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ON BIFURCATIONS OF ORIENTATION REVERSING DIFFEOMORPHISMS OF THE CIRCLE

Henryk Żołądek

We consider several-parameter families of orientation reversing diffeomorphisms of the circle.
In (1) Brunovský investigated arcs of preserving orientation diffeomorphisms of the circle. In the orientation reversing case we shall obtain stronger results than in the orientation preserving case. In particular, in (2) Guckenheimer proved non-genericity of structurally stable n-parameter families of orientation preserving diffeomorphisms of the circle for $n \geq 1$. Our goal in this paper is to investigate genericity of structurally stable families in the orientation reversing case. We shall prove the density of structurally stable families for n=1,2. Let $Diff_0^r(S^1)$ be the space of orientation reversing C^r-diffeomorphisms of the circle and let D^n be unit ball in $R^n, D^n = \{x \in R^n : |x| \leq 1\}$. Define $F^{n,r}$ as the space of C^r-maps from D^n to $Diff_0^r(S^1)$ with C^r-topology .

<u>Definition</u>

Two families ξ , $\eta \in F^{n,r}$ are called topologically conjugated iff there is a homeomorphism h of D^n and n-parameter continuous family h_μ of homeomorphisms of the circle, for which the following condition is satisfied

$h_u \circ \eta(u) = \xi(h(u)) \circ h_u$

The family $\xi \in F^{n,r}$ is called structurally stable iff there is an open neighbourhood U of ξ in $F^{n,r}$, such that for each $\eta \in U$ η and ξ are topologically conjugated.

Instead of n-parameter families of diffeomorphisms we may consider functions on $D^n \times R^1$ satisfying condition $f(u,x+1) = f(u,x)-1$ such that for every $(u,x) \in D^n x R^1$ $\frac{\partial f}{\partial x}(u,x) < 0$. Topology of this space is defined as the topology of uniform convergence with all derivatives to order r.We denote $f_u = f(u,.)$. It is a well known fact that every orientation reversing diffeomorphism of the circle has exactly two fixed points and that any other periodic orbit has period 2.

Further under "x is a periodic point of f_u" we shall understand"exp($2\pi ix$) is a periodic point of the diffeomorphism induced by f_u" i.e. a either $\exists n: f_u(x) = x+n$ and then $\exp(2\pi ix)$ is a fixed point , or $f_u^2(x) = x$ and then $\exp(2\pi ix)$ is periodic of period 2. For $f \in F^{n,r}$ there are two C^r- functions $x_0, x_1 : D^n \to R^1$, $x_1 - 1 < x_0 < x_1$ such that $f_u(x_o(u)) = x_0(u)$ and $f_u(x_1(u)) = x_1(u)-1$. If $x \in (x_1(u)-1, x_o(u))$ then $f_u(x) \in (x_o(u)$, $x_1(u))$ and if $x \in (x_o(u)-0 , x_1(u))$ then $f_u(x) \in (x_1(u)-1, x_o(u))$. We denote $G_0^n(f)$, $G_1^n(f)$ as n-dimensional C^r-submanifolds in $D^n \times R^1$ (with boundary) being the graphs of the functions x_0 , x_1 . Easy proof of the following two lemmas we leave to the reader.

Lemma 1.

Let $f: R^1 \to R^1$ of class C^{2k}, $k > 1$, be such that $f(0) = 0$, $f'(0) = -1$, $(f^2)''(0) = 0, \dots, (f^2)^{(2k-1)}(0) = 0$ then $(f^2)^{(2k)}(0) = 0$.

Lemma 2.

Let $f:R^1 \longrightarrow R^1$ of class C^k, $k \geq 2$, and $x \in R^1$ be such that $f(x_0) \neq x_0$, $f^2(x_0) = x_0$, $f'(x_0) \neq 0$.

Then $(f^2)'(x_0) = 1$, $(f^2)''(x_0) = 0, \dots, (f^2)^{(k)}(x_0) = 0$ imply

$(f^2)'(f(x_0)) = 1$, $(f^2)''(f(x_0)) = 0, \dots, (f^2)^{(k)}(f(x_0)) = 0$.

Lemma 3.

There is an open, dense subset $F_1^{n,r} \subset F^{n,r}$, $r \geq 2n+1$, such that for any $f \in F_1^{n,r}$:

a) the maps $G_0(f), G_1(f) : D^n \to R^{n+1}$ defined as follows:

$$G_{0(1)}(f)(u) = (f_u'(x_{0(1)}(u)) + 1, (f_u^2)'''(x_{0(1)}(u)), \dots, (f_u^2)^{(2n+1)}(x_{0(1)}(u)).$$

are transversal to $Q_j = \{(0,\dots,0,y_1,\dots,y_j) \in R^{n+1} : (y_1,\dots,y_j) \in R^j\}$ for $j \geq 0$.

b) the restrictions of the maps G_0, G_1 to $\partial D^n = S^{n-1}$ are transversal to Q_j.

Proof.

This lemma follows from Thom Transversality Theorem in the form given in (4). We define in jet bundle, $J^{2n+1}(D^n \times S^1, S^1)$, submanifolds C_j. Let (u,x) be coordinates in $D^n \times S^1$, y-coordinate in S^1 and let $\{u,x,y,a_{\alpha\beta}\}$, $\alpha \in N^n$, $\beta \in N$ $|\alpha| + \beta \leq 2n+1$ be coordinates in $J^{2n+1}(D^n \times S^1, S^1)$. $(a_{00} = y)$. C_j is defined by $y-x=0$, $a_{01} + 1 = 0$ and n-j equations $H_l(a_{01}, a_{02}, \dots, a_{02l+1}) = 0$, $l=1,\dots,n+1-j$. We need to define the functions H_l. Write $(f^2)^{(2l+1)}(x) = 0$ in form $K_l(f' \circ f(x), f'(x), \dots, f^{(2l+1)} \circ f(x), f^{(2l+1)}(x)) = 0$. Define $H_l(a_{01},\dots,a_{02l+1}) = K_l(a_{01}, a_{01}, \dots, a_{02l+1}, a_{02l+1})$. H_l is a polynomial such that $\frac{\partial H_1}{\partial a_{021+1}} = a_{01}+(a_{01})^{21+1}$. If $a_{01} = -1$ then this derivative is different from 0/is equal to -2/ . So C_j are submanifolds of class C^∞, in $J^{2n+1}(D^n \times S^1,S^1)$. From Thom Transversality Theorem it follows that the set of maps $f:D^n \times S^1 \to S^1$ of class C^∞, 2n+1- jets of which are transversal to all submanifolds C_j is dense in $C^\infty(D^n \times S^1,S^1)$. It is easy to see that such maps satisfy both conditions a),b) of Lemma /note that we are dealing with the fixed points/.

This proves the density of $F_1^{n,r}$ in $F^{n,r}$. Proof of the openess of $F_1^{n,r}$ in $F^{n,r}$ is simple and we left it to the reader.

Denote
$$G^{n-1}_{0(1)} = \{(u,x) \in G^n_{0(1)} : f'_u(x) = -1\}$$
$$G^{n-2}_{0(1)} = \{(u,x) \in G^{n-1}_{0(1)} : (f^2_u)'''(x) = 0\}$$
$$\cdots$$
$$G^{n-j}_{0(1)} = \{(u,x) \in G^{n-j+1}_{0(1)} : (f^2_u)^{(2j-1)}(x) = 0\} \quad \text{for } j=2,\ldots,n.$$

From Lemma 3.

It follows that these sets are submanifolds .

We also define $H^n(f) = \{(u,x): x\in(x_0(u),(x_1(u)) \cup (x_1(u)-1\ , x_0(u))\ , \quad f^2_u(x) = x\}$,

$$H^{n-1}(f) = \{(u,x) \in H^n(f): (f^2_u)'(x) = 1\}$$
$$\cdots$$
$$H^{n-j}(f) = \{(u,x) \in H^{n-j+1}(f) : (f^2_u)^{(j)}(x) = 0\} \quad \text{for } j=2,\ldots,n .$$

Now we investigate the set of periodic points of f near $x_0(u)$.

Proposition 4.

Let $r \geq 2n+1$ and $f \in F^{n,r}$.

Then there is $R > 0$ such that in

$V_0 = \{(u,x) : |x-x_0(u)| \leq R\}$,

the closures of H^{n-k} form submanifolds, which are transversal to G_0^n and the intersections of these submanifolds with are equal to G_0^{n-k-1}

Moreover, in $V_0 \setminus G_0^n$ the following properties hold :

a) the map $H(f) : V_0 \setminus G_0^n \longrightarrow R^{n+2}$ defined by the formula $H(f)(u,x) = \left[(f_u^2(x) - x\ ,\ (f_u^2)'(x)-1, (f_u^2)''(x), \dots, (f_u^2)^{(n+1)}(x))\right]$ is transversal to $P_1 = \{(0,\dots,0,y_1,\dots,y_1) : (y_1,\dots,y_1) \in R^1\}$ for $1 \geq 0$.

b) the restriction $H(f)$ to the set $(V_0 \setminus G_0^n) \cap (\partial D^n \times R^1)$ is transversal to P_j .

Proof.

If $(u_0,x_0) \in G_0^n \setminus G_0^{n-1}$ then x_0 is hyperbolic fixed point of f_{u_0} and there are no periodic points of period 2 of f_{u_0} (u close to u_0) near $x_0(u)$. Let $(u_0,x_0) \in G_0^{n-j}\ G_0^{n-j-1}$, $j > 0$. We can choose the system of coordinates in a neighbourhood of u_0 in the set of parameters and a n-parameter family of the diffeomorphisms $\phi_u : A \to B$. A isaneighbourhood of x_0 , B is a neighbourhood of 0 in R^1 , $\phi_u(x = x-x_0(u)$, such that in this coordinates the following conditions hold.

(0) $(f_u(x) = x \equiv x = 0\ ,\ u_0 = 0$,

(1) $(f_u)'(0) = -1 \iff u_1 = 0\ ,\ \frac{\partial}{\partial u_1}\left[f_u'(0)\right]_{u=0} \neq 0$

(2) $(f_u^2)'''(0) = 0 \iff u_2 = 0\ ,\ \frac{\partial}{\partial u_2}\left[(f_u^2)'''(0)\right]_{u=0} \neq 0$

....

(j) $(f_u^2)^{(2j-1)}(0) = 0 \iff u_j = 0\ ,\ \frac{\partial}{\partial u_j}\left[(f_u^2)^{(2j-1)}(0)\right]_{u=0} \neq 0$

(j+1) $(f_u^2)^{(2j+1)}(0) \neq 0$

For k=0 the first part of Proposition is true because by (0)

$f_\mu^2(x)-x = x.h_0(\mu,x)$, $h_0(\mu,x) = \int_0^1 (f_\mu^2)'(t_0.x)\,dt_0-1$, and thus H^n is given by the equation $h_0(\mu,x) = 0$ and by (1) the closure of H^n is a regular submanifold. $\frac{\partial h_0}{\partial x}(\mu,0) = \frac{1}{2}(f_\mu^2)''(0) = 0$ for $\mu_1 = 0$, hence $\overline{H^n}$ is transversal to G_0^n at $\overline{H^n} \cap G_0^n = \{(\mu,x) : x=0 , \mu_1=0\}$ $= G_0^{n-1}$.

Further we prove the following assertions by induction.

(i) for $1 \le j$ H^{n-1} is given by the equations $h_0(\mu,x) = 0,\dots,h_l(\mu,x) = 0$

(ii) $(f_\mu^2(x)-x)^{(l)} = \sum_{i=0}^{l} h_l\cdot c_{il}(\mu,x)+x^{l+1}h_l$, $l=1,\dots,j$, here c_{il} are some functions.

(iii) $\frac{\partial h_l}{\partial x} = \sum_{i=0}^{l} h_i \cdot d_{il}(\mu,x)+x.h_{l+1}$, $l=1,\dots,j-1$, d_{il} are some functions

(iv) $\mu_l = \sum_{i=0}^{l-1} h_i.b_{il}+x^2/\sum_{j=0}^{2l+1} B_{lj})$, $l=1,\dots,j$, here b_{il} are some functions,

$$B_{lj} = \sum_\alpha e_\alpha\cdot\int_0^1 \dots \int_0^1 Q^\alpha(t_1,\dots,t_{s_\alpha})\cdot g_j(P_1^\alpha\cdot\mu_1,\dots,P_n^\alpha\cdot\mu_n,P^\alpha.x)dt_1\dots dt_{s_\alpha}.$$

Here e_α-some functions, $Q^\alpha,P^\alpha,P_\nu^\alpha$-monomials of $t_1,\dots,t_s$ with coefficient equal to 1 , $g_j(\mu,x) = (f_\mu^2)^{(j)}(x)$.

(v) $h_l(\mu,x) = A_{2l+1} + x.(\sum_{j=0}^{2l+1} A_{lj})$, here A_{lj} takes the same form as B_{lj} and

$$A_{2l+1} = \int_0^1\dots\int_0^1 (t_0\dots t_{l-1})^2\cdot g_{2l+1}(t_1,\dots,t_l\cdot\mu_1,\dots,t_l\cdot\mu_l,\mu_{l+1},\dots,\mu_n,t_0\dots,t_l\, x)\, dt_0,\dots,dt_l$$

From our assumptions on f and from (v) it follows that the closure of H^{n-k} , $k < j$, is a regular submanifold.

$h_l(\mu,0) = 0 \iff A_{2l+1}(\mu,0) = 0$ and it is equivalent to $\mu_{l+1} = 0$. Thus $\overline{H^{n-k}} \cap G_0^n = G_0^{n-k-1}$. Because of $\frac{\partial h_l}{\partial x} = 0$ on G_0^{n-k-1} $\overline{H^{n-k}}$ is transversal to G_0^n .

Note that if $k=j$ then $g_{2n+1}(0,\dots,0) \neq 0$ and $H^{n-k} = \phi$ in a neighbourhood of $(0,0)$.

Now we prove the above assertions. Let $(i)_k,\dots,(v)_k$ be the assertion $(i),\dots,(v)$ for $1\le k$ it may happen that some of these are empty).

The scheme of the proof is :

I. $(iv)_k, (v)_k \Rightarrow (iv)_{k+1}$,

II. $(v)_k, (iv)_{k+1} \Rightarrow (iii)_k, (v)_{k+1}$;

III. $(iii)_k, (ii)_k \Rightarrow (ii)_{k+1}$;

IV. $(ii)_{k+1} \Rightarrow (i)_{k+1}$.

We prove only implications II, III, and IV because the proof of the first implication is similar to that of II.

II. $(v)_k, (iv)_{k+1} \Rightarrow (iii)_k, (v)_{k+1}$.

From $(v)_k$ one contains $\frac{\partial h_k}{\partial x} = \frac{\partial A_{2k+1}}{\partial x} + \sum_{j=0}^{2k+1} A_{kj} + x./ \sum_{j=0}^{2k+1} \frac{\partial A_{kj}}{\partial x}$

Let us observe that $A_{kj}(\mu,x) = 0$ for $\mu_1 = \mu_2 = \dots = \mu_{k+1} = x = 0$, thus $A_{kj} = \sum_{i=1}^{k+1} \mu_i \cdot \alpha_i(\mu,x) + x \cdot A'_{kj+1}$, A'_{kj+1} takes the same form as A_{kj+1} .

Now by Lemma 1. $\frac{\partial A_{kj}}{\partial x}(\mu,x)$ and $A'_{kj+1}(\mu,x)$ are equal to 0 for $\mu_1 = \mu_2 = \dots = \mu_{k+1} = x = 0$ and so they are in form

$\sum_{i=1}^{k+1} \mu_i\beta_i + x.A'_{kj+1} + x'A''_{kj+2}$ β_i are some functions and A''_{kj+1} , A''_{kj+2} takes the same form as A_{kj+1} or A_{kj+2} . From $(iv)_{k+1}$ it follows that $\frac{\partial h_k}{\partial x} = \frac{\partial A_{2k+1}}{\partial x} + \sum_{i=0}^{k} h_k\gamma_i + x^2 \cdot (\sum_{j=0}^{2k+3} \tilde{A}_{kj})$, $\tilde{A}_{kj}$ takes the same form as A_{kj} . One calculates that

$$\frac{\partial A_{2k+1}}{\partial x} = \sum_{i=1}^{k+1} \mu_i \cdot \delta_i(\mu,x) + x.A_{2k+3} \quad .$$

So $\frac{\partial h_k}{\partial x} = \sum_{i=1}^{k} h_i \cdot d_{ik+1} + x.A_{2k+3} + x^2 \cdot (\sum_{j=0}^{2k+3} A_{k+1j})$. Define $h_{k+1} = A_{2k+3} + x(\sum_{j=0}^{2k+3} A_{k+1j})$.

From this one obtains $(iii)_k$ and $(v)_{k+1}$.

III. $(iii)_k$, $(ii)_k \Rightarrow (ii)_{k+1}$. Proof is straightforward.

IV. $(ii)_{k+1} \Rightarrow (i)_{k+1}$.

If $(\mu,x) \in H^{n-k}$, i.e. $h_0(\mu,x) = \ldots = h_k(\mu,x) = 0$, then $(\mu,x) \in H^{n-k-1}$ iff $(f_\mu^2(x)-x)^{(k+1)} = 0$ but by $(ii)_{k+1}$ it may happen iff $h_{k+1}(\mu,x) = 0$.

Proof of the last part of proposition is easy and we omit it.

Lemma 5.

There is an open dense subset $F_2^{n,r} \subset F_1^{n,r}$, $r \geq 2n+1$, consisting of such f that :

a) the map $H(f):\{(u,x):x \in (x_0(u),x_1(u))\} \to R^{n+2}$ defined as follows $H(f)(u,x) = \left[f_u^2(x)-x,(f_u^2)'(x)-1,(f_u^2)''(x),\ldots,(f_u^2)^{(n+1)}(x)\right]$ is transversal to $P_j = \{(0,\ldots,0,y_1,\ldots,y_j) : (y_1,\ldots,y_j) \in R^j\}$ for $j \geq 0$.

b) the restriction of the map $H(f)$ to $\{(u,x) : u \in \partial D^n , x \in (x_0(u) , x_1(u))\}$ is transversal to P_j .

Proof of this Lemma is not difficult and we omit it.

The above considerations give no still answer to the question : are structurally stable families dense in $F^{n,r}$. For $n=1,2,3$ the answer is "yes". At first we give an idea of the proof. The submanifolds H^{n-j} and G^{n-j} are not so interesting as their projections on the set of the parameters. These projections give a stratification of the set of the parameters. If μ_1,μ_2 belong to one of the strata then $f_{\mu 1}$ and $f_{\mu 2}$ are topologically equivalent. The problem of genericity of structurally stable families reduces to the following: are families f such that for f_1 close to f the above stratifications are homeomorphic i.e. there is the homeomorphism h of D^n such that $h(\text{stratum}) = \text{stratum}$, dense in $F^{n,r}$. Some additional transversalities must be used in proving of structural stability of f. We give a proof of density for $n = 1,2$. Now we consider the case $n=1$.

By lemma 3 and lemma 5, we see that for $f \in F_2^{1,r}$, $r \geq 3$:

(i) f_{-1} and f_1 are structurally stable $(\partial D^1 = \{-1,1\})$,

(ii) if $(\mu,x) \in D^1 \times R^1$ is such that x is a fixed point of f_μ (for example $x = x_0(\mu)$) then

either x is hyperbolic,

or x is quasi-hyperbolic (i.e., $f'_\mu(x)=-1, (f^2_\mu)'''(x) \neq 0$),

$\frac{d}{d\lambda}(f'_\lambda(x_0(\lambda)))_{\lambda=\mu} \neq 0$

(iii) if (μ,x) is such that x is a periodic point of period 2 of f_μ then:

either x is hyperbolic,

or x is quasi-hyperbolic (i.e. $(f^2_\mu)'(x) = 1$, $(f^2_\mu)''(x) \neq 0$

and $\frac{d}{d\lambda}(f^2_\lambda(x))|_{\lambda=\mu} \neq 0$

We also know that the set of (μ,x) such that x is a quasi-hyperbolic point of f_μ is finite. By a small change of dependence of f on μ we obtain a family g such that (iv) for every $\mu \in D^1$ g_μ has at most one quasi-hyperbolic periodic point. Families $g \in F_2^{n,r}$ satisfying the last property (iv) are dense in $F^{n,r}$ (denote the set of them by $F_3^{1,r}$). It is not difficult to prove that every $f \in F_3^{1,r}$ is structurally stable (note that enough to consider a neighbourhoods of (μ,x) s.t. x is a quasi-hyperbolic periodic point of f_μ). I shall omit proof of this fact (is based on ideas of Sotomayor (3)). Now we consider the case $n=2$. By above facts we know that in the generic case $f|\partial D^2 x R^1$ is structurally stable, for $f \in F_2^{n,r}$. If (μ,x) is such that x is a hyperbolic periodic point of f_μ then in the generic case we need to obtain structural stability near (μ,x). Two cases are interesting:

(a) x_0 is a periodic point of f_{μ_0} of period 2 and

$$(f^2_{\mu_0})'(x_0) = 1 \ , \ (f^2_{\mu_0})''(x_0) = 0 \ ;$$

(b) x_0 is a fixed point of f_{μ_0} and $f'_{\mu_0}(x_0) = -1$, $(f^2_{\mu_0})'''(x_0) = 0$

Consider the case (a), for $f \in F_2^{n,r}$, $r \geq 5$, $n = 2$.

By lemma 5., we need only to assume that $(f^2_{\mu_0})'(x_0) = 0$ and the map $(\mu,x) \to [f^2_{\mu}(x)-x, (f^2_{\mu})'(x)-1, (f^2)''(x)]$ is regular at (μ_0,x_0)

Thus the map $(\mu,x) \to (f^2_{\mu}(x)-x \ , \ (f^2_{\mu})'(x)-1)$ is regular at (μ_0,x_0).

We can choose coordinates (μ,λ) in D^2 near μ_0 such that $\mu_0 = (0,0)$ and $\frac{\partial}{\partial u}(f^2_{\mu_0}(0))_{\mu=0} \neq 0$, $f^2_{\mu\lambda}(0) = 0 \iff \mu = 0$ and $\frac{\partial}{\partial \lambda}(f^2_{0\lambda})'(0)_{\lambda=0} \neq 0$, $(f^2_{\mu\lambda})'(0) = 1 \iff \lambda = 0$, (we put $x_0 = 0$)

By Weierstrass-Malgrange Preparation Theorem, we can assume that

$$f^2_{\mu\lambda}(x)-x = \mu . h_0(\mu,\lambda) + \lambda . h_1(\mu,\lambda) . x + (\mu . h_2 + \lambda h_{3(\mu,\lambda)}) . x^2 + \\ + h_4(\mu,\lambda,x) . x^3$$

Here $h_0(0,0) \neq 0$, $h_1(0,0) \neq 0$, $h_4(0,0,0) \neq 0$.

$f^2_{\mu\lambda}(x)-x = 0$ is an equation of H^2 .

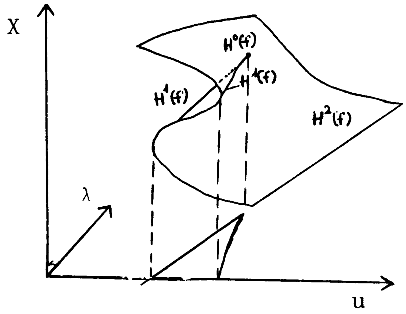
H^1 is given by a system of the equations $\begin{array}{l} f^2_{\mu\lambda}(x)-x=0 \\ (f^2_{\mu\lambda})'(x)-1=0 \end{array}$

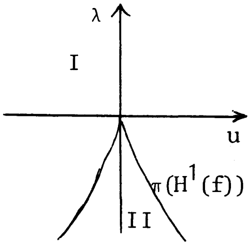
which is in our situation equivalent to the following system

$$\begin{cases} \mu = g_1(\mu,\lambda,x).x^3 \\ \lambda = g_2(\mu,\lambda,x).x^2 \end{cases}$$

here g_1 and g_2 are the fonctions depending on $h_0,\dots,h_4$ such that $g_1(0,0,0) \neq 0$ and $g_2(0,0,0) \neq 0$.

It is not difficult to prove that in general position the situation looks like at the picture and for g close to f the submanifolds $H^2(g)$, $H^1(g)$ and their projections are close to the analogous submanifolds and their projections as for f .





$f_{\mu\lambda}$ has one hyperbolic periodic point in the domain I , 3 hyperbolic periodic points in the domain II and one hyperbolic and one quasi-hyperbolic points in $\pi(H^1)$, all near 0 . Now we consider the case (b) .

We can choose the system of coordinates (μ,λ) and 2-parameter family of diffeomorphisms of R^1 such that after these changes

$$f^2_{\mu\lambda}(x)-x = \mu.h_1(\mu,\lambda).x + \mu.h_2(\mu,\lambda).x^2 + \lambda.h_3(\mu,\lambda)x3 + (\mu.h_4 + \lambda.h_4).x^4 + h_5(\mu,\lambda,x)x^5$$

and here $h_1(0,0) \neq 0$, $h_3(0,0) \neq 0$, $h_5(0,0,0) \neq 0$. It is true in generic situation, and it follows from Lemm 3. ($f'_{\mu\lambda}(0) = -1 \iff \mu = 0$ and $f_{\mu\lambda}(x) = x \iff x = 0$) in a neighbourhood of $(0,0,0)$.

$G^1_0(f)$ is equal to $\{\mu = 0 = x\}$

$H^2(f)$ is equal to $\{(\mu,\lambda,x) : \mu h_1 + \mu_n . h_2 x + \lambda h_3 x^2 + (\mu h_4 + \lambda h'_4)x^3 + h_5 x^4 = 0\}$

$H^1(f)$ is given by the system of equations

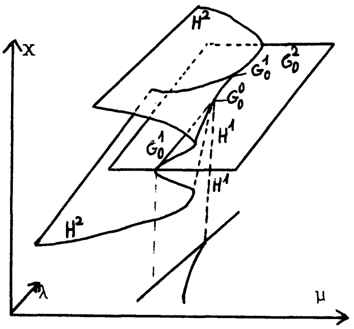
$$\begin{cases} \mu h_1 + \mu h_2 x + \lambda h_3 x^2 + (\mu h_4 + \lambda h_4) x^3 + h_5 x^4 = 0 \\ \mu(h_1 + 2h_2 x) + 3\lambda h_3 x^2 + 4(\mu h_4 + h'_4\lambda)x^3 + (5h_5 + \frac{\partial h_5}{\partial x} x)x^4 = 0 \end{cases}$$

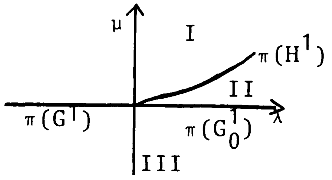
which is equivalent to the following system

$$\mu = x^4 . g_1(\mu,\lambda,x)$$

$$\lambda = x^2 . g_2(\mu,\lambda,x)$$

here g_1 , g_2 are the functions depending on $h_1,\dots,h_5$ such that $g_1(0,0,0) \neq 0$ and $g_2(0,0,0) \neq 0$. As in the case (a) it is not difficult to prove that in general position the situation looks like at the picture below ,





In the domain I $f_{\mu\lambda}$ has one fixed hyperbolic point and one hyperbolic periodic orbit of period 2, in the domain II $f_{\mu\lambda}$ has one fixed hyperbolic point and 2 hyperbolic periodic orbits of period 2, in the domain III f_{μ} has one fixed hyperbolic point ; on (H^1) $f_{\mu\lambda}$ has one hyperbolic fixed point and one quasi-hyperbolic periodic orbit of period 2 , if $\mu = 0, \lambda < 0$ then $f_{\mu\lambda}$ has one quasi-hyperbolic fixed point , if $\mu = 0, \lambda > 0$ then $f_{\mu\lambda}$ has one quasi-hyperbolic fixed point and one hyperbolic periodic orbit of period 2 .

The same situation we obtain for g close to f .

From the above considerations we know that there is only a finite set of points (μ, x) such that f_μ has a non-hyperbolic periodic orbit of type (a) or type (b). By a small change of the dependence of f on μ we can obtain a family g for which all the intersections and self-intersections of

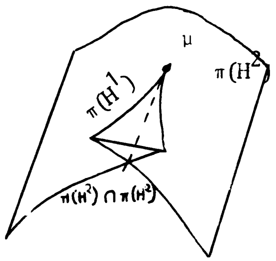
$\pi(H^1 \setminus H^0)$, $\pi(H^0)$, $\pi(G_0^1 \setminus G_0^0)$, $\pi(G_1^1 \setminus G_1^0)$, $\pi(G_0^0)$, $\pi(G_1^0)$ are transversal. It is easy to see that such family is structurally stable.

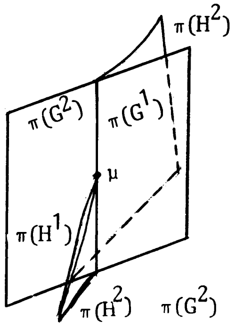
From above and from the considerations connecting the case $n=1$ one obtains the following.

Theorem 6.

For $n=1,2$, $r \quad 2n+1$, the structurally stable families of $F^{n,r}$ are dense in $F^{n,r}$.

We can ask what happens for $r < 2n+1$. Before I show how situation for 3-parameter generic families looks like. Below is given the stratification of the set of the parameters in a neighbourhood of a point μ such that f_μ has a non-hyperbolic periodic point.





We see that in this case $\pi(H^2 \setminus H^1)$ intersects itself (or intersects $\pi(G^2)$) arbitrary close to μ . It is not difficult to prove that in generic situation these intersections are transversal. As in the case $n=2$ we can prove of the density of structurally stable families in $F^{3,r}$, for $r \geq 7$. For $n > 3$ calculations are complicated in I don't know how to prove analogous to Theorem 6. for arbitrary n .

Now, I shall prove the following .

Theorem 7.

For $n=1$ or $n=2$, if $r < 2n+1$ there is in $F^{n,r}$ an open subset consisting of unstable families.

Proof.

We prove Theorem only in case $n=1$. Proof of Theorem in case $n=2$ is similar and is based on results connected with the case of $n=3$, $r \geq 7$.

Let in a neighbourhood of $(0,0)$ f has the form

$f(\mu.x) = (\mu-1)x$

Any g close to f in $F^{1,r}$ is in the form

$g(\mu,x) = c(\mu).(x-x_0)(\mu)) + d(\mu).(x-x_0)(\mu)) + 0(x-x_0)(\mu))^2 + x$

In a neighbourhood of $(\mu_0,x_0(\mu_0)),(c(\mu_0)) = -2)$ we can perturb g in $F^{1,2}$ to the following one

$g_1(\mu,x) = x+c(\mu).(x-x_0(\mu) + d(\mu).(x-x_0(\mu))^2 + e(\mu).(x-x_0(\mu))^3$

s.t. $(g^2_{1\mu_0})'''(x_0) = 0$

Let $q(\mu,\lambda,x) = g_1(\mu,x) \in F^{2.5}$. We can perturb q to $q_1 \in F^{2,5}_2$ and we bring the 1-parameter family $g_2(\mu,x) = q_1(\mu,\lambda,x)$ (λ=const). For some small λ we obtain a family such that for μ close to 0 $g_{2\mu}$ has one quasi hyperbolic periodic orbit, close to 0 , and thus the number of periodic orbits of period 2 is bigger than for g. This implies unstability of g .

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