Astérisque

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On the local structure of conformal mappings and holomorphic vector fields in C^2

ON THE LOCAL STRUCTURE OF CONFORMAL MAPPINGS AND HOLOMORPHIC VECTOR FIELDS IN C^2.

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The purpose of this note is to classify under topological conjugacy the local holomorphic diffeomorphisms which in a neighborhood of $0 \in C$ have the form

$$(1) \qquad f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots \qquad (z \in C, \quad \lambda^n = 1).$$

We relate also some implications of this in the topological behavior of the solutions of a holomorphic differential equation

$$(2) \qquad \frac{dz}{dT} = Z(z) \qquad (T \in C, \quad z \in C^2)$$

near a singular point $0 \in C^2$ under the following hypothesis. If λ_1, λ_2 denote the eigenvalues of $DZ(0)$, then

$$(3) \qquad \lambda_1 \neq 0 \neq \lambda_2 \quad \text{and} \quad \lambda_1/\lambda_2 \text{ is a rational number.}$$

The relation between (1) and (2) is based upon the general fact that the topology of the solutions of (2) is intimately connected with the holonomy of the invariant manifolds passing through $0 \in C^2$. This holonomy can be represented by local holomorphic diffeomorphisms which in the presence of (3) have the form (1).[1]

§1. Local Conformal Mappings.

A conformal mapping with a fixed point at $0 \in C$ can be expressed locally by a power series

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots. \qquad (\lambda \neq 0).$$

It is well known ([1] p.188) that for $|\lambda| \neq 1$ there is a holomorphic local diffeomorphism h, $h(0) = 0$, such that $h \circ f \circ h^{-1}(z) = \lambda z$. Our aim here is to determine the topological type of f in the case that $\lambda^n = 1$ for some $n \geq 1$. This is given in the next theorem

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[1] We are grateful to Paulo Sad for valuable discussions.

Theorem 1 [1]. Let f be a local conformal mapping

(1) $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$ ($\lambda^n = 1$, $n \geq 1$; $\lambda^m \neq 1$ for $0 < m < n$)

Then either the n-th iterate f^n is the identity, or there is a local homeomorphism h, h(0) = 0, and an integer k ≥ 1, such that

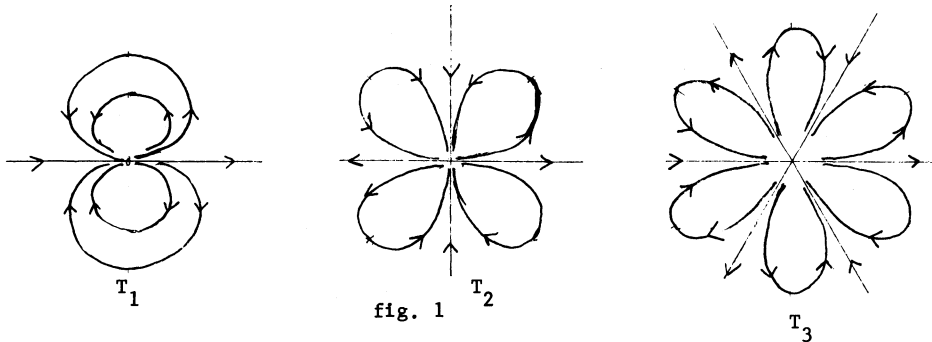$$h \circ f \circ h^{-1}(z) = f_{k,n}(z) = \lambda z(1 + z^{kn})$$

Clearly $f_{k,n}$ leaves invariant the union of kn lines in $\mathbb{C}$:

$$\{z : z^{kn} \in \mathbb{R}\}$$

It is the composition of

$$f_{kn,1} = T_{kn}(z) = z(1 + z^{kn})$$

(see fig.1) which leaves invariant each of 2 kn angular sectors of angular measure π/kn, and the rotation $z \to \lambda z$ over an angle $q\pi/n$, in case $\lambda = \exp(2q\pi i/n)$.

fig. 1

T_1 T_2 T_3

Proof. The following proof can best be read first for k = n = 1, then for k ≥ n = 1, then for n ≥ k = 1, and finally for the general case.

We first obtain a normal form for the initial part of the powerseries (1). Let us assume

(2) $f(z) = \lambda z(1 + b_m z^m + b_{m+1} z^{m+1} + \cdots)$, $b_m \neq 0$.

We try to get rid of the first non vanishing coefficient b_m by a holomorphic change of coordinate

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[1] We are grateful to N.H. Huiper who improved the exposition and rewrote the final version of the proof.

(3) $\qquad z = w(1 + \alpha w^m) = h^{-1}(w)$.

Neglecting terms of degree greater than $2m$, we have the inverse h :

(4) $\qquad w \equiv z(1 - \alpha z^m) \equiv h(z) \mod z^{2m+1}$

Thus we obtain

$$f \circ h^{-1}(w) \equiv \lambda[w(1+\alpha w^m)] \ (1+b_m w^m) \qquad\qquad \mod w^{m+2}$$

and

$$h \circ f \circ h^{-1}(w) \equiv \lambda w(1+\alpha w^m)(1+b_m w^m)(1-\alpha \lambda^m w^m)$$

$$\equiv \lambda w\{1 + [\alpha(1-\lambda^m)+ b_m]w^m\} \equiv \lambda w(1+b_m' w^m) \mod w^{m+2}$$

We can solve α from $b_m' = 0$ in case $m \neq 0 \mod n$. We repeat this process induc-
tively. Suppose we <u>get stuck</u> for the first time at $m = kn$. Then we may assume

$$f(z) = \lambda z(1+b_{kn} z^{kn}+ b_{kn+1} \ b^{kn+1} + \cdots)$$

A final change of coordinate $z = \alpha w$, where $\alpha^{-kn} = b_{kn}$, yields the (restricted)
<u>normal form</u> (again written in z)

(5) $\qquad f(z) = \lambda z(1+z^{kn}+ b_{kn+1} \ z^{kn+1} + \cdots)$

In case we <u>never get stuck</u>, then the powerseries $f(z)$ is equivalent in formal
powerseries to λz , and $(f^n)(z)$ is equivalent to the identity, z . But an
easy calculation shows that then (f^n) is the identity from the beginning. This
takes care of the first part of the theorem. In order to study (5) near $z = 0$ we
place the interesting point 0 at $\infty \in \mathbb{C} \cup \infty$ by a change of coordinate

(6) $\qquad z \rightarrow \dfrac{1}{z}$

For $kn > 1$ we compose this with the diffeomorphism of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ onto the
kn-fold covering $\mathbb{C}^{*[kn]}$ of $\mathbb{C}^*$ given by the formula

(7) $\qquad z \rightarrow z^{kn}$.

Each of the angular sectors mentioned above is then represented by one upper or
lower half-plane. And f is represented by a map that looks like a translation in
the coordinate space $\mathbb{C}^*$. We will use the base space $\mathbb{C}^*$ as local coordinate space
(variable w), but we must always remember the distinction between kn points of
$\mathbb{C}^{*[kn]}$ for each given $w \in \mathbb{C}^*$. We <u>denote</u>

(8) $\qquad \mathbb{C}_r^{*[kn]} = \{P \in \mathbb{C}^{*[kn]} , \ |w(P)| > r\}$

The composition of (6) and (7) is expressed by

$$(9) \qquad w = z^{-kn} \quad , \quad z = w^{-\frac{1}{kn}}$$

It transforms f in (5) into a well defined diffeomorphic embedding

$$G : \mathbb{C}_r^{*[kn]} \to C^{*[kn]}$$

(near ∞ , that is for large r). G is expressed in the $\mathbb{C}^*$-coordinate w by a multivalued function (because $w^{\frac{1}{kn}}$ is kn-valued) :

$$g(w) = [\lambda w^{-\frac{1}{kn}}(1+w^{-1}+ bw^{-1-\frac{1}{kn}} + \cdots)]^{-kn} = w[1-knw^{-1} + \cdots]$$

$$(10) \qquad g(w) = w-kn+cw^{-\frac{1}{kn}} + \cdots$$

For large $|w|$, g(w) tends to the <u>translation</u>

$$g_\tau (w) = w - kn$$

which is covered by a smooth diffeomorphism, the "<u>lifted translation</u>"

$$G_\tau \quad \text{of} \quad \mathbb{C}_r^{*[kn]} \quad \text{into} \quad \mathbb{C}^{*[kn]} \quad ,$$

which is unique by being near to G near ∞ . Also the derivative of g tends to the identity for $w \to \infty$.

Let φ be the C^∞-function

$$\varphi(t) = 0 \qquad \text{for} \quad t \le 0$$

$$\varphi(t) = 1 \qquad \text{for} \quad t \ge 1$$

and for $0 < t < 1$:

$$\varphi(t) = \int_o^t \exp \frac{1}{t(t-1)} \, dt \, / \int_o^1 \exp \frac{1}{t(t-1)} \, dt$$

Let $r_0, r_1, r_2, r_1-r_0, r_2-r_1, r_3-r_2$ be very large positive numbers $\gg (kn)^2$. We leave the suitable choice (see below) to the reader.

We glue G , restricted to $\mathbb{C}_{r_3}^{*[kn]}$, to G_τ , restricted to $\mathbb{C}_{r_0}^{*[kn]} \backslash \mathbb{C}_{r_2}^{*[kn]}$, to obtain a map G_η of $\mathbb{C}_{r_0}^{*[kn]}$ into $\mathbb{C}^{*[kn]}$, by defining its expression in the coordinate $w \in \mathbb{C}^*$ as follows :

$$g_\eta (w) = g_\tau (w) + \varphi \left(\frac{|w|-r_2}{r_3-r_2} \right) [g(w) - g_\tau (w)] \quad .$$

This map is a diffeomorphism if all the above-mentioned numbers are large enough.

As G_η and G are equal on $\mathbb{C}*_{r_3}^{\lceil kn \rceil}$ it suffices now to define a conjugacy between G_η and G_τ, H, such that

(11) $H \circ G_\tau \circ H^{-1} = G_\eta$

We will define H on some part called an "exagerated fundamental domain" $B \cup W$ of $\mathbb{C}*_{r_o}^{\lceil kn \rceil}$ such that for any point $P \in \mathbb{C}*_{r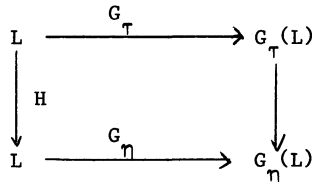_o}^{\lceil kn \rceil}$ there is a representative $G_\tau^\ell(P) \in B \cup W$ for some $\ell \in \mathbb{Z}$. Then H is defined everywhere by (11). But we have to check ambiguities. H will be the identity in

$$B = \mathbb{C}*_{r_o}^{\lceil kn \rceil} \setminus \mathbb{C}*_{r_1}^{\lceil kn \rceil}$$

Recall that G_τ and G_η are equal there ! For the _easiest case_ $k = n = 1$ we have $\mathbb{C}*^{\lceil kn \rceil} = \mathbb{C}*$. Let L be the imaginary axis $\{w : \text{real part of } w = 0\}$ and let W be the closed infinite strip between L and $G_\tau(L)$

(12) $\{w : -1 \le \text{Real}(w) \le 0\}$

We let the restriction $H|L$ be the identity. Then $H|G_\tau(L)$ is defined by the commutativity (see (11)) of the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\;\;G_\tau\;\;} & G_\tau(L) \\
\downarrow{\scriptstyle H} & & \downarrow \\
L & \xrightarrow{\;\;G_\eta\;\;} & G_\eta(L)
\end{array}
$$

We extend the homeomorphism H from $B \cup L \cup G_\tau(L)$ to $B \cup W$ by an arbitrary homeomorphism. Then H is determined by (11) on $\mathbb{C}*_{r_o}^{\lceil kn \rceil}$ and as G_τ is just a simple translation clearly no ambiguities for the global definition of H arise. In case $k > n = 1$ we take for W the part in $\mathbb{C}*_{r_1}^{\lceil kn \rceil}$ of the set of all points that cover the set (12) in the coordinate plane. $B \cup W$ is then obtained from $W \subset \mathbb{C}*_{r_o}^{\lceil kn \rceil}$ by attaching 2k half strips going to ∞ . In fig. 3 the case $k = 2$ is shown. $\mathbb{C}*^{\lceil 2 \rceil}$ is cut open along the positive real axis of $\mathbb{C}*$. In each of the two components, B has two extensions by pieces of W . The global definition of H goes as before. In case $n > k = 1$ we take for W the part of $\mathbb{C}*_{r_o}^{\lceil n \rceil}$ defined by

(13) $\{w : -n \le \text{Real}(w) \le 0\}$

but only in one of the connected components into which $\mathbb{C}*^{\lceil n \rceil}$ is cut by the posi-

tive real axis. G_τ^i maps this part W into the other covering components for $i = 1,...,n-1$. L is again the line and boundary part of W that covers the imaginary axis. H is taken to be the identity there. Then H is uniquely determined on the opposite boundary of W by the commutativity in

$$
\begin{array}{ccc}
L & \xrightarrow{\;\;G_\tau^n\;\;} & (G_\tau^n)(L) \\
\Big\downarrow{\scriptstyle H\,=\,id} & & \Big\downarrow{\scriptstyle H} \\
L & \xrightarrow{\;\;G_\eta^n\;\;} & G_\eta(L)
\end{array}
$$

We extend H from $B \cup L \cup G_\tau^n(L)$ to a homeomorphism defined on the strip $B \cup W$. Again because G_τ^n is a simple lifted translation (over distance n) then H is defined by (11) in all of $C_{r_o}^{*[n]}$ without ambiguity. See fig. 4.

The general case is just a little more complicated. For w we take in $C_{r_o}^{*[kn]}$ of a "union of strips" of width n covering

$$\{w : -n \le Real(w) \le 0\},$$

but only in those components of $(\mathbb{C}^{*[kn]}$ cut open$)$

$$C^{*[kn]} \setminus \{P : w(P) \text{ is positive real}\}$$

that correspond to

$$0 < \arg z < \pi k/n$$

in the z-plane.

In fig. 5 we have illustrated with the case $n = 3$, $k = 2$. We take H to be the identity on one side of these strips $(L \subset \partial W)$ and we extend as before without running into ambiguities.

For $\{\lambda = \exp 2\pi qi/n, k\}$ we have found that any two "G's" are conjugate near ∞ to G_τ and so any two "f's" are mutually topologically conjugate. The proof of the theorem is complete.
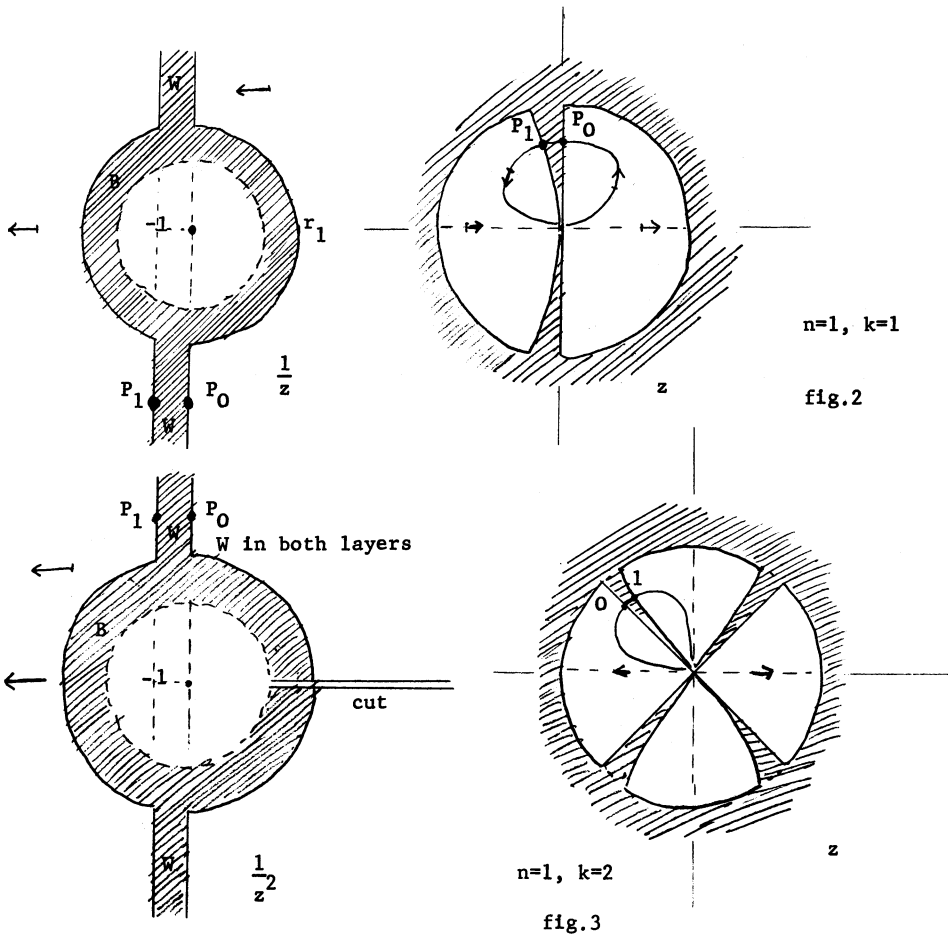
Remarks

1) Another proof of Theorem 1 follows from a theorem of Rodrigues-Roussarie [4] concerning the embedding of local diffeomorphisms of $\mathbb{R}^2$ in flows.

2) A local mapping $f : (\mathbb{C},0) \to (\mathbb{C},0)$ is said to be stable at $0 \in \mathbb{C}$ if for each neighborhood U of $0 \in \mathbb{C}$ there is a neighborhood $V \subset U$ of $0 \in \mathbb{C}$ such that

$f^n(V) \subset U$ for any $n = \pm 1, \pm 2, \ldots$. Then we have (see [1] p.185) that f is stable if and only if f is holomorphically conjugate to a linear rotation. On the other hand, we say that f is unstable at $0 \in \mathbb{C}$ if there is a neighborhood U of $0 \in \mathbb{C}$ such that for each $z \neq 0$ in U there is $n \in \mathbb{Z}$ such that $f^n(z) \notin U$. It was observed in [1] p.187 that it is not known whether there are unstable f with $f'(0)$ a root of unity. From Theorem 1 one obtains that f with $f'(0)$ a root of unity is never unstable.

Exagerated fundamental domains $B \cup W$

n=1, k=1

fig.2

n=1, k=2

fig.3

W   in one layer only

W

-2

cut

W

$\frac{1}{z^2}$

P_2 P_0 P_1, P_3 in other layer

2
0

4   3

z

n=2, k=1

fig.4

P_3 P_0

W

-3

W in two of
the six layers

W

$\frac{1}{z^6}$

$\lambda = e^{2\pi i/3}$, n=3, k=2

Fig.5

R U W

P_1

3
0

P_2

z
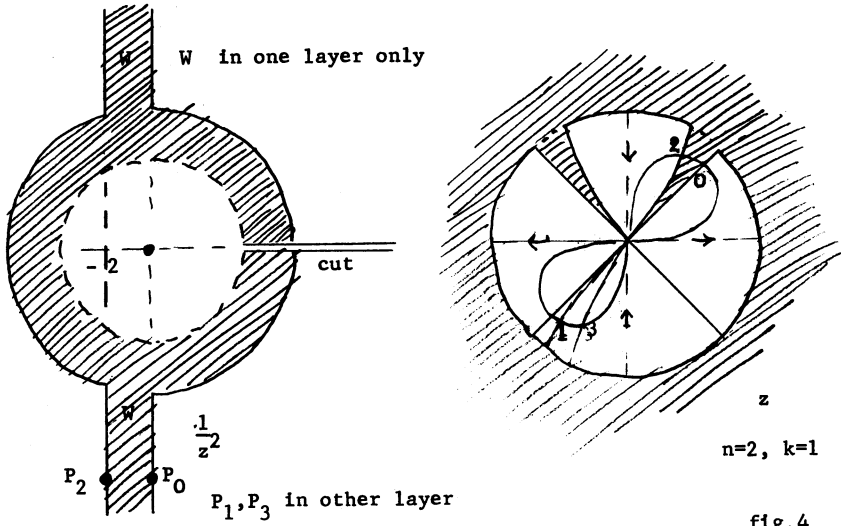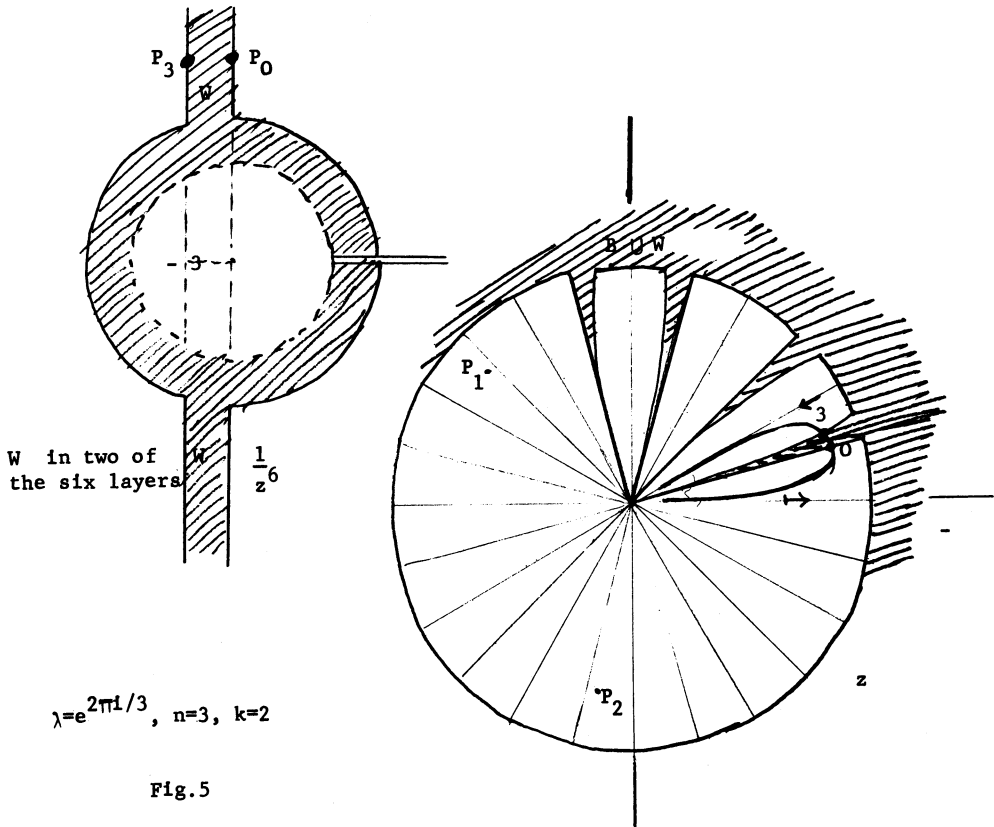
3) The iteration properties of conformal mappings near a fixed point were studied since long ago by Fatou [5] and Julia [6]. In particular Julia gave a description of the asymptotic properties of local mappings $f(C,0) \to (C,0)$ with $f'(0)$ a root of unity.

§2. Holomorphic Vector Fields in C^2.

Consider a holomorphic vector field Z with a singularity at $0 \in C^2$ and such that its linear part $A = DZ(0)$ has nonzero eigenvalues λ_1, λ_2. We wish to study the topology of the solutions of the differential equation

$$(2) \qquad \frac{dz}{dT} = Z(z) \qquad T \in C$$

in a neighborhood of $0 \in C^2$.

When λ_1/λ_2 is not real the topological structure of the solutions of (2) is well known. There is a real flow tangent to the solutions of (2) which is attracting at $0 \in C^2$ and transverse to a small 3-sphere centered at $0 \in C^2$. The intersection of the solutions of (2) with this sphere is an orientable line field, Morse-Smale type, with two periodic orbits linked with linking number one. Similarly when Z is linear the situation is easily understood. Here we will dwell with the more degenerate case $\lambda_1/\lambda_2 \in \mathbb{R}$ and Z non linear. To study Z is more convenient to reduce it to a simpler form by means of holomorphic changes of coordinates. In fact in many situations Z can be reduced to its linear part. This is the case for instance when λ_1/λ_2 is irrational and positive and also for a dense subset of irrational negative numbers, [7], [8]. On the other hand when λ_1/λ_2 is a rational number the best general reduced forms one can get are:

$$(12) \qquad \frac{dz}{dT} = \lambda_1 z_1 \frac{\partial}{\partial z_1} + (\lambda_2 z_2 + \sigma z_1) \frac{\partial}{\partial z_2} \qquad \text{when } \lambda_1 = \lambda_2.$$

$$(13) \qquad \frac{dz}{dT} = \lambda_1 z_1 \frac{\partial}{\partial z_1} + (\lambda_2 z_2 + a z_1^n) \frac{\partial}{\partial z_2} \qquad \text{when } \lambda_2 = n\lambda_1, \quad n > 1.$$

$$(14) \qquad \frac{dz}{dT} = (\lambda_1 z_1 + a\, z_1^{km+1} z_2^{kn} + R_1(z)) \frac{\partial}{\partial z_1} +$$

$$(\lambda_2 z_2 + b\, z_1^{km} z_2^{kn+1} + R_2(z)) \frac{\partial}{\partial z_2} \qquad \text{when } m\lambda_1 + n\lambda_2 = 0$$

where R_1 and R_2 are of order greater than $k(m+n)+1$.

The relation between the eigenvalues λ_1, λ_2 in (13) and (14) are called resonances. It is well known that the nonexistence of resonances implies in formal linearization of Z.

On a question of Dulac.

The study of the topology of solutions of holomorphic differential equations near a singular point $0 \in \mathbb{C}^2$, goes back to H. Dulac. In [9] he was interested among other things in finding solutions which admit $0 \in \mathbb{C}^2$ as a limit point. In the presence of resonances he succeeded in proving the existence of infinitely many solutions with $0 \in \mathbb{C}^2$ in their closure. By using Theorem 1 one can improve his result as follows

THEOREM 2 - Consider the differential equation

$$(2) \qquad \frac{dz}{dT} = Z(z) \qquad (Z(0) = 0, \quad T \in \mathbb{C})$$

such that the eigenvalues λ_1, λ_2 of $DZ(0)$ do not vanish and their quotient λ_1/λ_2 is a rational number. If Z is not linearizable, then any solution of (2) has $0 \in \mathbb{C}^2$ as an accumulation point.

Proof: Given any nonzero $a \in \mathbb{C}$ we define a real differential equation

$$(15) \qquad \frac{dz}{dt} = a \, Z(z) \qquad (t \in \mathbb{R}).$$

The solutions of (15) are contained in those of (2). To see this is enough to take $T = t \cdot a$.

When $\lambda_1/\lambda_2 > 0$ the theorem is evident since it is always possible to find $a \in \mathbb{C}$ such that all solutions of (15) converge to $0 \in \mathbb{C}^2$ as $t \to +\infty$. In fact, the eigenvalues of the linear part of (15) are $a\lambda_1$, $a\lambda_2$, so it is enough to take $a \in \mathbb{C}$ such that these numbers have negative real part.

When $\lambda_1/\lambda_2 < 0$, for any $a \in \mathbb{C}$ such that $\mathrm{Re}(\lambda_1 a) \neq 0 \neq \mathrm{Re}(\lambda_2 a)$ (15) has a saddle structure i.e. $\mathrm{Re}(\lambda_1 a)/\mathrm{Re}(\lambda_2 a) < 0$. Fix $a \in \mathbb{C}$ with this property. Then the stable and unstable manifolds S_1 and S_2 of $0 \in \mathbb{C}^2$ give invariant manifolds of (2). They are in fact complex curves cutting transversally at $0 \in \mathbb{C}^2$.

We associate to S_1 and S_2 their holonomy maps induced by the solutions of (2). The holonomy of S_1 has the following geometric interpretation. Consider in S_1 a little circle γ around $0 \in S_1$ and above γ a narrow fence Γ of real dimension three transverse to the solutions of (2). Clearly Γ is diffeomorphic to $D^2 \times S^1$.

Then the leaf intersection of (2) with Γ is a line field X with Y has a closed integral. The holonomy of S_1 is then represented by the germ of a local diffeomorphism f: $(\mathbb{C},0) \to (\mathbb{C},0)$ equivalent to a first return map of X near Y. Similarly the holonomy of S_2 is represented by a local map g: $(\mathbb{C},0) \to (\mathbb{C},0)$.

An easy computation using (14) shows that f and g have the following expressions

$$f(z) = \mu_1 z + A z^{kn+1} + \ldots \qquad (A \neq 0, \quad \mu_1 = e^{2\pi i \lambda_2/\lambda_1})$$

$$g(z) = \mu_2 z + B z^{km+1} + \ldots \qquad (B \neq 0, \quad \mu_2 = e^{2\pi i \lambda_1/\lambda_2})$$

Then since $\mu_1^n = 1$, Theorem 1 applies to f showing that any integral of X accumulates on Y. Now, since (15) has saddle structure, any solution of (2) different than S_2 intersects Γ and so accumulates in Y. Finally since S_1 and S_2 accumulate on $0 \in \mathbb{C}^2$ the proof is finished.

A more detailed study of the holonomies f and g and their relations allows to prove the following theorem.

THEOREM ([10]) - <u>Consider the differential equation</u>

(*) $\qquad \dfrac{dz}{dT} = Z(z) \qquad (z \in \mathbb{C}^2, \quad Z(0) = 0, \quad T \in \mathbb{C})$.

<u>Let</u> λ_1, λ_2 <u>be the eigenvalues of</u> $DZ(0)$ <u>and suppose that</u>

$$\lambda_1 \neq 0 \neq \lambda_2 \quad \underline{\text{and}} \quad \lambda_1/\lambda_2 \quad \underline{\text{is a rational number.}}$$

<u>Then</u> (a) <u>For Z linear and</u> $\lambda_1 = \lambda_2$ <u>the Jordan form of</u> Z <u>is a topological invariant of</u> (*).

(b) <u>For Z nonlinear and</u> $\lambda_1/\lambda_2 > 0$. <u>Then</u> $\lambda_1/\lambda_2 = n$ <u>or</u> $1/n$, Z <u>has the following normal form</u>

$$Z(z) = \lambda_1 z_1 \frac{\partial}{\partial z_1} + (\lambda_2 z_2 + z_1^n) \frac{\partial}{\partial z_2} \quad \underline{\text{for}} \quad \lambda_2 = n\lambda_1$$

<u>and</u> n <u>is a topological invariant.</u>

(c) <u>For Z nonlinear and</u> $\lambda_1/\lambda_2 < 0$. <u>Then</u> $m\lambda_1 + n\lambda_2 = 0$, $m,n \geq 0$. <u>Let</u> Z <u>be written as</u>

$$Z(z) = (\lambda_1 z_1 + a z_1^{km+1} z_2^{kn} + R_1(z))\frac{\partial}{\partial z_1} + (\lambda_2 z_2 + b z_1^{km} z_2^{kn+1} + R_2(z))\frac{\partial}{\partial z_2}$$

<u>where</u> $k \geq 0$ <u>and</u> R_1, R_2 <u>have order greater than</u> $k(m+n)+1$. <u>Then under nondegeneracy conditions</u> k <u>and</u> λ_1/λ_2 <u>are topological invariants of</u> (*).

The proof of (a) is immediate. The proof of (b) and (c) is more elaborate and consists in showing that the topology of Z is determined by the holonomies of the invariant manifolds passing through the origin.

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