $\theta > 1$ for which there is a $\lambda > 0$ such that $\|\lambda\theta^n\| \to 0$. A still unanswered question is whether $S = H$. This was considered by Thue [16] and Hardy [17], who showed that if $\|\lambda\theta^n\| = O(b^n)$ with $b < 1$, then $\theta \in S$. Hardy also pointed out that the only algebraic elements in H are the elements of S. Generalizations of this result were given by Vijayaraghavan in [17].

Until recently Pisot's result [13], that $\sum \|\lambda\theta^n\|^2 < \infty$ implies $\theta \in S$ was essentially the closest approach to a proof that $S = H$, but Cantor [7] has recently given a substantial improvement of this which is somewhat technical to describe here. Salem [14] used Pisot's result to prove that the set S is closed and hence is nowhere dense in $[1,\infty)$.

An interesting fact about H is that it is a countable set. Thus, if H contains any transcendental numbers then it does not do so for trivial reasons. We will see Pisot's [13] proof that H is countable in what follows. It should be mentioned that Vijayaraghavan [18] proved that the set of θ for which $\|\theta^n\| \to 0$ is countable by a somewhat different method.

2. E-sequences: Pisot's method of proof is to examine a certain interesting class of sequences of integers, now called E-sequences or Pisot sequences. To see

how these arise, suppose that $a_n = \lambda\theta^n + \varepsilon_n$, where $\lambda > 0$, $\theta > 1$, $a_n \in \mathbb{Z}$ and ε_n is bounded. We observe that

$$a_{n+1}a_{n-1} - a_n^2 = \lambda\theta^{n-1}(\theta^2\varepsilon_{n-1} - 2\theta\varepsilon_n + \varepsilon_{n+1}) + (\varepsilon_{n+1}\varepsilon_{n-1} - \varepsilon_n^2) ,$$

so that

$$\limsup |a_{n+1} - a_n^2/a_{n-1}| = \limsup |\theta^2\varepsilon_{n-1} - 2\theta\varepsilon_n + \varepsilon_{n+1}| = \delta \quad \text{,say.}$$

If $\delta < 1/2$, then eventually a_{n+1} is determined uniquely by a_n and a_{n-1}. By deleting some initial terms, we have that

$$\text{(1)} \qquad a_{n+1} = N(a_n^2/a_{n-1}) \quad , \quad n = 0,1,\ldots \quad ,$$

where $N(x) = [x + 1/2]$ = "the nearest integer to x". The formula (1) defines the E-sequence $E(a_0,a_1)$ for arbitrary integers $0 < a_0 < a_1$. Pisot showed that the limit $a_{n+1}/a_n \to \theta$ always exists, and this defines a certain set E. Clearly E is countable and contains H, ($\delta = 0$), so H is countable. On the other hand E is dense in $[1,\infty)$ so $E \neq S$, since S is nowhere dense by Salem's result.

One can show that $\lambda = \lim a_n/\theta^n$ exists if $\theta > 1$, and if one defines $\varepsilon_n = a_n - \lambda\theta^n$, then the above discussion shows that E is essentially characterized by the inequality

$$\text{(2)} \qquad \limsup |\theta^2\varepsilon_{n-1} - 2\theta\varepsilon_n + \varepsilon_{n+1}| \leq 1/2 \quad ,$$

in the sense that (2) is necessary for a_n to be an E-sequence, while (2) with strict inequality is sufficient for $\{a_{n-n_0}\}$, $n \geq n_0$ to be an E-sequence for some n_0.

In addition to the set S, E also contains the set T of Salem numbers which are real algebraic integers $\theta > 1$ such that all other conjugates lie within the unit circle, with at least one conjugate on the circle. This in fact implies that θ satisfies a reciprocal equation, so its conjugates are θ^{-1} and a certain set of numbers of modulus one [15]. To see that $E \supset T$, just choose $\lambda \in \mathbb{Z}(\theta)$ so that the other conjugates of λ are small enough so that (2) holds.

3. Recurrent E-sequences: The interesting question now is whether $E = S \cup T$, since this would tell us that T is dense in $[1,\infty)$ and hence that $\inf T = 1$, settling Lehmer's conjecture [12]. It would also imply that $H = S$, settling Pisot's conjecture.

One notes that the proof that $E \supset S \cup T$ shows somewhat more, namely that the corresponding E-sequence satisfies a linear recurrence relation, or equivalently that the generating function of the sequence is rational, so

$$\sum_{n=0}^{\infty} a_n z^n = A(z)/Q(z) \quad , \tag{3}$$

where A and Q are polynomials with integer coefficients, and $Q(0) = 1$. In [9], Flor shows that if $E(a_0,a_1)$ satisfies (3), then θ is in S or in T. We shall refer to these two possibilities as S-recurrence and T-recurrence.

In fact, in [13], Pisot already showed that $E(2,a_1)$ and $E(3,a_1)$ are S-recurrent with $\deg(Q) \le a_0$. For example $E(3,5) = 3, 5, 8, 13, 21, \ldots$ has degree 2. His proof distinguishes $E(a_0,a_1)$ according to the congruence class $a_1 \pmod{a_0^2}$. Cantor [6] has given the explanation of why this is natural, and has studied the families $E(a_0,ma_0^2 + b)$, giving conditions on a_0 and b in order that this sequence is S-recurrent for all $m \ge m_0$. The corresponding generating function is of the form $A(z)/(Q(z) - mzA(z))$.

However, Cantor and his student Galyean [5], by use of a computer algorithm designed for testing for linear recurrences showed that if $E(4,13)$ is recurrent, then $\deg(Q) \ge 100$, suggesting strongly that no such recurrence exists. In his thesis [10], Galyean found many examples of $E(a_0,a_1)$ satisfying no recurrence of degree ≤ 20, when $4 \le a_0 \le 10$.

4. Non-recurrent E-sequences: I was aware only of the example $E(4,13)$ when I proved [1] that indeed there are E-sequences which are non-recurrent, and in fact that the set of θ produced from such sequences is dense in $[(\sqrt{5} + 1)/2,\infty)$. The proof is rather amusing since it concentrates its attention on T-recurrence,

which one might expect to be the difficult case. The point is that, although we have very little quantitative information about T itself, T-recurrent sequences are so distinctive that non-T-recurrence is rather easily detected. In principle, S-recurrence causes no difficulty since one can work in intervals disjoint from S. However, as we shall see later, for specific E-sequences, S-recurrence is more difficult to handle because the intervals in the complement of S are extremely short for even moderately large θ.

To see how T-recurrence is dealt with, suppose then that $E(a_0,a_1)$ is T-recurrent, then, taking into account the structure of the conjugates of θ,

(4) $$a_n = \lambda\theta^n + \mu\theta^{-n} + \delta_n \quad , \quad n \geq n_0,$$

where δ_n is a linear combination of powers of numbers of modulus 1 and hence is almost periodic. Using (2) and the almost periodicity of δ_n, we find that, for all n, including negative n,

(5) $$|\theta^2\delta_{n-1} - 2\theta\delta_n + \delta_{n+1}| \leq 1/2 \ .$$

Furthermore, (4) can be used to define a_n for $n < n_0$, and since Q is reciprocal or antireciprocal, one finds that a_n is an integer for all n. Combining these two facts one then obtains a constructive extimate for n_0 (and this is where the condition $\theta > (\sqrt{5} + 1)/2$ seems unavoidable). For example, if $\theta > 2$ then $n_0 = 0$. Assuming then that $n_0 = 0$ (by shifting the sequence if necessary), the conditions that a_n be an integer for $n < 0$, combined with (5), produce various inequalities which must be satisfied by T-recurrent sequences. As a simple example, the condition that a_{-1} is an integer implies that

$$\|a_0^2/a_1\| \leq (1 + 2\theta)/(2\theta^2) + 1/a_1 \ .$$

This is an extremely restrictive condition for large θ, and shows that non-recurrent E-sequences produce a set of θ dense in $[1 + \sqrt{2}\ , \infty)$.

Applying this inequality to a family $E(a_0, ma_0^2 + b)$ with $b > 0$, we find that $\|a_0^2/a_1\| = a_0^2/a_1$ if $m \geq 2$, while on the other hand $(1+2\theta)/(2\theta^2) + 1/a_1$

is approximately $(a_0 + 1)/a_1$. Thus, within such a family, T-recurrence can only occur if $m = 0$ or 1. As another example consider $E(2m,7m)$ with $m \equiv 1 \pmod 7$. Then $\|a_0^2/a_1\| = 3/7$ while $(1 + 2\theta)/(2\theta^2) \to 16/49 < 3/7$ as $m \to \infty$. Thus, for sufficiently large m, none of these sequences is T-recurrent. Since $\theta \to 7/2 \notin S$ as $m \to \infty$, and since S is closed, it follows that $\theta \notin S$ for sufficiently large m, so $E(a_0,a_1)$ is not S-recurrent either.

5. Specific cases of non-recurrence: In spite of the ease of producing infinitely many non-recurrent E-sequences, one would still like to be able to answer the question of whether any specific $E(a_0,a_1)$ is recurrent or not. In his thesis [10], Galyean conjectured that if $E(a_0,a_1)$ is recurrent, then the degree of the recurrence is at most a_0. A proof of this would certainly provide the desired criterion. A result of this type seems reasonable when one considers that, in an E-sequence $\lambda \approx a_0$, and in order to make ε_n small enough for (2) to hold, it seems necessary to have the other conjugates of λ small. This in turn forces λ to be fairly large since the product of these numbers is at least as large as $1/\mathrm{disc}(\theta)$.

However, lacking such a quantitative result, we have based our proofs of non-recurrence for specific E-sequences on a different method. Proofs of non-T-recurrence are based on refinements of the ideas discussed above. It seems likely that the infinite set of necessary conditions for T-recurrence so obtained are also sufficient; this has certainly proved to be the case in practice. To prove non-S-recurrence we simply have to show that $\theta \notin S$, a constructively feasible procedure since S is closed and since we can generate arbitrarily good approximations to θ. The practical difficulties grow with θ so our success with this method is confined to $\theta < 2.5$. The main tool is a computer algorithm based on ideas of Dufresnoy and Pisot [8] and described in more detail in [3]. It is capable of finding all the elements in $S \cap (\alpha,\beta)$, provided this number is finite. The idea is that, if P is the minimal polynomial of θ, and $Q(z) = z^{\deg(P)}P(z^{-1})$, then

(6) $$f(z) = (\operatorname{sgn} P(0))P(z)/Q(z) = u_0 + u_1 z + \dots ,$$

where the u_n are integers and where $|f(z)| = 1$ on $|z| = 1$. The u_n are characterized by inequalities obtained from Schur's algorithm:

(7) $$w_n(u_0,\dots,u_{n-1}) \le u_n \le w_n^*(u_0,\dots,u_{n-1}) .$$

If in addition $\alpha \le \theta \le \beta$, then there are additional inequalities

(8) $$v_n(u_0,\dots,u_{n-1};\alpha) \le u_n \le v_n^*(u_0,\dots,u_{n-1};\beta) .$$

These lead to the search of a finite tree if $S \cap (\alpha,\beta)$ is a finite set.

An instructive example is the sequence E(10,22), with $\theta = 2.190327956\dots$ The criteria for T-recurrence are easily shown to be violated. A search of a small interval containing θ shows that $\operatorname{dist}(\theta,S) = .905 \times 10^{-8}$, the closest point of S being a root of the following 32nd degree polynomial:

P = 1 -2 0 0 -1 -2 0 0 -2 0 0 0 -1 2 0 0 1 2 0 0 2 0 0 0 1 -2 0 0-1-1 0 0 -1

(notation: a b c ... means $ax^k + bx^{k-1} + \dots$). Thus E(10,22) is non-recurrent. From Galyean's thesis, we find that E(10,22) is predicted to a_{21} by the generating function $(10 + 2z + 4z^2 + 9z^3)/(1 - 2z - 2z^4)$. However the polynomial $z^4 - 2z^3 - 2$, in addition to a root $\phi = 2.190327947$, has roots $\gamma,\bar{\gamma}$ with $|\gamma| \simeq 1.0157$. Hence this is not the generating function of an E-sequence. The fact that $|\gamma|^{44} < 2$ makes it clear how this sequence can masquerade as an E-sequence for many terms. Intuitively, it appears that E(10,22) is diverted away from nearby S-numbers of small degree by the presence of this "pseudo"-S-number of degree 4. Since $a_0 = 10$ is apparently too small to allow E(10,22) to satisfy a recurrence of high degree, the sequence is unable to satisfy any recurrence whatsoever.

An extremely interesting example of this type, mentioned in [5], is E(6,16) which is connected with the polynomial $P(z) = z^5 - 3z^4 + z^3 - z - 1$, which has roots at $\phi = 2.699\dots$ and $\gamma, \bar{\gamma}$ with $|\gamma| \simeq 1.007$. This polynomial turns out to be a limit point of polynomials with the same properties. Since $\operatorname{dist}(\phi,S) < 10^{-46}$, we have as yet been unable to show E(6,16) is not S-recurrent.

There are in addition many other examples of non-recurrence which are not explainable by this mechanism. For example, the non-recurrence of $E(7,15)$ seems to be explained by our arbitrary choice of "rounding up" in the definition of $N(x)$. For details of this and other examples, the reader may consult [3].

6. <u>Concluding Remarks</u>: Space has not permitted a discussion of the new characterization of T given in [2], nor the application of the above-mentioned computer algorithm to questions concerning the distribution of T in the real line, but this is adequately described in [3].

As far as applications of E-sequences to finding T-numbers, as suggested in [5], it seems that a more fruitful type of sequence to use is given by the following non-linear recurrence:

$$a_{n+2} = N(a_{n+1}(a_{n+1} + a_{n-1})/a_n - a_n) \quad , \quad n = 1,2,\ldots$$

If one takes $a_0 = 0$, $a_1 > 0$ and $a_2 \geq 2a_1+1$, then one obtains all Salem numbers as limits of the ratios a_{n+1}/a_n . The criterion for T-recurrence is now valid for all $\theta > 1$, because the inequality (5) is replaced by a more amenable form. Some details concerning these sequences are to be found in [4].

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