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DOWNCROSSINGS AND THE MARKOV PROPERTY OF LOCAL TIME

by

J.B. WALSH

Paul Lévy suggested that the Brownian local time at zero could be gotten as a limit of the number of downcrossings of the interval $(0,\varepsilon)$ mutiplied by ε, as $\varepsilon \to 0$. This was proved in [3], and a shorter proof has been given in [1]. The method is interesting for several reasons, not the least of which is that it works equally for any continuous local martingale. We refer the reader to [2] for a treatment of local time for semi-martingales from this viewpoint.

Our aim in this article is to use the local-time-as-a-limit-of-upcrossings to approach the Markov properties of the local time discovered by F. Knight and D. Ray.

This approach is quite computational, but in fact the computations are conceptually simple and usually come down to summing an appropriate geometric series. It has the advantage of providing the exact distributions of the quantities involved. In addition, the downcrossings themselves have some rather curious properties, so while the path to the final results may the tedious in some stretches, it affords unexpected views in others.

In sections one to three we study the Brownian local time in the simplest case, in which the process starts at 1 and is stopped when it first hits 0. We then treat the general case in section four, in which we treat the local time for a general regular diffusion. This is perhaps not the most efficient way to do it, because all the ideas needed in the general case are already needed in the special case, making the second half of the article somewhat of a repetition of the first. However, we have let the first part stand, feeling that the theorems are most easily understood in an uncluttered special case, and leaving us free to concentrate on finding the infinitesimal generator in the most general case.

A historical note : the fundamental theorems on the local time which we prove - Theorems 3.3 and 4.1 - are due to Knight and Ray, and to Ray respectively. The only things with any claim to novelty in the article are the approach and the theorems on the downcrossing processes.

1. THE UPCROSSING AND DOWNCROSSING PROCESSES

Let X be a diffusion on the line with a possibly finite lifetime ζ. We suppose that X is canonically defined on the space $(\Omega, \mathcal{F}, P)$ of right-continuous functions on $[0,\infty)$ to $\mathbb{R} \cup \{\delta\}$ - where δ is the cemetary - which are continuous until the lifetime ζ and equal δ from then on. Let θ_t be the translation operator, P_t the semi group, and P^x the **distribution of** X given **that** $X_o = x$. The first-hitting time of a point a is

$$T_a = \inf\{t > 0 : X_t = a\}$$

Let $x < y$ and define stopping times $S_o, S_1, \dots$ by

$$S_o = T_y, \; S_1 = S_o + T_x \circ \theta_{S_o}, \; S_2 = S_1 + T_y \circ \theta_{S_1},$$

and by induction

$$S_{2n} = S_{2n-1} + T_y \circ \theta_{S_{2n-1}}, \; S_{2n+1} = S_{2n} + T_x \circ \theta_{S_{2n}}.$$

Then X_{S_n} equals x for odd n, and y for even n. It makes an upcrossing of (x,y) between S_{2n-1} and S_{2n}, and a downcrossing between S_{2n} and S_{2n+1}, so the total number of downcrossings, D_{xy}, of (x,y) is

$$D_{xy} = \sup \{n : S_{2n-1} < \infty\}.$$

Let $Y_t = X_{t \wedge T_y}$ and $Z_t = X_{t \wedge T_x}$, and define $Y_n = Y \circ \theta_{S_{2n-1}}$, and $Z_n = Z \circ \theta_{S_{2n-2}}$, n=1,2,... We call Y_n and Z_n the n^{th} upcrossing and downcrossing processes respectively.

If $X_o \in (x,y)$, it is not clear whether we should call the process from $t=0$ until $t=S_o$ an upcrossing or a downcrossing. We avoid this inessential ambiguity by assuming that $X_o \equiv a$ for some real a, and we will only consider intervals which do not contain a.

This agreed, we set $Y_o(t) = X_{t \wedge T_y}$ in case $a \leq x$, whereas if $a \geq y$, we set $Z_o(t) = X_{t \wedge T_x}$. We then define the upcrossing field $\mathcal{U}_{xy}$ by

$$\mathcal{U}_{xy} = \sigma\{Y_n, n=1,2,\ldots\}$$

Remarks

1°). If (x,y) and (x',y') are intervals not containing z, with $x \leq x'$ and $y \leq y'$, then $\mathcal{U}_{xy} \subset \mathcal{U}_{x'y'}$. This is because it is impossible for an upcrossing of (x,y) to occur during a downcrossing of (x',y') : each upcrossing of (x,y) occurs either during an upcrossing of (x',y') or, if $a < x'$, before time $T_{y'}$. Thus all upcrossings of (x,y) are made by the upcrossing processes Y_o, Y_1,... of the upper interval. (By the same token, all downcrossings of (x',y') are made by the downcrossing processes of the lower interval (x,y)).

2°). D_{xy} is $\mathcal{U}_{xy}$-measurable. Indeed

$$\{D_{xy} \geq n\} = \{S_{2n-1} < \infty\} = \{Y_n(0) \in \mathbb{R}\} \in \mathcal{U}_{xy}$$

3°). Warning : in case $a \geq y$, the process Z_1 is a sub-process of Z_o, for both end at time T_x.

The processes Z_n and Y_n are defined only on the sets $\{S_{2n-2} < \infty\}$ and $\{S_{2n-1} < \infty\}$ respectively. If $D_{xy} \equiv \infty$, it follows from the strong Markov property that all of the processes Z_n and Y_n will be independent. If D_{xy} is finite, we have the following conditional statement.

PROPOSITION 1.1

Let $N \geq 1$ and suppose $a \notin (x,y)$. Then, conditioned on the event $\{D_{xy} = N\}$, we have that

i). $Z_1,\dots,Z_N$ are iid and independent of $\mathcal{U}_{xy}$;

ii). Z_o is independent of $Z_1,\dots,Z_N$ and of $\mathcal{U}_{xy}$;

iii). each Z_i has the distribution of $X_{t \wedge T_x}$, conditioned on $\{T_x < \infty\}$.

Proof : let $\Lambda_1,\dots,\Lambda_N$ and $\Gamma_1,\dots,\Gamma_N$ be events in path space, that is, elements of $\mathcal{F}$. We will write $\{Y_i \in \Gamma_i\}$ instead of $\{\omega : Y_i(.,\omega) \in \Gamma_i\}$. Let $\Gamma = \bigcap_{i=1}^{N} \{Y_i \in \Gamma_i\}$. We must show that

$$(1.1) \qquad P^a\{Z_i \in \Lambda_i,\ i=1,\dots,N \ ;\ \Gamma | D_{xy} = N\}$$
$$= P^a\{\Gamma | D_{xy}=N\} \prod_{i=1}^{N} P^y\{X_{\cdot \wedge T_x} \in \Lambda_i | T_x < \infty\}.$$

Now $D_{xy} = N$ iff $S_o < \infty,\dots,S_{2N-1} < \infty$, $S_{2N+1} = \infty$. Remembering that $Z_j = Z\circ\theta_{S_{2j-2}}$ and $Y_j = Y\circ\theta_{S_{2j-1}}$, we can write the left-hand side of (1.1) as

$$P^a\{D_{xy}=N\}^{-1} \ P^a\{S_o < \infty,\ Z\circ\theta_{S_o} \in \Lambda_1,\ S_1 < \infty,\ Y\circ\theta_{S_1} \in \Gamma_1,\dots$$
$$\dots Z\circ\theta_{S_{2n-2}} \in \Lambda_N,\ S_{2N-1} < \infty\ ,\ Y\circ\theta_{S_{2N-1}} \in \Gamma_N,\ S_{2N+1} = \infty\}.$$

Apply the strong Markov property successively to $S_{2N-1}, S_{2N-2},\dots,S_1$, noting that $X_{S_{2j}}=y$ and $X_{S_{2j-1}}=x$.

$$= P^a\{D_{xy}=N\}^{-1} \ P^a\{S_o< \infty\}\ P^y\{Z \in \Lambda_1, S_1< \infty\}\ P^x\{Y \in \Gamma_1,\ S_o < \infty\} \dots$$
$$\dots P^y\{Z \in \Lambda_N,\ S_o < \infty\}\ P^x\{Y \in \Gamma_N,\ S_1= \infty\}$$

Collect all terms containing $\mathcal{Z}$:

$$= \prod_{i=1}^{N} P^y\{\mathcal{Z} \in \Lambda_i,\ S_1<\infty\} \left[\frac{P^a\{S_o < \infty\}}{P^a\{D_{xy}=N\}} \left(\prod_{i=1}^{N-1} P^x\{Y \in \Gamma_i,\ S_o < \infty\}\right) P^x\{Y \in \Gamma_N, S_1 = \infty\} \right]$$

Taking $\Lambda_1 = \ldots = \Lambda_N = \Omega$ above, we identify the term in brackets as $P^a\{\Gamma | D_{xy}=N\}\ P^y\{S_1 < \infty\}^{-N}$. But $Z_t = X_{t \wedge T_x}$, so

$$P^y\{X_{\cdot \wedge T_x} \in \Lambda_i | \ T_x < \infty\} = P^y\{\mathcal{Z} \in \Lambda_i,\ S_1 < \infty\} P^y\{S_1 < \infty\}^{-1},$$

and we conclude that the above equals the right-hand side of (1.1), which is therefore proved. The statement for $\mathcal{Z}_o$ follows upon replacing $\mathcal{Z}_1$ by $\mathcal{Z}_o$ above and making the obvious modifications to account for the fact that $\mathcal{Z}_o$ starts from a rather than y. q e d

2. THE BASIC CALCULATIONS

We will be encountering geometric distributions fairly often in what follows, so it is convenient to record the following.

LEMMA 2.1

Let X *be an integer-valued random variable with*

$$P\{X \geq n\} = \begin{cases} 1 & n=0 \\ cr^{n-1} & n=1,2,\ldots \end{cases} \qquad c \leq 1,\ r < 1.$$

Then $E\{X\} = \frac{c}{1-r}$, $\mathrm{Var}\{X\} = \frac{c(1+r-c)}{(1-r)^2}$

and $E\{e^{sX}\} = \frac{1-c+(c-r)e^s}{1-r\ e^s}$.

Most of the calculations we will make below involve nothing deeper than the following lemma.

LEMMA 2.2

Let $b < x' < y'$ and let $\{M_t, t \geq 0\}$ be a continuous martingale with initial value $z > b$, such that $T_b < \infty$ a.s. Let $D_{x'y'}$ be the number of downcrossings of (x',y') before T_b. Then

i). $P\{D_{x'y'} \geq n\} = \frac{y' \wedge z - b}{y'-b} \left(\frac{x'-b}{y'-b}\right)^{n-1}$, $n \geq 1$

ii). $E\{D_{x'y'}\} = \frac{y' \wedge z - b}{y'-x'}$

iii). $\mathrm{Var}\{D_{x'y'}\} = \frac{y' \wedge z - b}{y'-x'}$

iv). $E\{e^{s D_{x'y'}}\} = 1 - \frac{(y' \wedge z - b)(1-e^s)}{y'-b-(x'-b)e^s}$, $\mathrm{Re}\ e^s < \frac{y'-b}{x'-b}$

Proof : once (i) is proved, (ii), (iii), and (iv) follow from Lemma 2.1.

To prove (i), note that for M to make one downcrossing of (x',y'), it must first reach y' before hitting b, an event which is certain if $z > y'$ and which has probability $z-b/y'-b$ if $z < y'$. Once at y', it is sure to make at least one downcrossing.

Next, in order to make $n+1$ downcrossings, it must first make n, with probability p_n, say. It will then be at x, and must first return to y without hitting x, and event of probability $x'-b/y'-b$. It is then sure to make at least one more downcrossing, since it eventually reaches b. Thus

$$p_{n+1} = p_n \frac{x'-b}{y'-b}.$$

Since $p_1 = y' \wedge z - b/y'-b$, (i) follows by induction. q e d

We now specialize our diffusion X. Let X be a Brownian motion with $X_o \equiv 1$, stopped when it first hits the origin. (In fact, it is not necessary that X be a Brownian motion ; everything we do below is valid if X is any diffusion on natural scale).

If $x < y$ and if $D_{xy}(t)$ is the number of downcrossings of (x,y) in the interval $[0,t]$, then $2(y-x)\, D_{xy}(t)$ converges a.s. as $y \downarrow x$ to the standard

Brownian local time at x. ([1], [2], and [3]).

Rather than looking at this for a fixed t, we want to look at this at the time T_o, the first hit of zero. Accordingly, we will write D_{xy} **instead** of $D_{xy}(T_o)$ for the total number of upcrossings of (x,y) by X, and we will define

$$M_{xy} = (y-x)\, D_{xy}$$

(We are normalizing by $y-x$ instead of $2(y-x)$, so M_{xy} will converge to half the standard Brownian local time as $y \downarrow x$).

Before doing this let us note that we can derive many of the properties of $D_{xy}(t)$ from those of $D_{xy}(T_o)$. Indeed, $D_{xy}(t) = D_{xy}(T_o)$ if $t > T_o$, and, on $\{t < T_o\}$,

$$D_{xy}(t) = D_{xy}(T_o) - D_{xy}(T_o)\circ\theta_t - \gamma,$$

where $\gamma = 0$ or 1 is there to account for the possibility that t falls during a downcrossing, which would not be counted in either $D_{xy}(t)$ or $D_{xy}(T_o)\circ\theta_t$. But now if $\lim\limits_{y\downarrow x} (y-x)\, D_{xy}(T_o)$ exists a.s., so does $\lim\limits_{y\downarrow x} (y-x)\, D_{xy}(T_o)\circ\theta_t$ by the Markov property, hence so does $\lim\limits_{y\downarrow x}(y-x)\, D_{xy}(t)$. Similarly, if this limit is continuous in x at time T_o, it must be continuous at time t, too.

PROPOSITION 2.3

Let (x,y) and (x',y') be intervals not containing 1, such that $x \leq x'$ and $y \leq y'$. Then

(i) M_{xy} is L_p-bounded for y in compacts, all $p > 0$;

(ii) $M_{x'y'}$ is conditionally independent of $\mathcal{U}_{xy}$ given M_{xy} ;

(iii) $E\{M_{x'y'} \mid \mathcal{U}_{xy}\} = M_{xy} + y'\wedge 1 - y\wedge 1$;

(iv) $\mathrm{Var}\{M_{x'y'} \mid \mathcal{U}_{xy}\} = (x'-x + y'-y)\, M_{xy} + (y'\wedge 1 - y\wedge 1)(x'-y + y'-y'\wedge 1)$

Proof : once (iii) has been proved, (i) follows, since by Lemma 2.2 (iv), all moments of M_{xy} exist, and so for $p \geq 1$, if $x < y \leq k$, Proposition 2.3 (iii) implies :

(2.1) $$E\{|M_{xy} - y \wedge 1|^p\} \le E\{|M_{k\,k+1} - 1|^p\} < \infty.$$

To prove (ii), consider the cases $y \le 1$ and $y > 1$ separately. Suppose first that $y > 1$ and let $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ be the successive downcrossing processes, as defined in §1. Each downcrossing of (x',y') takes place during a downcrossing of (x,y) ; it is not possible for one to happen during an upcrossing. Thus, if $V_1, V_2, \ldots$ are the downcrossings of (x',y') by $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ respectively, then

(2.2) $$D_{x'y'} = V_1 + V_2 + \ldots$$

But the $\mathcal{Z}_i$ are iid (Proposition 1.1) and independent of $\mathcal{U}_{xy}$, and the V_i are functions of the $\mathcal{Z}_i$, so that $D_{x'y'}$ must be conditionally independent of $\mathcal{U}_{xy}$ given D_{xy}, which proves (ii) in this case. Moreover, the exact distribution of the V_i are given by Lemma 2.2, with $z=y$ and $b=x$.

(2.3) $$E\{V_i\} = \frac{y-x}{y'-x}\ , \quad \mathrm{Var}\{V_i\} = \frac{(y-x)(x'-x+y'-y)}{(y'-x')^2}$$

Since D_{xy} is $\mathcal{U}_{xy}$-measurable

$$E\{D_{x'y'} \mid \mathcal{U}_{xy}, D_{xy}=N\} = \sum_{i=1}^{N} E\{V_i \mid D_{xy}=N\}$$

$$= N\frac{y-x}{y'-x'} = \frac{y-x}{y'-x'} D_{xy}$$

Since $M_{xy} = (y-x)\, D_{xy}$,

(2.4) $$E\{M_{x'y'} \mid \mathcal{U}_{xy}\} = M_{xy} \quad \text{if } y > 1.$$

The V_i are independent, so their variances add, and

$$\mathrm{Var}\{D_{x'y'} \mid \mathcal{U}_{xy}, D_{xy} = N\} = N\frac{(y-x)(x'-x+y'-y)}{(y'-x')^2}\ ,$$

so that

(2.5) $$\mathrm{Var}\{M_{x'y'} \mid \mathcal{U}_{xy}\} = (x'-x+y'-y)\, M_{xy} \quad \text{if } y > 1$$

In case $y < 1$, the same analysis holds except that there can now be some downcrossings of (x',y') before it reaches y. This means that the process

$\mathcal{Z}_o$ must be taken into account, and $\mathcal{Z}_1$ ignored (see remark 3, §1) since all downcrossings of $\mathcal{Z}_1$ are also downcrossings of $\mathcal{Z}_o$. Thus

$$D_{xy} = V_o + V_2 + V_3 + \dots .$$

As before, $D_{x'y'}$ will be conditionally independent of $\mathcal{U}_{xy}$, so that

$$E\{D_{x'y'} \mid \mathcal{U}_{xy}, D_{xy} = N\} = E\{V_o\} + (N-1)\, E\{V_2\}.$$

Taking $z=1$ in Lemma 2.2.

$$E\{V_o\} = \frac{y'\wedge 1 - x}{y'-x'}, \quad \mathrm{Var}\{V_o\} = \frac{(y'\wedge 1 - x)\,(x'-x+y'-y'\wedge 1)}{(y'-x')^2}.$$

Since the expectation and variance of $V_2,\dots,V_N$ are given by (2.5)

$$E\{D_{x'y'} \mid \mathcal{U}_{xy}\} = \frac{y'\wedge 1 - y}{y'-x'} + \frac{y-x}{y'-x'}\, D_{xy}$$

$$\mathrm{Var}\{D_{x'y'} \mid \mathcal{U}_{xy}\} = \frac{(y - y'\wedge 1)(x'-y+y'-y'\wedge 1)}{(y'-x')^2} + \frac{(y-x)(x'-x+y'-y)}{(y'-x')^2}\, D_{xy}$$

In terms of M_{xy}, if $y > 1$ this is

(2.6) $$E\{M_{x'y'} \mid \mathcal{U}_{xy}\} = y'\wedge 1 - y + M_{xy}$$

(2.7) $$\mathrm{Var}\{M_{x'y'} \mid \mathcal{U}_{xy}\} = (y - y'\wedge 1)\;(x'-y + y' - y\wedge 1) + (x'-x + y'-y)D_{xy}$$

Equations (2.4)-(2.7) prove (ii) and (iii).

q e d

<u>Remark</u> : $\{M_{xy},\ 0 \le x < y,\ 1 \notin (x,y)\}$ is a two-parameter process. If we partially order its parameter set by "$\prec$", where $(x,y) \prec (x'y')$ if $x \le x'$, $y \le y'$, then Proposition 2.3 tells us that it is a Markov process in the sense that $M_{x'y'}$ is conditionally independent of $\mathcal{U}_{xy}$ - and hence of the "past" before (x,y) - given M_{xy}. It is natural to ask whether this process satisfies Levy's Markov property : that is, given a nice subset A of the parameter set, is it true that the process M_{xy} for $(x,y) \in A$ is independent of M_{xy} for (x,y) outside of A, given $\{M_{xy},\ (x,y) \in \partial A\}$? The answer is no. While it is not hard to show that this is true for sets A of the form $\{(x,y) : 0 \le x < y \le y_o\}$, it doesn't hold for those of the form $\{(x,y) : x \le x_o,\ 0 \le x < y \le y_o\}$.

3. THE RESULTS

With the basic calculations out of the way, we can draw some conclusions. The following, for example, is an immediate consequence of Proposition 2.3.

THEOREM 3.1

$\{M_{xy}-y\wedge 1,\ \mathcal{U}_{xy},\ 0\leq x<y,\ 1\notin(x,y)\}$ is a two-parameter martingale. In particular, it is a martingale in either parameter when the other is fixed.

Since $M_{xy}-y\wedge 1$ is an L^p-bounded martingale in y for each fixed x, the martingale convergence theorem allows us to define the local time L_x at x by

$$L_x = \lim_{y\downarrow x} M_{xy}.$$

Define $\mathcal{U}_x = \bigcap_{y>x} \mathcal{U}_{xy}$. Note that the fields $\mathcal{U}_x$ are increasing and that L_x is $\mathcal{U}_x$-measurable. Then we have :

COROLLARY 3.2

$\{L_x-x\wedge 1,\ \mathcal{U}_x,\ x\geq 0\}$ is a martingale, locally bounded in L^p for all $p>0$, whose associated increasing process is

$$A_x = 2\int_o^x L_y\,dy.$$

Proof : set $N_x = L_x-x\wedge 1$; it is an L_p-bounded martingale for all $p\geq 1$ (just let $y\downarrow x$ in Theorem 3.1). We can then go to the limit in Proposition 2.3 (iv) to see that

$$(3.1)\qquad E\{N_{x'}^2 - N_x^2\mid \mathcal{U}_x\} = \mathrm{Var}\{L_{x'}\mid \mathcal{U}_x\}$$

$$= 2(x'-x)M_{xy} + (x'\wedge 1-x\wedge 1)(x'-x+x'-x\wedge 1).$$

To identify A_x as the associated increasing process, note that

$$E\{2\int_x^{x'} L_y\,dy\mid \mathcal{U}_x\} = 2\int_x^{x'}(E\{N_y\mid \mathcal{U}_x\} + y\wedge 1)dy$$

$$= 2(x'-x)N_x+2\int_x^{x'} y\wedge 1\,dy$$

$$= 2(x'-x)(L_x-x\wedge 1) + 2\int_x^{x'} y\wedge 1\,dy,$$

which equals the right-hand side of (3.1), verifying that $N_x^2 - A_x$ is a martingale.

q e d

We are now in a position to prove the Markov property of the local time L_x as a function of x. The following is the simplest case of theorems by Ray and Knight.

THEOREM 3.3 (D. Ray, F. Knight)

The process $\{L_x, x \geq 0\}$ is a continuous strong Markov process on $[0,\infty)$, absorbed at zero, and having infinitesimal generator $\mathfrak{C}$: if $f \in D(\mathfrak{C}) \cap C^{(2)}$,

$$\mathfrak{C}f(t) = \begin{cases} t f''(t) + f'(t) & \text{if } 0 < t < 1 \\ t f''(t) & \text{if } t \geq 1. \end{cases} \tag{3.2}$$

Proof : it is nearly clear from Proposition 2.3 that L is a Markov process, but to pass from nearly clear to clear will take some work. We will do this by brute force, and calculate the characteristic function of L_x. We claim that if $0 < x < x'$ and Re $s \leq 0$

$$E\{e^{sL_{x'}} \mid \mathcal{U}_x\} = \frac{1+(x'-x'\wedge 1)s}{1-(x'-x\wedge 1)s} \, e^{\frac{sL_x}{1-(x'-x)s}} \tag{3.3}$$

The Markov property follows immediately since, as the right-hand side of (3.3) depends only on L_x, $L_{x'}$ must be conditionally independent of $\mathcal{U}_x$, and hence of $L_{x''}$ for $x'' \leq x$, given L_x.

To see (3.3), let $\varepsilon > 0$, and consider the intervals $(x, x+\varepsilon)$ and $(x', x'+\varepsilon)$, where $x+\varepsilon < x'$. Let $\mathcal{Z}_0, \mathcal{Z}_1, \ldots$ be the downcrossing processes of the lower interval $(x, x+\varepsilon)$. Suppose $x \geq 1$. Then

$$D_{x', x'+\varepsilon} = V_1 + V_2 + \ldots$$

and, given $D_{x\, x+\varepsilon} = N$, $V_1, \ldots, V_N$ are iid and independent of $\mathcal{U}_{x,x+\varepsilon}$. Apply Lemma 2.2 (iii) with $b=x$, $z=x+\varepsilon$ and $y'=x'+\varepsilon$:

(3.4) $E\{e^{sM_{x',x'+\varepsilon}} | \mathcal{U}_{x,x+\varepsilon}\} = E\{e^{\varepsilon s D_{x',x'+\varepsilon}} | \mathcal{U}_{x,x+\varepsilon}\}$.

On $\{D_{x\ x+\varepsilon}\} = N$ this is

$$= E\{e^{\varepsilon s V_1}\}^N = \left[1 - \frac{\varepsilon(1-e^{s\varepsilon})}{\varepsilon+(x'-x)(1-e^{\varepsilon s})}\right]^{\frac{1}{\varepsilon} M_{x,x+\varepsilon}}$$

As $\varepsilon \to 0$, $M_{x,x+\varepsilon} \to L_x$ while the quotient is

$$\frac{-s\varepsilon}{1-(x'-x)s} + o(\varepsilon),$$

so the right-hand side of (3.3) tends to

$$\text{(3.5)} \qquad e^{\frac{sL_x}{1-(x'-x)s}}$$

In case $x' < 1$, we reason as before that $D_{x',x'+\varepsilon} = V_o + V_2 + \ldots$ so that

$$e\{^{sM_{x'\ x'+\varepsilon}} | \mathcal{U}_{xy}\} = \frac{E\{e^{\varepsilon s V_o}\}}{E\{e^{\varepsilon s V_1}\}} E\{e^{\varepsilon s V_1}\}^{D_{x,x+\varepsilon}}.$$

The expression involving V_1 is just as above, with its limit given by (3.5). Let $z=1$ in Lemma 2.2 (iii) to see that the quotient is

$$\frac{E\{e^{\varepsilon s V_o}\}}{E\{e^{\varepsilon s V_1}\}} = \frac{1-(x'-x'\wedge 1)s}{1-(x'-x)s} + O(\varepsilon)$$

which, together with (3.5), gives (3.3).

To see L_x is continuous, note that, if

$$\Phi(s) = E\{e^{s(L_{x'}-L_x)} | \mathcal{U}_x\} = \frac{1+(x'-x'\wedge 1)s}{1-(x'-x\wedge 1)s}\, e^{\frac{(x'-x)s^2 L_x}{1-(x'-x)s}},$$

then $E\{(L_{x'}-L_x)^4 | \mathcal{U}_x\} = \Phi^{(4)}(0)$. We can compute the fourth derivative, which is most easily done by considering the cases $x < x' \leq 1$ and $x \geq 1$ separately rather than by differentiating the formula as is, and we find that in both cases,

$$\Phi^{(4)}(0) = 4(x'-x)^2 L_x^2 + \text{higher powers of } (x'-x).$$

Since L_x is L^p-bounded for x in compacts for all p, we conclude that for $0 \le x < x' \le k$, there is a constant c such that $E\{(L_{x'} - L_x)^2\} \le c(x'-x)^2$.

This implies by a well-known theorem of Kolmogorov that there is a version of L_x which is continuous in x.

Now we can calculate the transition function $P_z(u,v)$ from (3.2). Indeed, (3.2) says that if $f(v) = e^{sv}$, $s < 0$, that if $z = x'-x$ and if $x' < 1$, for instance,

$$P_z f(u) = \frac{1}{1-zs} e^{\frac{su}{1-zs}},$$

which is continuous in u. It follows that $u \to P_z f(u)$ is continuous if f is a linear combination of exponentials, and, since these are dense in C_o, for all $f \in C_o$. In short, P_z is a Feller semi-group, and the process is strongly Markov on $0 \le x < 1$. Similarly, it is also a Feller process - with a different semi-group - on $x \ge 1$.

Finally, the generator is easily determined by a stochastic integral argument. If f is bounded and twice continuously-differentiable, then

$$f(L_{x+h}) - f(L_x) = \int_x^{x+h} f'(L_y)\, dL_y + \frac{1}{2}\int_x^{x+h} f''(L_y) dA_y$$

where A_y is the increasing process of Corollary 3.2. This is

$$= \int_x^{x+h} f'(L_y)\, d(L_y - y \wedge 1) + \int_x^{x+h} f'(L_y) dy \wedge 1 + \int_x^{x+h} f''(L_y) L_y\, dy$$

Take the expectation given $\mathcal{U}_x$. As $L_y - y \wedge 1$ is a martingale the first integral has zero expectation, so

$$\mathcal{G} f(L_x) = \lim_{h \downarrow o} \frac{1}{h} E\{f(L_{x+h}) - f(L_x) | \mathcal{U}_x\} = \lim_{h \downarrow o} E\left\{\int_x^{x+h} f'(L_y) dy \wedge 1 + \int_x^{x+h} f''(L_y) L_y\, dy \,|\, \mathcal{U}_x\right\}$$

$$= L_x f''(L_x) + f'(L_x) I_{\{x < 1\}},$$

proving (3.2).

q e d

4. THE EXTENSION TO GENERAL DIFFUSIONS

Let X be a regular diffusion on the line with lifetime $\zeta < \infty$, and scale function $s(x)$. We assume that there is an a such that

(4.1) $$X_o = a \quad \text{a.s., and} \quad X_{\zeta-} = 0 \quad \text{a.s.}$$

and also that $s(0)=0$. Define

$$f_x(y,z) = P^y\{T_z < T_x\}$$

and

$$p(y,z) = P^y\{T_z < \infty\}$$

By the definition of the scale function, if $0 \notin [x,z]$,

(4.2) $$f_x(y,z) = \frac{s(y) - s(x)}{s(z) - s(x)} .$$

If $x < y < z$, $p(z,x) = p(z,y)\, p(y,x)$; if $y > 0$, $p(z,y) = 1$, for by (4.1), the process must pass y in order to die at the origin. We have treated the case where $\zeta = \inf\{t : X_{t-} = 0\}$ in the first part of this article, so we will now suppose that $\zeta > T_o$. Thus, set

$$p(x) = \begin{cases} 1 & x > 0 \\ p(0,x) & x \leq 0. \end{cases}$$

Then for $y > x$,

(4.3) $$p(y,x) = p(x)/p(y)$$

(There is a similar function, p_+, such that if $x < y$, $p(x,y) = p_+(y)/p_+(x)$, but we will not need to use it).

In order to relate s and p, notice that if $x-h < x < 0$, the process can pass from x to $x-h$ either by going there directly, or by first going to zero, then passing to $x-h$. Thus

$$p(x,x-h) = f_o(x,x-h) + f_{x+h}(x,0)\, p(0,x-h).$$

Use (4.2), (4.3) and a little algebra to see that

$$\frac{p(x)-p(x-h)}{s(x)-s(x-h)} = -\frac{p(x)(1-p(x-h))}{s(x-h)} .$$

Let $h \downarrow 0$ to see that

$$\text{(4.4)} \qquad \frac{dp}{ds} = \frac{p(p-1)}{s} \qquad x < 0$$

which has the solution

$$\text{(4.5)} \qquad p(x) = \begin{cases} 1 & \text{if } x > 0 \\ \dfrac{1}{1-cs(x)} & \text{if } x < 0. \end{cases}$$

Since $p(x)$ decreases as x decreases, $c \geq 0$. The parameter c determines the amount of time X spends at 0 before ζ. If ζ is the first hitting time of 0, $c=\infty$; if $c=o$, the process is never killed, and if $0 < c < \infty$, the process spends some time at the origin before ζ.

Let a^+ and a^- be the positive and negative parts of a, and define

$$\delta(x) = \begin{cases} 0 & \text{if } x \geq a^+ \\ 1 & \text{if } a^- \leq x < a^+ \\ 2 & \text{if } x < a^- \end{cases}$$

As before, let D_{xy} be the total number of downcrossings of (x,y) by X. Define the local time L_x at x by

$$L_x = \lim_{y \downarrow x} (y-x)D_x$$

if it exists. We are going to follow the methods of the first three sections to determine the properties of the local time process. Although we will not be ready to prove it for some time, the final result will be the following, which is a minor modification of a theorem of D. Ray [6].

THEOREM 4.1

Suppose $s \in C^{(2)}$, and let $m=\inf_t\{X_t\}$. Then the process $\{L_x, x > m\}$ is an inhomogeneous diffusion on $\mathbb{R}^+$, absorbed at 0, with infinitesimal generator $\mathfrak{G}_x$: if $f \in C^{(2)}$ is bounded,

$$(4.6) \qquad \mathfrak{C}_x f(u) = uf''(u) + \left(\delta(x) - \frac{(p^2(x)s'(x))'u}{p^2(x)s'(x)}\right)f'(u), \quad u > 0.$$

We will be following the same path in this section that we did in sections 1-3, so we will be able to treat some of the calculations in less detail. However, the total length of the treatment is increased since at each stage we have to handle the three special cases $x < a^-$, $a^- \leq x < a^+$, and $x \geq a^+$ separately.

The reason these three cases require special handling is apparent if we look at the paths of X_t. If $\ell_x = \sup\{t : X_t=x\}$, we can divide the path into three sections : from 0 to T_x, from T_x to ℓ_x, and from ℓ_x to ζ. The process behaves differently in each of these intervals. If, now, $x < x'$, and x,x' $x,x' \in (-\infty,0)$ all three of these time-intervals can contribute to $L_{x'}$; if $0 < x,x' < a$, there is no contribution from (T_x,ζ), i.e. $L_\zeta = L_{T_x}$, and if $x,x' \in (a,\infty)$, there is no contribution from either $(0,T_x)$ or from (T_x,ζ). Thus, it is not so much that we need to treat the space-intervals $(-\infty,0)$, $(0,a)$ and (a,∞) separately as it is that we need to handle L_{T_x} , $L_{\ell_x}-L_{T_x}$, and $L_\zeta-L_{\ell_x}$ separately.

The following two results generalize Lemma 2.2 and Proposition 2.3.

<u>LEMMA 4.2</u>

$$P^a\{D_{xy} \geq n\} = p(a,y)p(y,x)(p(y,x)p(x,y))^{n-1}.$$

<u>Proof</u> : to have one downcrossing, the process must first reach y, then x,which has probability $p(a,y)p(y,x)$. If the process has already finished n down-crossings and is at x, then to make another, it must first go from x to y, and then return to x, which it does with probability $p(x,y)p(y,x)$. The lemma follows by induction.

LEMMA 4.3

Let (x,y) and (x',y') be intervals with $x \le x'$ and $y \le y'$. Suppose neither a nor zero is in the interval $[x,y']$, and let V_{xy} be the number of downcrossings of the interval (x',y') before time T_x. Then for $n \ge 1$,

(a) $P^y\{V_{x'y'} \ge n | T_x < \zeta\} = f_x(y,y')p(y',y)[f_x(x',y')p(y',x')]^{n-1}$

(b) If $a > y'$,

$P^a\{V_{x'y'} \ge n | T_x < \zeta\} = [f_x(x',y')p(y',x')]^{n-1}$

(c) If $0 > y'$, $P^y\{V_{x'y'} \ge n | T_x \ge \zeta\} = [f_x(x',y')p(y',x')]^n$.

Proof : in order to have at least one downcrossing the process must first reach y' without hitting x, then reach x' before ζ, which has probability $p_x(y,y')p(y',x')$. If the process has just completed $n-1$ downcrossings, it is at x'. To make another, it must first hit y' without hitting x, then reach x', which has probability $r = f_x(x',y')p(y',x')$. Thus by induction

(4.7) $P^y\{V_{x'y'} \ge n\} = f_x(y,y')p(y',x')r^{n-1}$.

Having finished n downcrossings, the process is again at x'. To have $T_x < \zeta$, it must go from x' to x, which has probability $p(x',x)$.
Since $P^y\{T_x < \zeta\} = p(y,x)$,

(4.8) $$P^y\{V_{x'y'} \ge n | T_x < \zeta\} = \frac{f_x(y,y')p(y',x')p(x',x)}{p(y,x)} r^{n-1}.$$

After simplifying, (e.g. $p(y',x')p(x',x) = p(y',x)$) this gives (a).

If $a > y'$, the same derivation holds except that y is replaced by a, and the probability of reaching y' initially is $p(a,y')$ instead of $f_x(y,y')$ in (4.7) and (4.8).

To prove (c), note that if the process has just finished n downcrossings, it is at x' ; then $T_x > \zeta$ if the process never reaches x, an event of probability $1-p(x',x)$, while $P^y\{T_x > \zeta\} = 1-p(y,x)$. Combining this with (4.7), we get

$$P^y\{V_x \geq n \mid T_x > \zeta\} = \frac{1-p(x',x)}{1-p(y,x)} f_x(y,y')\ p(y',x')r^{n-1}.$$

But notice that

$$1-p(y,x) = f_x(y,x')\ (1-p(x',x))$$

since, if the process is at y, in order to have $T_x > \zeta$, it must first hit x' - since it is at 0 at time $\zeta-$ then not hit x. This makes the above expression

$$= \frac{f_x(y,y')}{f_x(y,x')}\ p(y',x') = f_x(x',y')p(y',x')r^{n-1} = r^n.$$

q e d

<u>NOTATION</u> : we will often write $\hat{x}$ instead of $s(x)$.

<u>COROLLARY 4.3</u>

<u>Under the same conditions</u>

(a) $$E^y\{V_{x'y'} \mid T_x < \zeta\} = \frac{p(x)p(y)(\hat{y}-\hat{x})}{p(x')p(y')(\hat{y}'-\hat{x}')} ;$$

(b) $$E^a\{V_{x'y'} \mid T_x < \zeta\} = \frac{p(x)(\hat{y}'-\hat{x})}{p(x')(\hat{y}'-\hat{x}')}$$

(c) $$E^y\{V_{x'y'} \mid T_x \geq \zeta\} = \frac{p(x)}{p(y')}\ \frac{(\hat{x}'-\hat{x})}{(\hat{y}'-\hat{x}')}$$

(d) $$\mathrm{Var}^y\{V_{x'y'} \mid T_x < \zeta\} = \frac{p(x)p(y)(\hat{y}-\hat{x})}{(p(x')p(y')(\hat{y}'-\hat{x}'))^2} \times \ldots$$

$$\ldots \left[2p(x)p(y')(\hat{y}'-\hat{x})-p(x')p(y')(\hat{y}'-\hat{x}')-p(x)p(y)(\hat{y}-\hat{x})\right]$$

(e) $$\mathrm{Var}^y\{V_{x'y'} \mid T_x > \zeta\} = \mathrm{Var}^a\{V_{x'y'} \mid T_x < \zeta\} = \frac{p(x)^2\ (\hat{x}'-\hat{x})(\hat{y}'-\hat{x})}{p(x')p(y')(\hat{y}'-\hat{x}')^2}$$

<u>Proof</u> : in a sense, there is nothing to prove, since (a)-(e) follow from Lemma 4.2 and Lemma 1.1. However, there are some identities which we use to simplify the expressions which we should point out. Let $r=f_x(x',y')p(y',x')$ and notice that

$$p(x',x) = f_{y'}(x',x) + f_x(x',y')p(y',x')\ p(x',x)$$

which just expresses the fact that the process can go from x' to x either directly, without hitting y', or by first hitting y', then returning to x', then to x. Thus

$$(4.9) \qquad 1-r = \frac{f_{y'}(x',x)}{p(x',x)}$$

If we apply the formulae of Lemma 1.1 with r as above, and with c equal to $f_x(y,y')\,p(y',y)$, 1, and r respectively, the expectations in (a), (b) and (c) become respectively

$$(4.10a) \qquad \frac{f_x(y,y')p(y',y)p(x',x)}{f_{y'}(x',x)}$$

$$(4.10b) \qquad \frac{p(x',x)}{f_{y'}(x',x)}$$

$$(4.10c) \qquad \frac{f_x(x',y')p(y',x')p(x',x)}{f_{y'}(x',x)}$$

Turning to the variances, in (d) we write $1+r-c = 2-(1-r)-c$ to get

$$(4.10d) \qquad \left(\frac{p(x',x)}{f_{y'}(x',x)}\right)^2 f_x(y,y')p(y',y)\left[2 - \frac{f_{y'}(x',x)}{p(x',x)} - f_x(y,y')p(y',y)\right]$$

If either $c=1$, or $c=r$, $c(1+r-c) = r$, so the final two variances equal $r(1-r)^{-2}$, i.e.

$$(4.10e) \qquad \left(\frac{p(x',x)}{f_{y'}(x',x)}\right)^2 f_x(x',y')\ p(y',x').$$

If we write (4.10a) - (4.10e) in terms of $p(x)$ and $s(x) = \hat{x}$, we get (a)-(e).

q e d

For $x < y$, define

$$M_{xy} = p(x)\ p(y)\ (s(y) - s(x))D_{xy}.$$

We assume that $a \geq 0$, a being the initial value of the process, and leave it to the reader to make the necessary modifications in case $a < 0$. The following generalizes Theorem 3.1.

THEOREM 4.4

Let $x_o \leq 0$. The following are two-parameter martingales which are locally bounded in L^p for $1 \leq p < \infty$.

(a) $\{M_{xy}, \mathcal{U}_{xy}, a \leq x < y\}$;

(b) $\{M_{xy} - \hat{y}, \mathcal{U}_{xy}, 0 \leq x < y \leq a\}$;

(c) $\{M_{xy} - \int_{x_o}^{x} p^2(u)\,d\hat{u} - \int_{x_o}^{y} p^2(u)\,d\hat{u}, \mathcal{U}_{xy}, x_o \leq x < y \leq 0\}$,

on the set $\{\inf_t X_t < x_o\}$.

Proof : the M_{xy} are geometric random variables, hence are in L^p for fixed x and y. Once it is established that (a)-(c) are martingales, local boundedness in L^p is immediate since in case (a), for example, $|M_{xy}|^p$ will be a sub-martingale, so if $x < y < k$, $E\{|M_{xy}|^p\} \leq E\{|M_{k,k+1}|^p\} < \infty$.

Let (x,y) and (x',y') be intervals such that $a \leq x \leq x'$ and $y \leq y'$. Let $Z_1, Z_2, \ldots$ be the successive downcrossing processes of (x,y) and let V_i be the number of downcrossings of (x',y') by Z_i. Both the initial and final values of X lie below x, so all downcrossings of (x',y') occur during downcrossings of (x,y). Thus

$$D_{x'y'} = V_1 + V_2 + \ldots$$

Given that $D_{xy} = N$, $V_1, \ldots, V_N$ are iid and independent of $\mathcal{U}_{xy}$, with a distribution given by Lemma 4.3a). Since $p(y',y) = p(y',x') = 1$

$$E\{V_j | D_{xy} = N\} = \frac{\hat{y} - \hat{x}}{\hat{y}' - \hat{x}'} N,$$

so

$$E\{D_{x'y'} | \mathcal{U}_{xy}\} = \frac{\hat{y} - \hat{x}}{\hat{y}' - \hat{x}'} D_{xy}$$

and

$$E\{M_{x'y'} | \mathcal{U}_{xy}\} = (\hat{y} - \hat{x}) D_{xy} = M_{xy}.$$

In case (b), $a > y'$ so there is at least one downcrossing of (x',y') before T_x, and we must consider the initial dowcrossing process $\mathcal{Z}_o$ instead of $\mathcal{Z}_1$ (see remark 3, §1). Then

$$D_{x'y'} = V_o + V_2 + \dots .$$

The distribution of V_o is given by Lemma 4.3 (b), and that of $V_2,\dots,V_N$ is the same as above. Thus

$$E\{D_{x'y'} | D_{xy} = N\} = \frac{\hat{y}'-\hat{x}}{\hat{y}'-\hat{x}'} + (N-1)\,\frac{\hat{y}-\hat{x}}{\hat{y}'-\hat{x}'}$$

leading to

$$E\{M_{x'y'} | \mathcal{U}_{xy}\} = \hat{y}'-\hat{y} + M_{xy}.$$

Finally, in case (c), $X_{\zeta-} = 0 > y'$, so the last "downcrossing" of (x,y) is incomplete. If $D_{xy} = N$, then $\mathcal{Z}_{N+1}$ starts from y, and is killed before hitting x, so that

$$D_{xy} = V_o + V_2 + \dots + V_N + V_{N+1},$$

and the distribution of V_{N+1} is given by Lemma 4.2 c. The distributions of V_o and V_2 are as before, except that $p(x)$ is no longer necessarily one, so

$$E\{D_{x'y'} | D_{xy} = N\} = \frac{p(x)(\hat{y}'-\hat{x})}{p(x')(\hat{y}'-\hat{x}')} + (N-1)\,\frac{p(x)p(y)(\hat{y}-\hat{x})}{p(x')p(y')(\hat{y}'-\hat{x}')}$$
$$+ \frac{p(x)\ (\hat{x}'-\hat{x})}{p(y')(\hat{y}'-\hat{x}')}.$$

Thus

$$E\{M_{x'y'} | \mathcal{U}_{xy}\} = M_{xy} + p(x)p(y')(\hat{y}'-\hat{x})$$
$$+ p(x')p(x)(\hat{x}'-\hat{x}) - p(x)p(y)\ (\hat{y}-\hat{x}).$$

We can simplify this. Since $p(x) = 1/1+c\,\hat{x}$,

$$(4.11) \qquad \int_x^y p^2(u)\,d\hat{u} = p(y)\ p(x)\ (\hat{y}-\hat{x})$$

so the above equals $\int_x^{y'} + \int_x^{x'} - \int_x^{y} = \int_x^{x'} + \int_y^{y'} p^2(u)\,d\hat{u}$, giving finally

$$E\{M_{x'y'} | \mathcal{U}_{xy}\} = M_{xy} + \int_x^{x'} + \int_y^{y'} p^2(u)d\hat{u}. \qquad \text{q e d}$$

For fixed x, the processes of Theorem 4.4 are martingales in y, so that

$$M_x \overset{\text{def}}{=} \lim_{y \downarrow x} M_{xy},$$

exists a.s. and in L^p for all $p > 1$. As the processes are also martingales in x, so are their limits. We can unify them as follows. Set $\mathcal{U}_x = \bigcap_{y > x} \mathcal{U}_{xy}$, and define

$$\delta(x) = \begin{cases} 0 & \text{if } x \geq a \\ 1 & 0 \leq x < a \\ 2 & x < 0 \end{cases}$$

Then for any x_o,

$$(4.12) \qquad \{M_x - \int_{x_o}^{x} \delta(u)\, p^2(u)d\hat{u},\ \mathcal{U}_x,\ x \geq x_o\}$$

is a martingale on the set $\{\inf X_t < x_o\} = \{M_{x_o} > 0\}$. Now M_x is closely related to the local time L_x:

$$L_x = \lim_{y \downarrow x} (y-x)D_{xy} = \lim_{y \downarrow x} \frac{y-x}{p(x)p(y)(\hat{y}-\hat{x})} M_{xy}$$

It follows that L_x exists if $s(x)$ is differentiable, and

$$(4.13) \qquad L_x = (p^2(x)\ s'(x))^{-1} M_x.$$

Now an elementary calculation with stochastic integrals gives us

$$dL = (p^2s')^{-1}\ dM - M_x \frac{(p^2s')'}{(p^2s')^2} dx$$

$$= (p^2s')^{-1}\ (dM - \delta p^2s'dx) + (\delta - \frac{(p^2s')'}{p^2s'} L_x)dx$$

or

$$L_x - L_{x_o} - \int_{x_o}^{x} (\delta(u) - \frac{(p^2s')'}{p^2s'} L_u)du = \int_{x_o}^{x} (p^2s')^{-1}\ (dM - \delta p^2s'\ dx).$$

The last term is a martingale, which brings us to

THEOREM 4.5

Suppose $s \in C^{(2)}$. Then for any x_o

$$\{L_x - \int_{x_o}^{x} (\delta(u) - \frac{(p^2 s')'}{p^2 s'} L_u)du,\ \mathcal{U}_x,\ x \geq x_o\}$$

is a martingale on the set $\{L_{x_o} > 0\}$. Its increasing process is

$$\langle L\rangle_x = 2\int_{x_o}^{x} L_u\, du.$$

Proof : we have just shown that the process is a martingale. To identify its increasing process, we will identify the process $\langle M\rangle_x$ first. Notice that $\langle M\rangle_x$ can be characterized as a continuous increasing process with the property that if $x < x'$,

$$(4.14) \qquad \mathrm{Var}\{M_{x'} \mid \mathcal{U}_x\} = E\{\langle M\rangle_{x'} - \langle M\rangle_x \mid \mathcal{U}_x\}.$$

As in the proof of Theorem 4.4, let $x < y < x' < y'$ and suppose all four lie in one of the intervals $[a,\infty)$, $[0,a]$, or $[-\infty,0]$. We have to treat all three cases separately. We will refer to them as cases (a), (b) and (c) respectively, and we will use the notation of the proof of Theorem 4.4.

Suppose $D_{xy} = N$ and that $y-x$ and $y'-x'$ are small. Then Corollary 4.3e gives

$$\mathrm{Var}(V_o) = \frac{p(x)^2}{p(x')p(y')} \frac{(\hat{x}'-\hat{x})(\hat{y}'-\hat{x})}{(\hat{y}'-\hat{x}')^2}$$

in cases (b) and (c), and in case (c), $\mathrm{Var}(V_{N+1}) = \mathrm{Var}(V_o)$. If $1 \leq j \leq N$

$$\mathrm{Var}(V_j) = \frac{2p(x)p(y)\ (\hat{y}-\hat{x})}{[p(x)p(y')(\hat{y}'-\hat{x}')]^2} \left[p(x)p(y')(\hat{y}'-\hat{x}) + o(1)\right].$$

The V_i are conditionally independent given $\mathcal{U}_{xy}$, so that the variances add, hence in cases (a), (b) and (c) respectively, $\mathrm{Var}\{D_{x'y'} \mid \mathcal{U}_{xy}\}$ is equal to

$$D_{xy}\ \mathrm{Var}\{V_1\}\ ,\ \mathrm{Var}\{V_o\} + (D_{xy}-1)\ \mathrm{Var}\{V_1\}$$

and

$$\mathrm{Var}\{V_o\} + (D_{xy}-1)(\mathrm{Var}\{V_1\} + \mathrm{Var}\{V_{N+1}\}$$

This gives us $\mathrm{Var}\{ D_{x'y'} | \mathcal{U}_{xy}\}$ in all three cases. Now multiply by $[p(x')p(y')(\hat{y}'-\hat{x}')]^2$, and let $y \downarrow x$ and $y' \downarrow x'$ to see that in cases (a), (b), and (c) respectively, that $\mathrm{Var}\{M_{x'} | \mathcal{U}_x\}$ equals

(4.15a) $\quad 2p(x')\ p(x)(\hat{x}'-\hat{x})M_x$

(4.15b) $\quad 2p(x')p(x)(\hat{x}'-\hat{x})M_x + [p(x)p(x')(\hat{x}'-\hat{x})]^2$

(4.15c) $\quad 2p(x')p(x)(\hat{x}'-\hat{x})M_x + 2[p(x)p(x')(\hat{x}'-\hat{x})]^2.$

We claim $\langle M\rangle_x = 2\int_{x_o}^{x} p^2(u)\ M_u\ ds(u)$ for all $x \geq x_o$.

Indeed,

$$E\{2\int_x^{x'} p^2(u)M_u\ ds(u) | \mathcal{U}_x\} = 2\int_x^{x'} E\{M_u | \mathcal{U}_x\}\ p^2(u)\ ds(u)$$
$$= 2\int_x^{x'}(M_x + \int_x^u \delta(v)p^2(v)ds(v))p^2(u)ds(u).$$

Now δ is constant on (x',x), so

$$= 2\ M_x \int_x^{x'} p^2(u)ds(u) + \delta(x)\ (\int_x^{x'} p^2(u)ds(u))^2$$

by (4.11) this is

$$= 2\ M_x\ p(x')p(x)(\hat{x}'-\hat{x}) + \delta(x)[p(x')p(x)(\hat{x}'-\hat{x})]^2.$$

Compare this with (4.15a-c) to verify (4.12).

But now that we know $\langle M\rangle$, we get $\langle L\rangle$ immediately since, by (4.13)

$$d\langle L\rangle_x = (p^2 s')^{-2}\ d\langle M\rangle_x$$
$$= 2(p^2 s')^{-2}\ p^2 s'\ M_x\ dx$$
$$= 2\ L_x\ dx$$

q e d

We can now prove Theorem 4.1. We proved the Markov property in a simpler setting in §3, and a similar argument can be used here, or we can just refer to [7, Theorem 2.3]. We are interested primarily in the infinitesimal generator. To find this, let $f \in C^{(2)}$ be bounded. Then for any x, on the set $\{L_x > 0\}$

$$f(L_{x+h})-f(L_x) = \int_x^{x+h} f'(L_y)dL_y + \frac{1}{2}\int_x^{x+h} f''(L_y)d\langle L\rangle_y$$

$$= \int_x^{x+h} f'(L_y)(dL_y - (\delta(y) - \frac{(p^2s')'}{p^2s'}L_y)dy)$$

$$+ \int_x^{x+h} [f'(L_y)(\delta(y) - \frac{(p^2s')'}{p^2s'}L_y) + f''(L_y)L_y]dy,$$

The first integral is with respect to a martingale, so it has expectation zero, and

$$G_x f(L_x) = \lim_{h\downarrow 0}\frac{1}{h} E\{f(X_{x+h})-f(L_x) \mid \mathcal{U}_x\} = L_x f''(L_x) + (\delta(x) - \frac{(p^2s')'}{p^2s'} L_x)f''(L_x),$$

which verifies (4.6), and we are done.

Example : it is clear from (4.6) that $\{L_x, x \geq m\}$ is not time-homogeneous, for $\delta(x)$ is not constant. It will, however, be homogeneous on each of the intervals $(-\infty,0)$, $(0,a)$, and (a,∞) if $(p^2s')'/p^2s'$ is constant there. This was the case in Theorem 3.3 since $p\equiv 1$ and $s(x)\equiv x$ there. It is true of any diffusion on natural scale for $x > 0$ for the same reason. Another celebrated case, due to D. Ray, is the following. Let B_t be a Brownian motion and S an exponential random variable with parameter α , independent of B_t. Let $\bar{B}_t$ be B_t killed at S. Finally, let X_t be the diffusion $\bar{B}_t$, conditioned on $B_S=0$. (This conditioning can be made rigorous via Doob's h-path processes. The semi-group of $\bar{B}_t$ is $\bar{P}_t = e^{-\alpha t} P_t$, where P_t is the Brownian semi-group, and the semi-group of X is the h-path transform of $\bar{P}_t$ for $h(x) = e^{\sqrt{2\alpha}\, x} \wedge e^{-\sqrt{2\alpha}\, x}$. This is a $\bar{P}$-excessive function which has a pole at the origin, and which is invariant away from the origin). The infinitesimal generator G of X is the h-transform of the generator of $\bar{B}$.

$$\mathcal{G}f(x) = \frac{1}{2h(x)}(hf)'' - \alpha f$$

$$= \frac{1}{2} f''(x) - 2\sqrt{2\alpha}\ \ \text{sgn}\ x\ f'(x)$$

The scale function s satisfies $\mathcal{G}s=0$; such a function which vanishes at 0 is

$$s(x) = \begin{cases} e^{2\sqrt{2\alpha}\,x} - 1 & \text{if } x \geq 0 \\ 1-e^{-2\sqrt{2\alpha}\,x} & \text{if } x \leq 0 \end{cases}$$

and it follows that for some $c > 0$,

$$p(x) = \begin{cases} 1 & x \geq 0 \\ \dfrac{1}{1-cs(x)} & x \leq 0 \end{cases}$$

To identify c, it is easily seen from Lemma 4.2 that

$$E^o\{L_o\} = \lim_{x \to o} \frac{x}{s(x)}\, s(x)\, E\{D_{ox}\} = \frac{1}{cs'(o)}$$

$$= \frac{1}{2c\sqrt{2\alpha}} \,.$$

Next calculate $E^o\{L_o\}$ from another standpoint. Let $\overline{E}$ be the expectation operator for $\overline{B}$, and E the expectation operator for X. Now X is identical to the process $\overline{B}$, killed at its last exit from 0 (5), so

$$\overline{E}\{L_o(S)\} = E\{L_o(\zeta)\} = E\{L_o\}$$

But as $\frac{1}{2}|\overline{B}_t| - L_o(t)$ is a martingale,

$$\overline{E}\{L_o(S)\} = \frac{1}{2}\,\overline{E}\{|\overline{B}_S|\}$$

$$= \int_o^\infty \int_o^\infty \frac{x}{\sqrt{2\pi\, t}}\, e^{-\frac{x^2}{2t}}\ \alpha e^{-\alpha t}\, dx\, dt$$

$$= \frac{1}{2\sqrt{2\alpha}}$$

Comparing this with the above, we see $c=1$, hence

$$p(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ e^{2\sqrt{2\alpha}\, x} & \text{if } x \leq 0 \end{cases}$$

so that $p^2 s' = 2\sqrt{2\alpha}$, and the generator of L_x is

$$\mathcal{C}_x f(u) = u \frac{d^2}{du^2} + (\delta(x) - 2\sqrt{2\alpha}\, u) \frac{d}{du}.$$

For the interpretation of this in terms of Bessel processes, see [6] or [8].

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