R. C. VAUGHAN

A survey of some important problems in additive number theory

Astérisque, tome 61 (1979), p. 213-222
<http://www.numdam.org/item?id=AST_1979__61__213_0>

Société Mathématique de France
Astérisque 61 (1979) p. 213-222

A SURVEY OF SOME IMPORTANT PROBLEMS IN ADDITIVE NUMBER THEORY

by

R.C. VAUGHAN

-:-:-:-

Since many aspects of additive number theory were covered by Halberstam's adress [27] to the recent meeting on Additive Number Theory in Bordeaux, I shall content myself by adumbrating just two of the principal areas which have interested me particularly. These are problems dealing with

(A) sums of kth powers,

(B) sums of primes.

One of the fascinating aspects of these problems is the interplay between them and other areas of analytic number theory.

A. - The typical problem involving sums of kth powers is

1. - <u>Waring's problem</u>, regarding which there is an excellent survey article by Ellison [24]. Let $g(k)$ denote the smallest s such that for every $n \geq 1$ there exist $x_i \geq 0$ such that $n = x_1^k + \ldots + x_s^k$. The problem of evaluating g has been essentially solved for all k except $k = 4$. It is thought that

$$g(k) = 2^k + [(\tfrac{3}{2})^k] - 2 . \tag{1}$$

It is classical that this holds whenever $k \neq 4, 5$ and

$$\{(\tfrac{3}{2})^k\} < 1 - 2^{-k}[(\tfrac{3}{2})^k] . \tag{2}$$

Mahler [38] has shown that (2) has at most a finite number of exceptions, and Stemmler [50] has verified that (2) holds for $k \leq 200\,000$. Incidently, this has lead to interesting questions concerning distribution modulo 1, see Mahler [39].

More recently, Chen [3,4,5] has shown that (1) also holds when $k=5$. This leaves $k=4$. Here there has been considerable recent progress. The upper bound for $g(4)$ has been reduced first to 34, then to 30 and 23 and finally to 22 by Dress [22], Dress [23], Thomas [51] and Thomas [52] respectively. It is trivial that $g(4) \geq 19$.

2. - The more interesting and challenging problem is that of the estimation of $G(k)$, the smallest s such that every sufficiently large integer is the sum of at most s kth powers of positive integers. So far only $G(2)$ and $G(4)$ are known. If one defines $\Gamma(k)$ to be the least s such that for every q, n the congruence $x_1^k + \ldots + x_s^k \equiv n \pmod q$ is soluble, then one has $G(k) \geq \max(k+1, \Gamma(k))$. One might guess that equality occurs. The current of play for small values of k is as follows ;

$G(2) = 4$, Lagrange [34],
$G(3) \leq 7$, Linnik [35], Watson [66],
$G(4) = 16$, Davenport [12],
$G(5) \leq 23$, $G(6) \leq 36$, Davenport [13, 14],
$G(7) \leq 53$, Davenport's method (the claim $G(7) \leq 52$ of Sambasiva Rao [46] is fallacious),
$G(8) \leq 73$, Narasimhamurti [43].

For larger k, the principle results in the last thirty years have been

$G(k) < k(3\log k + 11)$, Vinogradov [63, 64],
$\left.\begin{array}{l} G(k) < k(3\log k + 9) \quad (k = 2^m), \\ G(k) < k(3\log k + 7) \quad (k \neq 2^m), \end{array}\right\}$ Tong [53],
$G(k) < k(3\log k + 5.2)$, Chen [2],
$G(k) < k(2\log k + 4\log\log k + 2\log\log\log k + 13)$, Vinogradov [65].

This last result is superior to Chen's only when $k > 6\,103\,975\,350$. More recently, the method described in Vaughan [59] gives $G(9) \leq 91$, $G(10) \leq 107$, $G(11) \leq 122$, $G(12) \leq 137$, $G(13) \leq 153$, $G(14) \leq 168$, $G(15) \leq 184$, $G(16) \leq 200$, $G(17) \leq 216$ and $G(k) < k(3\log k + 4.2)$.

3. - Homogeneous additive equations. - Davenport and Lewis [15] have shown that there is an $s(k)$ such that if $s \geq s(k)$, then for every $c_1, \ldots, c_s$ (with $c_1 c_2 < 0$

if k is even) the equation $c_1x_1^k + \ldots + c_sx_s^k = 0$ has a non-trivial solution in integers $x_1, \ldots, x_s$. They showed that it is possible to take $s(k) \leq k^2 + 1$ when $k \leq 6$ or $k \geq 18$ giving partial verification of Artin's conjecture that any form of odd degree represents 0 non-trivially whenever $s \geq k^2 + 1$.

Vaughan [59] has partly filled the gap by showing that $s(k) \leq k^2 + 1$ is permissible when $11 \leq k \leq 17$.

4. - Simultaneous homogeneous additive equations. - Davenport and Lewis [17] have treated the system

$$\begin{cases} c_{11}x_1^k + \ldots + c_{1n}x_n^k = 0, \\ \vdots \qquad\qquad \vdots \\ c_{r1}x_1^k + \ldots + c_{rn}x_n^k = 0. \end{cases}$$

There are many open questions in connection with this. Earlier [16], they had studied pairs of additive cubics

$$(2) \quad \begin{cases} c_1x_1^3 + \ldots + c_nx_n^3 = 0, \\ d_1x_1^3 + \ldots + d_nx_n^3 = 0. \end{cases}$$

They showed that if $n \geq 18$, then there is a non-trivial solution of (2), and that there exist $c_1, d_1, \ldots, c_{15}, d_{15}$ such that (2) with $n = 15$ has only the trivial solution. Cook [10] has replaced the 18 by 17 and Vaughan [57] has reduced this to 16, the best possible.

For related matters see Davenport and Lewis [18].

5. - Vinogradov's mean value theorem. - Let $I(X, s, k)$ denote the number of solutions of

$$\begin{cases} x_1 + \ldots + x_s = y_1 + \ldots + y_s \\ x_1^2 + \ldots + x_s^2 = y_1^2 + \ldots + y_s^2 \\ \cdot \qquad \cdot \quad \cdot \qquad \cdot \\ x_1^k + \ldots + x_s^k = y_1^k + \ldots + y_s^k \end{cases}$$

with $0 < x_i, y_i \leq X$. Karatsuba and Korobov [31] have shown that

$$I(X, s, k) < C(k, \ell)\, X^{2s - \frac{1}{2}(k+1) + \delta}$$

with $\delta = \frac{1}{2}k(k+1)(1-\frac{1}{k})^{\ell}$ whenever $s \geq k^2+k\ell$. For an earlier account of this see Vinogradov's book [63, 64]. There are a number of important applications.

The value of $C(k,\ell)$ is not usually very important in additive number theory, but the contrary is true in the applications to multiplicative number theory.

Recently Bombieri has shown that it is possible to take $\delta = \frac{1}{2}k^2(1-\frac{1}{k})^{\ell}$ whenever $s \geq k\ell$.

B. - The archetypal problem concerning sums of primes is Goldbach's problem. This stems from two letters from Goldbach to Euler in 1742 in which he conjectures that every even natural number is the sum of two primes and that every integer greater than 2 is the sum of three primes. He included unity as a prime. There have been three lines of attack on these problems.

1. - Direct applications of sieve methods. - There are excellent surveys of earlier work in Halberstam and Roth [29] and Halberstam and Richert [28]. The most recent result is the celebrated theorem of Chen [6, 7] to the effect that for $n>n_0$ either $2n=p+p_1$ or $2n=p+p_1p_2$. There are shorter proofs by Ding, Pan and Wang [21], and Ross [44]. Ross [45] has also shown that the primes can be restricted in various ways. Graham [26] has made n_0 effectively computable.

2. - Indirect applications of sieve methods. - This stems from Shnirel'man [47, 48]. He showed that there exists a constant C such that if $n>n_0$, then $n=p_1+\ldots+p_s$ with $s \leq C$. His C is very large, and the method was later superceded by the more powerful Hardy-Littlewood-Vinogradov method (see below). However, alternative lines of approach are always of interest in connection with difficult problems. In recent times Chechuro and Kuzjashev [1], and Siebert [49] obtained $C=10$ by this method. This is improved to $C=6$ in Vaughan [56]. This last paper contains a brief survey of previous work via this method.

Perhaps more interesting is the fact that this method readily yields a C_0 such that every $n>1$ can be written as the sum of at most C_0 primes. The most recent work in this direction is

$C_0 = 6\times 10^9$, Klimov [32],
$C_0 = 115$, Klimov, Pil'tai and Sheptitskaya [33],

$C_o = 75$, Deshouillers [19],
$C_o = 27$, Vaughan [58].

This last paper contains two different methods, in one of which the calculations are easier. However the more difficult method would permit a smaller C_o provided certain calculations could be carried out. Deshouillers [20] has thereby obtained $C_o = 26$.

3. - The Hardy-Littlewood-Vinogradov method. - By obtaining non-trivial estimates for

$$(3) \qquad \sum_{p \leq N} e^{2\pi i \alpha p} \text{ when } |\alpha - \tfrac{a}{q}| \leq q^{-2}, \ (a,q) = 1, \ (\log N)^A < q \leq N(\log N)^{-A},$$

Vinogradov [62] gave an unconditional proof that every sufficiently large odd integer is the sum of at most three primes. Linnik [36, 37] (see also Chudakov [9]), Montgomery [40] and Vaughan [60] have given different ways of estimating (3).

Immediately following Vinogradov's work, Chudakov [8], van der Corput [11] and Estermann [25] all showed that if $E(x) = |\{n \leq x : 2n \nmid p + p'\}|$, then $E(x) = O_A(x \log^{-A} x)$. Thix was later improved to $O(x \exp(-c\sqrt{\log x}))$ and $O(x^{1-\delta})$ by Vaughan [54] and Montgomery and Vaughan [42] respectively.

For another question connected with Goldbach's problem, see Montgomery and Vaughan [41], and Vaughan [55].

Let me conclude by emphasizing the interaction between this subject and others of analytic number theory. Recently the ideas contained in Vaughan [60] have been used

(a) to give (Vaughan [61]) a new and simple proof of Bombieri's prime number theorem,

(b) by Heath-Brown and Patterson [30] as an aid in their resolution of Kummer's problem concerning cubic Gaussian sums, to the effect that the arguments are uniformly distributed modulo 2π.

-:-:-:-

REFERENCES

[1] E.F. CHECHURO and A.A. KUZJASHEV, The representation of large integers by sums of primes, Studies in Number Theory, n° 3, 45-50, Izdat. Saratov Univ., Saratov 1969.

[2] J.R. CHEN, On Waring's problem for nth powers, Acta Math. Sinica, 8 (1958), 253-257 ; Chinese Math. - Acta 8 (1966), 849-853 (1967).

[3] J.R. CHEN, Waring's problem for g(5), Sci. Record (Peking) (N.S.) 3 (1959), 327-330.

[4] J.R. CHEN, Waring's problem for g(5) = 37, Sci. Sinica 13 (1964), 335.

[5] J.R. CHEN, Waring's problem for g(5) = 37, Sci. Sinica 13 (1964), 1547-1568.

[6] J.R. CHEN, On the representation of a large even integer as the sum of a prime and the product of at most two primes, Kexue Tongbao (Foreign Lang. Ed.) 17 (1966), 385-386.

[7] J.R. CHEN, On the representation of a larger even integer as the sum of a prime and the product of at most two primes, Sci. Sinica 16 (1973), 157-176.

[8] N.G. CHUDAKOV, On the density of the set of even numbers which are not representable as a sum of two odd primes, Izv. Akad. Nauk SSSR Ser. Nat. 2 (1938), 25-40.

[9] N.G. CHUDAKOV, On Goldbach-Vinogradov's theorem, Ann. of Math. (2) 48 (1947), 515-545.

[10] R.J. COOK, Pairs of additive equations, Michigan Math. J. 19 (1972), 325-331.

[11] J.G. van der Corput, Sur l'hypothèse de Goldbach pour presque tous les nombres pairs, Acta Arithmetica 2 (1937), 266-290.

[12] H. DAVENPORT, On Waring's problem for fourth powers, Ann. of Math. 40 (1939), 731-747.

[13] H. DAVENPORT, On sums of positive integeral kth powers, Amer. J. Math. 64 (1942), 189-198.

[14] H. DAVENPORT, On Waring's problem for fifth and sixth powers, Amer. J. Math. 64 (1942), 199-207.

[15] H. DAVENPORT and D.J. LEWIS, Homogeneous additive equations, Proc. Roy. Soc. Ser. A 274 (1963), 443-460.

[16] H. DAVENPORT and D.J. LEWIS, Cubic equations of additive type, Philos. Trans. Roy. Soc. Ser. A 261 (1966), 97-136.

[17] H. DAVENPORT and D.J. LEWIS, Simultaneous equations of additive type, Philos. Trans. Roy. Soc. Ser. A 264 (1969), 557-595.

[18] H. DAVENPORT and D.J. LEWIS, Two additive equations, Number Theory (Proc. Symp. Pure Math., vol. XII, Houston, Texas, 1967), 74-98, Amer. Math. Soc. Providence, R.I., 1969.

[19] J.-M. DESHOUILLERS, Amélioration de la constante de Šnirelman dans le problème de Goldbach, Sém. Delange-Pisot-Poitou 1972/73, Fasc. 2, n° 17, Paris, 1973.

[20] J.-M. DESHOUILLERS, Sur la constante de Šnirel'man, Sém. Delange-Pisot-Poitou 1975/76, Fasc. 2, exp.n° G16, Paris, 1977.

[21] X.-X. DING, C.-D. PAN and Y. WANG, On the representation of every large even integer as a sum of a prime and an almost prime, Sci. Sinica 28 (1975), 599-610.

[22] F. DRESS, Amélioration de la majoration de $g(4)$ dans le problème de Waring : $g(4) \leq 34$, Sém. Delange-Pisot-Poitou 1969/70, fasc. 1, exp. 15, Paris, 1970.

[23] F. DRESS, Sur le problème de Waring pour les puissances quatrièmes, C.R. Acad. Sci. Paris, Sér. A, 272 (1971), 457-459.

[24] W.J. ELLISON, Waring's problem, Amer. Math. Monthly 78 (1971), 10-36.

[25] T. ESTERMANN, On Goldbach's problem : Proof that almost all even positive integers are sums of two primes, Proc. London Math. Soc. (2) 44 (1938), 307-314.

[26] S.W. GRAHAM, Applications of sieve methods, Dissertation, University of Michigan, Ann Arbor, 1977.

[27] H. HALBERSTAM, Additive Number Theory, Proc. Springer Lecture Notes (à paraître).

[28] H. HALBERSTAM and H.-E. RICHERT, Sieve methods, Academic Press, London, 1974.

[29] H. HALBERSTAM and K.F. ROTH, Sequences, vol. 1, Clarendon Press, Oxford, 1966.

[30] D.R. HEATH-BROWN and S.J. PATTERSON, paper in preparation.

[31] A.A. KARATSUBA and N.M. KOROBOV, A mean-value theorem, Dokl. Akad. Nauk SSSR 149 (1963), 245-248.

[32] N.I. KLIMOV, Apropos the computations of Shnirel'man's constant, Volzh. Mat. Sb. Vyp. 7 (1969), 32-40.

[33] N.I. KLIMOV, G.Z. PIL'TAI and T.A. SHEPTITSKAYA, Estimation of the absolute constant in the Goldbach-Shnirel'man problem, Issled. Teor. Chisel, Saratov 4 (1972), 35-51.

[34] J. L. LAGRANGE, Nouv. Mém. Acad. Roy. Soc. Berlin 1770 (1772), 123-133.

[35] Ju. V. LINNIK, On the representation of large numbers as sums of seven cubes, Mat. Sb. 12 (54) (1943), 218-224.

[36] Ju. V. LINNIK, On the possibility of a unique method in certain problems of "additive" and "distributive" prime number theory, Dokl. Akad. Nauk SSSR 48 (1945), 3-7.

[37] Ju. V. LINNIK, A new proof of the Goldbach-Vinogradov theorem, Mat. Sb. 19 (61)(1946), 3-8.

[38] K. MAHLER, On the fractional parts of the powers of a rational number II, Mathematika 4 (1957), 122-124.

[39] K. MAHLER, An unsolved problem on the powers of 3/2, J. Austral. Math. Soc. 8 (1968), 313-321.

[40] H. L. MONTGOMERY, Topics in multiplicative number theory, Lecture Notes in Math. 227 (1971), Berlin and New York.

[41] H. L. MONTGOMERY and R.C. VAUGHAN, Error terms in additive prime number theory, Quart. J. Math. Oxford (2) 24 (1973), 207-216.

[42] H. L. MONTGOMERY and R.C. VAUGHAN, The exceptional set in Goldbach's problem, Acta Arithmetica 27 (1975), 353-370.

[43] V. NARASIMHAMURTI, On Waring's problem for 8th, 9th and 10th powers, J. Indian Math. Soc. 5 1941), 122.

[44] P.M. ROSS, On Chen's theorem that each large even number has the form p_1+p_2 or $p_1+p_2p_3$, J. London Math. Soc. (2) 10 (1975), 500-506.

[45] P.M. ROSS, A short intervals result in additive prime number theory, J. London Math. Soc. (2) 17 (1978), 219-227.

[46] K. SAMBASIVA RAO, On Waring's problem for smaller powers, J. Indian Math. Soc. 5 (1941), 117-121.

[47] L.G. SHNIREL'MAN, On additive properties of numbers, Izv. Donskowo Politeh. Inst., 14 (1930), 3-28.

[48] L.G. SHNIREL'MAN, Uber additiven Eigenschaften von Zahlen, Math. Ann. 107 (1933), 649-690.

[49] H. SIEBERT, Darstellung als Summe von Primzahlen (Diplomarbeit, Marburg, 1968).

[50] R.M. STEMMLER, The ideal Waring theorem for exponents 401-200 000, Math. Comp. 18 (1964), 144-146.

[51] H.E. THOMAS Jr., A numerical approach to Waring's problem for fourth powers, Dissertation, University of Michigan, Ann Arbor, 1973.

[52] H.E. THOMAS Jr., Waring's problem for twenty two biquadrates, Trans. Amer. Math. Soc., 193 (1974).

[53] K.-C. TONG, On Waring's problem, Advancement in Math. 3 (1957), 602-607.

[54] R.C. VAUGHAN, On Goldbach's problem, Acta Arithmetica 22 (1972), 21-48.

[55] R.C. VAUGHAN, A new estimate for the exceptional set in Goldbach's problem, Analytic Number Theory (Proc. Symp. Pure Math., vol. XXIV, St. Louis, Missouri, 1972), 315-320, Amer. Math. Soc. Providence, R.I., 1973.

[56] R.C. VAUGHAN, A note on Shnirel'man's approach to Goldbach's problem, Bull. London Math. Soc., 8 (1976), 245-250.

[57] R.C. VAUGHAN, On pairs of additive cubic equations, Proc. London Math. Soc. (3) 34 (1977); 354-364.

[58] R.C. VAUGHAN, On the estimation of Shnirel'man's constant, J. fur Reine Angew. Math., 290 (1977), 93-108.

[59] R.C. VAUGHAN, Homogeneous additive equations and Waring's problem, Acta Arithmetica, 33 (1977), 231-253.

[60] R.C. VAUGHAN, Sommes trigonométriques sur les nombres premiers, C.R. Acad. Sci. Paris, Sér. A, 258 (1977), 981-983.

[61] R.C. VAUGHAN, An elementary method in prime number theory, Acta Arithmetica, to appear.

[62] I.M. VINOGRADOV, Some theorems concerning the theory of primes, Recueil Math. 2 (44), 2 (1937), 179-195.

[63] I.M. VINOGRADOV, The method of trigonometrical sums in the theory of numbers, Trav. Inst. Math. Steklov 23 (1947).

[64] I.M. VINOGRADOV, The method of trigonometrical sums in the theory of numbers, translated from the Russian, revised an annotated by K.F. Roth and A. Davenport, Interscience, London, 1954

[65] I.M. VINOGRADOV, On an upper bound for G(n), Izv. Akad. Nauk SSSR 23 (1959), 637-642.

[66] G.L. WATSON, A proof of the seven cube theorem, J. London Math. Soc. 26 (1951), 153-156.

-:-:-:-

R.C. VAUGHAN
Mathematics Department
Imperial College
LONDON