WOLFGANG KRIEGER

On topological Markov chains

Astérisque, tome 50 (1977), p. 193-196
<http://www.numdam.org/item?id=AST_1977__50__193_0>

Société Mathématique de France
Astérisque 50 (1977) p.193-196

On Topological Markov Chains

Wolfgang Krieger

The method of the dimension has been of use in the theory of C^*-algebras (in the present context see in particular [2]). We apply this method here to the classification of topological Markov chains. For this we consider countable locally finite groups of homeomorphisms of zero dimensional separable σ-compact spaces. We say that a countable locally finite group G of homeomorphisms is ample if every homeomorphism in G has an open fixed point set and if every homeomorphism that can be pieced together from finitely many elements of G is already contained in G. Thus ample groups are something like the full groups of ergodic theory [1,3,4]. An ample group G on a σ-compactum X acts on the Boolean ring C_X of compact and open subsets of X. Denote the resulting orbit space by $C_X|G$. This orbit space carries an algebraic operation where for $\alpha, \beta, \gamma \in C_X|G$ one has $\gamma = \alpha \dot{+} \beta$ if there exist $A \in \alpha$, $B \in \beta$ and $C \in \gamma$ such that $A \cap B = \emptyset$ and $C = A \cup B$. The quotient map $\delta_G : C_X \to C_X|G$ is called the dimension of G. Its range $C_X|G$ generates the cone of an ordered abelian group $K_0(G)$, the dimension group of G. The spacial classification of ample groups is achieved by the ranges of their dimensions [4].

Consider now an expansive homeomorphism T of a zero dimensional compactum X. Let d be a metric of X. Denote by F_T the set of homeomorphisms U of X such that

$$\lim_{|n| \to \infty} \sup_{x \in X} d(T^n U T^{-n} x, x) = 0 .$$

F_T is an ample group. The elements of F_T are precisely those homeomorphisms U of X for which there is a bound on the number of coordinates of any point that are changed under the action of U, where T is represented as a subshift by means of some generator. Thus the elements of F_T can be called the uniformly finite dimensional homeomorphisms of T. A dimension δ_T and a dimension group $K_0(T)$ are now defined as δ_{F_T} and $K_0(F_T)$. T induces an automorphism σ_T on the range of its dimension, and the pair $(C_X | F_T, \sigma_T)$ is an invariant of T. Observe that we have here the C^*-algebra, which is the crossed product of F_T with $C_{\mathbb{C}}(X)$, associated to T, and T induces an automorphism of this C^*-algebra. In this way we have connected to T a non-commutative dynamical system.

Let now Σ be a finite symbol space, and let $q \in \mathbb{Z}_+^{\Sigma\times\Sigma}$ be a transition matrix. q gives rise to a topological Markov chain $T(q)$ on a shift space $X(q)$. The pair $(C_{X(q)} \mid F_{T(q)} , \sigma_{T(q)})$ is an invariant of shift equivalence (concerning shift equivalence see [5]). Certain AF algebras are associated to topological Markov chains. E.g. the C^*-algebras that are associated to the full shifts belong to Glimm's class. The divisible envelope of an ordered abelian group can be completed. If this completion is carried out for the dimension group of an irreducible and aperiodic Markov chain then one obtains $\mathbb{R}$, and the range of the canonical injection on the dimension range coincides with the range of the probability measure of maximal entropy on the closed and open sets. One can give examples of aperiodic and irreducible Markov chains with identical ζ-functions whose probability measures of maximal entropy have distinct ranges on the closed and open sets.

We obtain another dimension for a topological Markov chain $T(q)$ by considering an ample group $F_T(x)$ of uniformly finite dimensional homeomorphisms on the stable manifold $W^s(x)$ of any point $x \in X(q)$. We call this dimension the past dimension of $T(q)$. A future dimension is produced on the unstable manifolds. $T(q)$ induces an automorphism $\sigma^{(p)}_{T(q)}$ of $C_{W^s(\cdot)} / F_T(\cdot)$. The resulting algebraic structure can be described as follows. Consider on $\mathbb{Z}^\Sigma \times \mathbb{Z}$ the equivalence relation $\Lambda(q)$ where $(c,i)\, \Lambda(q)\, (c',i')$ if there exists a $j < i,i'$ such that $q^{i-j}\, c = q^{i'-j}\, c'$. An addition on $\mathbb{Z}^\Sigma \times \mathbb{Z} / \Lambda(q)$ is given by

$$[(c,i)]_{\Lambda(q)} + [(d,i)]_{\Lambda(q)} = [(c+d,i)]_{\Lambda(A)}, \quad c,d \in \mathbb{Z}^\Sigma, \quad i \in \mathbb{Z} ,$$

and a cone in $\mathbb{Z}^\Sigma \times \mathbb{Z} / \Lambda(q)$ consists of the elements $[(c,i)]_{\Lambda(q)}$ such that for some $j \leq i$ $q^{i-j}\, c \geq o$.
An automorphism σ_q of the resulting ordered abelian group is given by

$$\sigma_q : [(c,i)]_{\Lambda(q)} \to [(c,\, i-1)]_{\Lambda(q)} \qquad , \quad c \in \mathbb{Z}^\Sigma,\ i \in \mathbb{Z} .$$

The pair $(\mathbb{Z}^\Sigma \times \mathbb{Z} / \Lambda(q),\, \sigma_q)$ is isomorphic to the pair $(C_{W^s(\cdot)} / F_{T(q)}(\cdot),\, \sigma^{(p)}_{T(q)})$. This pair contains a considerable amount of information. E.g. it is a complete invariant of shift equivalence.

References

1. H.A. Dye: On groups of measure preserving transformations I, Amer.J.Math. 81 (1959) 119-159

2. G.A. Elliott: On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J.of Algebra 38 (1976) 29-44

3. W. Krieger: On non-singular transformations of a measure space I, Z.Wahrscheinlichkeitstheorie 11 (1969) 83-97

4. W. Krieger: On a Dimension for a Class of Homeomorphism Groups, to appear

5. R.F. Williams: Classification of subshifts of finite type, Ann.of Math. 98 (1973) 120-153; Erratum 99 (1974)380-381

Wolfgang Krieger
Institut für Angewandte Mathematik
der Universität Heidelberg
6900 Heidelberg