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Astérisque, tome 50 (1977), p. 151-192
<http://www.numdam.org/item?id=AST_1977__50__151_0>

Société Mathématique de France
Astérisque 50 (1977) p.151-192

RECENT PROGRESS ON ERGODIC THEOREMS

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0. Introduction

The principal purpose of this paper is to give a survey of the significant progress that has been achieved concerning numerous very different problems on ergodic theorems during the last couple of years. The extensive bibliography indicates the broad spectrum of questions that have been treated in the research literature. The theory of contractions in L_1 and in particular the celebrated Chacon-Ornstein theorem have repeatedly been discussed in books because of their interest for the study of Markov processes, see e.g. Foguel [46] and Revuz [127], but many other investigations on ergodic theorems are only available in the form of the original research articles and it is very difficult for newcomers to find out what has been done. Because of this situation, Antoine Brunel and I have dedided to write a book on ergodic theorems, which should cover the main results concerning the entire spectrum of directions, e.g. mean ergodic theory, pointwise ergodic theorems for groups of transformations and contractions, local ergodic theorems, ergodic theorems for subsequences and weighted averages, Kingman's subadditive ergodic theory, vector-valued ergodic theorems, etc. We can devote only some of our time to this project, so that it will take us a while. Therefore, it seems worthwhile to write this preliminary report. Undoubtedly I have overlooked a number of results in this report and I would be grateful if authors of papers on ergodic theorems would send them to us. I shall state various open problems and it would be nice if some of them were solved in the near future. I also use this opportunity to announce some new results which have been obtained jointly with Antoine Brunel. They are given in theorems 1.5., 4.2., 4.3., 4.4. .

The material in this survey will be subdivided as follows:

1. Measure preserving transformations and contractions in L_1
2. Contractions in L_p $(1 < p < \infty)$; Akcoglu's ergodic theorem
3. Local ergodic theorems
4. Groups and semigroups
5. Mean ergodic theorems
6. Ergodic theorems for subsequences and weighted averages
7. Vector valued ergodic theorems
8. Kingman's subadditive ergodic theory.

As this article had to be ready for the conference in Warszawa I did not find the time to read all the quoted articles carefully and to search the literature systematically. Therefore the readers should go back to the referenced papers and also look up further references there.

A supplement mentions an important non-commutative ergodic theorem of C.Lance.

1. Measure preserving transformations and contractions in L_1

The most amazing new ergodic theorem surely is the ergodic theorem of Furstenberg [58]. Here it is not possible to discuss any of the new methodology, but in a survey I should at least point out the importance and novelty of his theory. The main result is:

Theorem 1.1. [58]:

If τ is a measure preserving invertible transformation of a probability space $(\Omega, \mathcal{B}, \mu)$, $A \in \mathcal{B}$ a set with $\mu(A) > 0$ and k an integer with $k \geqq 2$, then there exists $n \geqq 1$ with $\mu(A \cap \tau^{n} A \cap \tau^{2n} A \cap \ldots \cap \tau^{(k-1)n} A) > 0$.

This is an ergodic theoretic result analogous to a very deep recent theorem of Szemerédi: Any subset of the integers having positive asymptotic density contains arithmetic progressions of arbitrary length. Furstenberg also develops a technique which permits to derive Szemerédi's theorem from theorem 1.1. In the weakly mixing case Furstenberg shows

$$\lim_{(N-M)\to\infty} (N-M)^{-1} \sum_{n=M+1}^{N} \mu(A_1 \cap \tau^{n} A_2 \cap \tau^{2n} A_3 \cap \ldots \cap \tau^{(k-1)n} A_k)$$

$$= \mu(A_1) \cdot \mu(A_2) \ldots \mu(A_k).$$

In particular this answers the question if weak mixing implies 2-fold weak mixing positively. But the main difficulty in the proof of theorem 1.1. is the passage to transformations that are not weakly mixing.

A new observation in a very classical context has been made by D.Ornstein [122]. If τ is measure preserving in a finite measure space $(\Omega, \mathcal{B}, \mu)$, the dominated ergodic theorem tells us that $\sup_{n\geq 1} n^{-1} \sum_{k=o}^{n-1} f \circ \tau^k = f^*$ is integrable if $|f|\log^+|f|$ is integrable. Ornstein showed that for ergodic τ and $f \geq 0$ the converse is true. Y.Derriennic [38] has proved an extension to conservative ergodic transformations in a σ-finite measure space.

A rather nice converse for the condition $f \in L_1$ in Birkhoff's ergodic theorem has been proved by J.Aaronson [1]:

<u>Theorem 1.2.</u>:

If τ is an ergodic measure preserving transformation of a probability space $(\Omega, \mathcal{B}, \mu)$, $f \geq 0$, $\int f \, d\mu = \infty$ and (b_n) a sequence of positive reals then either

$\limsup b_n^{-1} \sum_{k=o}^{n-1} f \circ \tau^k = \infty$ a.e. or

$\liminf b_n^{-1} \sum_{k=o}^{n-1} f \circ \tau^k = 0$ a.e., or both.

The theorem of Aaronson is a special case of an analogous theorem on σ-finite measure spaces given in the same paper.

Another paper relating to Birkhoff's theorem is due to Nuber [120]. He gives conditions for measure preserving transformations which are necessary and sufficient for the conclusion of Birkhoff's theorem to hold constructively in the sense of Bishop.

Most of the other results in this section will be convergence theorems for contractions in L_1, Harris processes, quasicompact transition kernels, etc. and related material e.g. on existence of invariant measures. Much of this theory has been described in books [46], [127], so that here few remarks and references shall suffice.

A new type of question is studied in the paper [75] by R.Isaac. It is motivated by an example of Burkholder [26], who showed that the ergodic ratios may diverge after an application of a single conditional expectation: There exists an ergodic τ in a probability space $(\Omega,\mathcal{B},\mu)$, a sub-σ-algebra $\mathcal{B}_o \subset \mathcal{B}$ and an $f \in L_1$ so that $\limsup E(n^{-1} \sum_{k=o}^{n-1} f \circ \tau^k | \mathcal{B}_o) = +\infty$. Isaac studies ratios of the form $E(\sum_1^n f \circ \tau^k | \mathcal{B}_o)/E(\sum_1^n g \circ \tau^k | \mathcal{B}_1)$.

F.Papangelou [124] has described a martingale approach to the study of the convergence of the n-step transition probabilities, for Markov processes with a finite invariant measure.

As an example of the many new convergence theorems for Markov kernels and contractions in L_1 I want to discuss the beautiful "zero-two"-law of Ornstein and Sucheston [123]. The background has been a theorem of Orey asserting that for aperiodic recurrent Markov chains with n-step transition probabilities $p_{ij}^{(n)}$ $\lim_{n\to\infty} \sum_{j\in I} |p_{i_1j}^{(n)} - p_{i_2j}^{(n)}| = 0$ for all $i_1, i_2 \in I$ = {countable state space}. Now let the σ-algebra $\mathcal{B}$ in Ω be countably generated and $\{\omega\} \in \mathcal{B}$ for all $\omega \in \Omega$. Let $P^{(n)}(\omega,A)$ be the n-step transition probability from ω to A and assume that $\mu(A) = 0$ implies $P^{(1)}(\omega,A) = 0$ μ - a.e.. Let $\|\cdot\|$ denote the total-variation norm.

Identifying under the Radon-Nikodym isomorphism L_1 with the space of μ-continuous finite signed measures φ on $\mathcal{B}$, $P^{(1)}$

induces the usual contraction in L_1 by

$$T\varphi(A) = \int P^{(1)}(\omega,A)\varphi(d\omega).$$

<u>Theorem 1.3</u>. [106]:

If $P^{(n)}$ is as above and the induced contraction in L_1 is conservative and ergodic, and if

$$h(\omega) = \lim_n ||P^{(n)}(\omega,\cdot) - P^{(n+1)}(\omega,\cdot)||,$$

then either $h(\omega) = 0$ μ - a.e. or $h(\omega) = 2$ a.e. In the first case

$$\lim_n ||T^n\varphi|| = 0$$

for all $\varphi \in L_1$ with $\int \varphi \, d\mu = 0$.

This theorem essentially contains Jamison and Orey's generalization of Orey's theorem to the case of Harris processes. In subsequent papers S.R.Foguel [47],[48], and Y.Derriennic [40] have extended the zero-two-law, and given applications.

Fong and Sucheston [55] have also investigated the case when T is no longer a contraction in L_1. An interpretation of such operators can e.g. be given for branching processes. T is called power-bounded if $\sup_n ||T^n|| < \infty$. If T is positive and power-bounded, a theorem of Sucheston says that Ω decomposes into the remaining part Y, having the property that $\liminf \int T^n f > 0$ for all $0 \neq f \in L_1^+$ with support in Y, and the disappearing part $Z = \Omega \setminus Y$, which is such that $\lim \int |T^n f| = 0$ for all integrable f with support in Z. There exists an $e \in L_\infty^+$ with $\{e > 0\}=Y$

and $T^*e = e$. Fong and Sucheston prove that
(1) $\int |T^n f| \cdot e \, d\mu \to 0$ implies (2) $\int |T^n f| d\mu \to 0$.
They also give a new proof of the Jamison-Orey theorem via the "filling scheme" investigated by H.Rost [128]. Another nice theorem in [55] says that for any conservative positive contraction T in L_1 the class of functions $\{f - Tf : f \in L_1^+\}$ is dense in $G := \{f - Tf : f \in L_1\}$. This permits to strengthen Chacon's "resolution of positive operators". Chacon had shown that L_1 is the direct sum of the closure of G and $I := \{h \in L_1 : Sh = h\}$, where S is the adjoint T* of T, naturally extended to L_1.

An unaveraged convergence theorem for L_p-norms with $p > 1$ is due to Michael Lin [113]. If $P^{(n)}$ is as above and has a σ-finite invariant measure $\tilde{\mu}$ equivalent to μ, and if $P^{(n)}(\omega, A) \to 0$ a.e. for some A with $\tilde{\mu}(A) > 0$, then $\| T^n f \|_p \to 0$ for every $f \in L_p(\tilde{\mu})$. Also in [110] Lin studies very similar questions.

Many papers have been published on the convergence of ratios $Q_n = \langle T^n f_1, h_1\rangle / \langle T^n f_2, h_2\rangle$ with $f_i \in L_1^+, h_i \in L_\infty^+$ (strong ratio limit theorems) and of ratios of averages $\tilde{Q}_n = \sum_{i=o}^{n} \langle T^i f_1, h_1\rangle / \sum_{i=o}^{n} \langle T^i f_2, h_2\rangle$ (ratio limit theorems). Usually, the process is assumed to be recurrent in the sense of Harris, i.e. it is a Harris process. Métivier [116] proves convergence of $\tilde{Q}_n$ for any pair of functions h_1, h_2 which is "special" in the sense of Neveu [119], compare also Lin [112]. M.Lin [107] has a generalization of Orey's strong ratio limit theorem. Also Foguel and Lin [50] investigate ratio limit theorems.

Chacon's general ergodic theorem for ratios $\sum_{k=o}^{n} T^k f / \sum_{k=o}^{n} p_k$ where T is a contraction in L_1 and (p_k) an "admissible" sequence, i.e. a sequence such that $|Tg| \leq p_{k+1}$ if $|g| \leq p_k$, has been proved in a continuous-time setting by Tsurumi [146]. Gologan [59] has a short argument to deduce Chacon's theorem in the real-valued case from the case of positive contractions.

A. Ionescu-Tulcea and M. Moretz [74] have shown that the ratios of the Chacon-Ornstein theorem fail to converge on the Z-part of Sucheston's decomposition, they converge on the Y-part. Fong and Sucheston [57] have an analogous result for continuous time. Y. Kubokawa [102] has extended the Dunford-Schwartz ergodic theorem on the convergence of $n^{-1} \sum_{k=o}^{n-1} T^k f$ under the assumptions $\sup_n \| n^{-1} \sum_{k=o}^{n-1} T^k \|_1 < \infty$ and $\sup_n \| n^{-1} \sum_{k=o}^{n-1} T^k \|_\infty \leq 1$.

H. Dinges [42] has proposed to look at convergence theorems like that of Chacon-Ornstein in such a way that the joint distributions of the sequences should be emphazised, as in Birkhoff's ergodic theorem. In contrast, in the Chacon-Ornstein theorem, ratios $q_n = (f_o+f_2+\ldots+f_k)/(p_o+p_1+\ldots+p_k)$ are considered and there is a specific condition how the f_k and p_k arise: $Tf_k = f_{k+1}$ and $Tp_k = p_{k+1}$. This is not in any apparent way a condition on the joint distributions of the f_k and the p_k. After some hard analysis it turned out that the following condition could be used to prove convergence of ratios q_n:

Let P_n be the joint distribution of $(f_n, p_n, f_{n+1}, p_{n+1}, f_{n+2}, p_{n+2}, \ldots)$, then $\int h \, d P_n \geq \int h \, d P_{n+1}$ for all non-negative, positively homogeneous, convex functions h and all $n \in \mathbb{N}$ should hold. However, it then was shown by Rost [129] and Engmann [44] that under such a condition on the sequence, q_n did arise from a positive contraction. Thus conditions as the one above could not lead to new ergodic theorems; they could, however, be used to characterize the existence of certain operators. This program is carried out in [44] also for the sequences of operators in an ergodic theorem of Cuculescu and Foias [37].

D.Ornstein [121] has obtained a generalization of the Chacon-Ornstein theorem in a very abstract setting: Let L be a Riesz-space, i.e. a vector-lattice of real-valued measurable functions on $(\Omega, \mathcal{B}, \mu)$. Let T be a positive linear operator in L. $\bar{f}$ is called a modification of f if $f - h + Th = \bar{f}$ and $f \geq 0$, $\bar{f} \geq 0$, $h \geq 0$. Ornstein's main result is:

<u>Theorem 1.4.</u>:
For $f, g \in L^+$ either $\lim_{n\to\infty}(\sum_0^n T^i f)/(\sum_0^n T^i g)$ exists a.e. on $\{\sum_0^\infty T^i g > 0\}$ or there are arbitrarily large modifications of some fixed $f_0 \geq 0$, i.e. there exists $0 \neq g_0 \in L^+$ and modifications $\bar{f}_i$ of f_0 such that $\bar{f}_i > i\, g_0$.

Antoine Brunel [18] has given equivalent conditions to the existence of arbitrarily large modifications and he has proved in this abstract setting a theorem which contains Chacon's theorem for ratios with admissible sequences in the denominator in the case of positive operators.

On the other hand for many asymptotic statements the class of positive contractions of L_1 is already too general and restrictions must be imposed. A bounded linear operator T is called quasicompact, if $\|T^n - K\| < 1$ for some $n > 0$ and some compact linear operator K. This is equivalent to the classical Doeblin-condition. Horowitz [70] studied the quasicompactness of an ergodic conservative contraction T and of certain induced contractions. Lin [112] derives equivalent conditions for quasi-compactness and gives applications to Harris processes.

Brunel [19] also studies such contractions T in L_1, the existence of σ-finite invariant measures, and the validity of an inequality

$$\sup_n \|\sum_{j=o}^{n} T^{*j} f\|_\infty \leq C \|f\|_\infty$$

for functions $f \in L_\infty$ with $\int f \, d\mu = 0$ which have a "bounded" support. This continues work of Ornstein and Métivier. A.Brunel and D.Revuz [24] have further investigated quasicompact operators, including their spectral theory, Neveu's special functions for Harris processes and a related inequality. M.Lin [111] shows that, for Markov operators, quasicompactness is equivalent to uniform ergodicity with finite dimensional fixed points space and for ergodic transition probabilities strong convergence of the averages is also equivalent to quasicompactness.

V.I.Istratescu [79], [77] has studied a class of operators extending the class of quasicompact operators by looking at measures of non-compactness.

A problem on positive contractions in L_1 closely related to the convergence theorems, is the problem of existence of finite or σ-finite invariant measures, see [46]. An important reference is the article of Neveu [118], who derived many necessary and sufficient conditions for the existence of finite invariant measures and who introduced weakly wandering functions. I now want to discuss some recent results of Brunel and myself which unify and strengthen some of the known criteria. Let T be positive contraction of $L_1(\mu)$. A weakly wandering function h is an element of L_∞^+ for which there exists a strictly increasing sequence $0 = k_0 < k_1 < k_2 < \dots$ of integers with $\|\sum_{\nu=0}^{\infty} T^{*k_\nu} h\|_\infty < \infty$. If the even stronger condition

$$\sup_{j\geq 0} \|\sum_{k_\nu \geq j} T^{*k_\nu - j} h\|_\infty < \infty$$

holds, h is called m-weakly wandering. m-weakly wandering functions have been introduced in the unpublished thesis of Brunel.

<u>Theorem 1.5.</u>:

There exists a disjoint decomposition $\Omega = \tilde{C} \cup \tilde{D}$, uniquely determined up to nullsets, such that $\tilde{C}$ is contained in the conservative part C, $T^* 1_{\tilde{C}} = 1_{\tilde{C}}$ and

(i) there exists a $p_0 \in L_1^+$ with $Tp_0 = p_0$ and $\{p_0 > 0\} = \tilde{C}$, and

(ii) there exists an m-weakly wandering $h_0 \in L_\infty^+$ with $\{h_0 > 0\} = \tilde{D}$.

The decomposition is, of course, Neveu's decomposition into the strongly conservative part $\tilde{C}$ and its complement. The novelty lies in the last equation $\{h_o > 0\} = \tilde{D}$ which not only makes the theorem more symmetric, but also permits to derive the different criteria of Neveu in a simple way. Now observe that $\sum_{i=o}^{n} T^{*k_i}h \leq c$ implies

$$\limsup_{m\to\infty} \|m^{-1} \sum_{i=o}^{m-1} T^{*i} h\|_\infty \leq (n+1)^{-1} c,$$

because

$$(n+1) \sum_{i=o}^{m-1} T^{*i}h \leq \sum_{i=o}^{m-1} T^{*i}(\sum_{\nu=o}^{n} T^{*k_\nu} h) + (n+1)k_n \|h\|_\infty$$

$$\leq mc + (n+1)k_n\|h\|_\infty.$$

Therefore $\lim_{m\to\infty} \|m^{-1} \sum_{i=o}^{m-1} T^{*i}h_o\|_\infty = 0$ and also theorem E, p.45 in [46] follows directly.

Fong [52] has studied the existence of finite invariant measures in the power-bounded case. Y.Derriennic and M.Lin [41] have obtained Suchestons decomposition $\Omega = Y \cup Z$ under the weaker condition $\sup_{n\geq 1} \|n^{-1} \sum_{k=o}^{n-1} T^k\|_1 < \infty$. In this case one gets $\lim_{n\to\infty} n^{-1}\sum_{k=o}^{n-1} \|T^k f\|_1 = 0$ for functions $f \in L_1$ with $\{f \neq 0\} \subset Z$. They also discuss finite invariant measures and convergence theorems in this setting.

A.Brunel [21] has shown that a positive contraction T in L_1 possesses a finite invariant measure if and only if all convex combinations $\sum_{i=o}^{\infty} a_i T^i$ with $a_i \geq 0$, $\sum_{i=o}^{\infty} a_i = 1$ are conservative. An application of this has been given by Brunel and Revuz [23]. S.R.Foguel and B.Weiss [51] have a new elegant proof of Brunel's criterium. They also show that for convex $A(s) = \sum_{i=o}^{\infty} a_i s^i$ with $A'(1) < \infty$ $A(T)$ is conservative if T is conservative, and that in the case $A'(1) = \infty$ there exists a conservative T for which A(T) is dissipative.

Other papers studying finite invariant measures for semigroups $\{T_t, t \geq 0\}$ of positive contraction are due to Lin [106] (who carries over many other theorems from discrete to continuous time as well), and Falkowitz [45]. Horowitz [71] investigates convex combinations of semigroups $\{T_t, t \geq 0\}$ of the form $Qf = \int_0^\infty \varphi(t)T_t f\, dt$ ($f \in L_\infty, \varphi(t) > 0$, $\varphi(t) \downarrow 0$, $\int t\, \varphi(t)dt < \infty$).

2. Contractions in L_p $(1<p<\infty)$; Akcoglu's Ergodic Theorem:

If T is a contraction in L_2 von Neumann's classical ergodic theorem implies norm-convergence of the ergodic ratios $g_n = n^{-1} \sum_{k=o}^{n-1} T^k f$. The question for almost everywhere convergence remained open for a long time. In 1962 Burkholder [25] gave an example for which g_n did not converge a.e. On the other hand, A.Ionescu-Tulcea [72] proved convergence a.e. if T was a positive invertible isometry of L_p $(1 < p < \infty)$. In this case T was induced by a nonsingular point transformation. In the sixties much effort has been devoted to the problem of a.e.-convergence of g_n for positive contractions T in L_p $(1 < p < \infty)$ and $f \in L_p$. Some progress has been obtained, notably by R.V.Chacon, who solved the problem under some additional conditions weaker than those of A.Ionescu-Tulcea, but the full problem remained as hard as ever. Finally, Akcoglu [4] succeeded in solving the problem by introducing "dilations" of positive contractions:

Theorem 2.1. [4]:

If T is a positive contraction in L_p and $f \in L_p$ then g_n converges a.e.

The principal difficulty was the proof of the following maximal inequality: If $f^* = \sup_{n\geq 1} n^{-1} \sum_{k=o}^{n-1} |T^k f|$, then $\|f^*\|_p \leq p(p-1)^{-1} \|f\|_p$, which had already been proved in the special case of positive invertible isometries by A.Ionescu-Tulcea. A special case of the following "dilation theorem" did permit a passage from isometries to contractions:

Theorem 2.2. [11]:

Let $T : L \to L$ be a positive contraction on an L_p-space L. Then there exists another L_p-space B and a positive invertible isometry $Q : B \to B$ so that $DT^n = PQ^n D$ for all $n = 0,1,2,...$ where $D : L \to B$ is a positive isometric imbedding of L into B and $P : B \to B$ is a positive projection.

This theorem with a non-constructive proof is contained in the paper [11] of Akcoglu-Sucheston. In [4] Akcoglu needed and proved such a result only in the finite dimensional case under some additional conditions. The general finite-dimensional case follows from the results of Akcoglu-Sucheston in [12]. W.B.Johnson observed that the general dilation theorem follows already from the finite-dimensional case by some general techniques in Banach spaces due to D.Dacunha-Castelle and J.L.Krivine, but the proof in [11] uses in part different arguments.

A dilation theorem for positive contractions in L_1 has been proved by Akcoglu [3]. A different approach to this theorem has been presented in [89] by Kern, Nagel and Palm.

Akcoglu and Sucheston [9] have shown that there exist even unitary operators T in L_2, for which g_n may diverge, if the assumption of positivity is dropped.

It is known from the work of Chacon [29] that g_n in general diverges, if T is a positive contraction in L_1 and $f \in L_1$. For positive contractions in L_1 the ratios $r_n = \sum_{o}^{n} T^i f / \sum_{o}^{n} T^i g$ $(g > 0,\ f \in L_1)$ converge a.e. For contractions in L_p $(1 < p < \infty)$ the convergence behaviour is just the opposite. Akcoglu [2] has shown that r_n in general diverges for positive contractions T in L_p $(1 < p < \infty)$. H.Fong and M.Lin [54] have constructed a class of more elementary examples. They also show that under mild assumptions, a positive operator acting on a linear lattice of measurable functions induces a decomposition $\Omega = Y \cup Z$ such that Y supports a subinvariant measure and the convergence of the ratios fails on every non-null subset of Z. Also R.Sato [132] studies such a decomposition for semigroups in L_p. In [135] R.Sato investigates the existence of finite invariant measures for positive operators in L_p with $\sup_{n \geq 1} \|n^{-1} \sum_{k=o}^{n-1} T^k\|_p < \infty$.

The proof of A.Ionescu-Tulcea in [72] has been given via a reduction to the case of contractions in L_1; it depends on many results. Therefore a recent direct proof of her maximal inequality for positive invertible isometries due to de la Torre [145] seems to be of considerable importance.

The dilation theorem has been successfully applied by Akcoglu and Sucheston for the proof of an ergodic theorem for subsequences. This will be discussed in section 6.

3. Local Ergodic Theorems:

Let $\{\tau_t,\ t \in \mathbb{R}^1\}$ be a measurable flow of measure-preserving transformations in $(\Omega, \mathcal{B}, \mu)$. In the thirties, N.Wiener proved that $M_\varepsilon f = \varepsilon^{-1} \int_0^\varepsilon f \circ \tau_t dt$ converges to f a.e. as $\varepsilon \to 0 + 0$. Such results are called local ergodic theorems. The author [97] and D.Ornstein [121] have independently started the study of such results on an operator theoretic level by proving

Theorem 3.1.:

If $\{T_t,\ t \geq 0\}$ is a strongly continuous semigroup of positive contractions in L_1 and $f \in L_1$ the averages $M_\varepsilon f = \varepsilon^{-1} \int_0^\varepsilon T_t f\, dt$ converge a.e. to $T_0 f$.

Under mild measurability conditions a semigroup $\{T_t, t\geq 0\}$ is always strongly continuous for $t > 0$, i.e. $\|T_{t+h} f - T_t f\|_1 \to 0$ as $h \to 0$. However, it need not be continuous for $t = 0$. Akcoglu and Chacon [5] have proved the a.e. convergence of $M_\varepsilon f$ assuming only the continuity of $\{T_t, t > 0\}$. In this case the limit may be different from $T_o f$, in fact D.Ornstein pointed out that the convergence of $M_\varepsilon f$ to $T_o f$ implies continuity at $t = 0$. The result of Akcoglu and Chacon is much deeper and mathematically more satisfactory, though the examples without continuity at 0 do not seem to play a role in practice. The discontinuity can only be produced if there is a part of the space Ω, The "initially dissipative" part D_o such that $T_t f$ is 0 on D_o for all $t > 0$.

Th.R.Terrell [143] has extended theorem 3.1. to n-parameter commutative semigroups. In this case, $M_\varepsilon f$ is replaced by

$$M_\varepsilon^{(n)} f = \varepsilon^{-n} \int_o^\varepsilon \int_o^\varepsilon \cdots \int_o^\varepsilon T_{t_1,t_2,\ldots,t_n} f \, dt_1 \ldots dt_n$$

and $\{T_{t_1,\ldots,t_n}, t_i \geq 0\}$ is a strongly continuous semigroup of positive contractions in L_1. He also proved convergence with the positivity assumption replaced by the assumption $\|T_{t_1,\ldots,t_n}\|_\infty \leq 1$ for all $t_1,\ldots,t_n$, where

$$\|T\|_\infty = \sup\{\|Tf\|_\infty : f \in L_1 \cap L_\infty, \|f\|_\infty \leq 1\}.$$

In [144] he further investigated the equivalence of the local ergodic theorem without positivity with a maximal inequality. Y.Kubokawa [104] and Claude Kipnis [93] finally removed the positivity assumption in Theorem 3.1. altogether. They independently constructed a minimal semigroup $\{S_t, t \geq 0\}$ of positive contractions majorizing a given semigroup $\{T_t, t \geq 0\}$ of contractions in L_1. This is the continuous analogue of the modulus $\tilde{T}$ of a bounded linear operator in L_1 as studied by R.V.Chacon and the author [30]. A new difficulty compared with the construction of the individual moduli $\tilde{T}_t$ arizes, since they do not form a semigroup in general. Toughly speaking S_t is the supremum of all products $\tilde{T}_{t_1} \cdot \tilde{T}_{t_2} \dots \tilde{T}_{t_r}$ with $t_1 + t_2 + \dots + t_r = t$, r finite. The result of Kipnis is slightly more precise in that he does not make use of the continuity of the semigroup $\{T_t, t \geq 0\}$ in the construction. Moreover, he has related theorems for resolvants.

H.Fong and L.Sucheston [57] have proved a local ergodic theorem under a weakened boundedness condition. Their theorem (like the one of Akcoglu-Chacon [5]) is spelled out in a ratio form, which, however, is not very different from considering $M_\varepsilon f$.

Y.Kubokawa [103] proved the a.e. convergence of $M_\varepsilon f$ for strongly continuous semigroups of positive bounded operators, dropping the contraction hypothesis. It seems to be an open problem if the contraction property and the positivity can be dropped simultaneously.

Y.Kubokawa [101] made a very exciting contribution by proving a local ergodic theorem for operators in L_p $(1<p<\infty)$. He showed that $M_\varepsilon f$ converges a.e. for strongly continuous semigroups of bounded positive operators in L_p. He also proved convergence for $f \in L_p$ without positivity under the assumptions $\|T_t\|_\infty \leq 1$ and $\|T_t\|_1 \leq 1$. J.R.Baxter and R.V.Chacon [15] proved a local ergodic theorem for contractions in L_p with $\|T_t\|_\infty \leq 1$.

McGrath [65] has an n-parameter generalization of the L_p-local ergodic theorem of Kubokawa and some new proofs of results above. In [62] he obtains a local ergodic theorem for n-parameter semigroups of not necessarily positive isometries of L_1.

Further related results can be found in the papers [63], [64] of McGrath and [134] of R.Sato.

Some authors have proved "Abelian" local ergodic theorems, i.e. theorems on convergence of $\lambda R_\lambda f$ $(\lambda \to \infty)$ where $R_\lambda f = \int_0^\infty e^{-\lambda t} T_t f \, dt$. Such theorems can also be obtained as direct consequences of the local ergodic theorems above.

There are still many open problems on local ergodic theorems: E.g., it would be interesting to know if $M_\varepsilon f$ converges a.e. for not necessarily positive bounded operators in L_p $(1 < p < \infty)$ without further restrictions, or at least for contractions in L_p. The n-parameter version of the result of Kubokawa-Kipnis is an open problem. In all the above results except theorem 3.1. it is an interesting open problem, whether the assumption of continuity of the semigroup at 0 can be dropped as in the paper of Akcoglu-Chacon. R.Sato [133] shows that for nonpositive contractions in L_1 with $\|T_t\|_\infty \leq 1$ a semigroup $\{T_t, t>0\}$ continuous for $t > 0$ can be made continuous at $t = 0$. Finally, nothing general has been done for contractions in L_∞. If $\{\tau_t, t \geq 0\}$ is a semigroup of nonsingular transformations of $(\Omega, \mathcal{B}, \mu)$ and $f \in L_\infty$, then $\varepsilon^{-1} \int_0^\varepsilon f \circ \tau_t dt$ converges a.e. as observed by the author [96]. This is a special semigroup of contractions in L_∞ and there may well be a general theorem.

4. Groups and semigroups

Questions on norm-convergence of averages for groups and semigroups of operators in Banach spaces are usually discussed in the context of the decomposition theorems of mean ergodic theory. We refer the reader to the book and lecture notes of K.Jacobs [80], [81].

I begin with the discussion of n-parameter commutative semigroups. The following theorem of A.Brunel [20] allows to assert pointwise convergence for contractions in L_1, which also contract the L_∞-norm; this contains the case of measure preserving point mappings.

Theorem 4.1.:

If $T_1, T_2, \ldots, T_n$ are commuting contractions of L_1 with $\|T_i\|_\infty \leq 1$ $(i=1,\ldots,n)$ the averages

$$M_K f = K^{-n} \sum_{\nu_1=0}^{K-1} \cdots \sum_{\nu_n=0}^{K-1} T_1^{\nu_1} \cdots T_n^{\nu_n} f$$

converge a.e. for $f \in L_1$ as $K \to \infty$.

The limit is also described in [20]. This is a discrete analogue of a theorem of Dunford-Schwartz, but it seems to have the advantage that it implies the continuous version for semigroups $\{T_{t_1,\ldots,t_n};\ t_i \geq 0\}$ while the converse implication does not seem to hold.

A.Brunel and the author have applied the constructions in

[20] to obtain the following n-parameter generalization of Akcoglu's ergodic theorem:

Theorem 4.2.:

If $T_1,\dots,T_n$ are positive commuting contractions in L_p $(1 < p < \infty)$ and $f \in L_p$ the averages $M_K f$ converge a.e.

By similar techniques we have also extended the stochastic ergodic theorem of the author [95]:

Theorem 4.3.:

If $T_1, T_2,\dots,T_n$ are commuting contractions of L_1 and $f \in L_1$, the averages $M_K f$ converge stochastically, i.e. for all $\varepsilon > 0$ and $A \in \mathcal{B}$ with $\mu(A) < \infty$

$\mu\{A \cap \{|M_K f - \bar{f}| > \varepsilon\}\} \to 0$, where $\bar{f}$ is the limit function.

Clearly theorem 4.2. and 4.3. imply their continuous time analogues.

One might expect an analogous n-parameter version of the Chacon-Ornstein theorem, but such a theorem does not even hold for $T_1 = T_2$:

Theorem 4.4.:

There exist bounded functions $f,g > 0$ on $\Omega = \mathbb{Z}$, integrable with respect to the counting measure, such that the ratios

$$\frac{\sum_{i=1}^{k} \sum_{j=1}^{k} T_1^i T_2^j f}{\sum_{i=1}^{k} \sum_{j=1}^{k} T_1^i T_2^j g}$$

diverge in Ω, where $T_1 f(\omega) = f(\omega+1)$ and $T_2 = T_1$.

A.A.Tempelman [141] has announced ergodic theorems for semi-groups of measure preserving transformations. He has supplied proofs and further extensions in [151]. In the meantime, T.Bewley [16] and J.Chatard [31] have studied the subject. Also F.P.Greenleaf [66] and W.R.Emerson [43] have closely related results. Some of them discuss also norm-convergence.

For almost sure convergence the natural setting seems as follows: Let G be a locally compact amenable group and λ left Haar measure. A summing sequence $\{S_n\}$ is an increasing sequence of compact subsets $S_n \subset G$ with $\lambda(S_n) > 0$ such that $G = \bigcup_{n=o}^{\infty} S_n$ and $\lambda(S_n)^{-1} \lambda(g\, S_n \,\Delta\, S_n) \to 0$ for all $g \in G$. Such a sequence exists if and only if G is σ-compact. If $\{\tau_g, g \in G\}$ is a group of measure preserving transformations of $(\Omega, \mathcal{B}, \mu)$ one considers the averages

$$A_n f := \int_{S_n} f \circ \tau_g \, \lambda(dg)$$

Unfortunately, they need not converge for all summing sequences $\{S_n\}$. It is necessary to impose a supplementary condition, see e.g. [43]. In the above papers the condition

(*) $\quad \lambda(S_n^{-1} S_n) < K\lambda(S_n)$ for some $K > 0$ and all $n \in \mathbb{N}$

is shown to suffice. One gets:

Theorem 4.5.:

If G is a locally compact amenable group, $\{\tau_g, g \in G\}$ a group of measure preserving transformations such that the map $(g,\omega) \to \tau_g\omega$ is measurable with respect to the product σ-algebra in $G \times \Omega$ and $\{S_n\}$ a summing sequence satisfying (*), then $A_n f$ converges a.e. for $f \in L_1$.

Finally, I want to mention that the problem of existence of finite invariant measures for amenable semigroups of contractions in L_1 has been discussed by U.Sachdeva [131]. The case of semigroups of nonsingular point mappings is due to Granirer [61].

The reader should also consult the article [36] of J.P.Conze and Dang Ngoc, about which I learned only recently.

5. Mean Ergodic Theorems:

Mean ergodic theory has received new stimulation mainly through its connection with the study of quasicompact Markov operators mentioned in section 1 and weak mixing properties which played a role for ergodic theorems for subsequences.

M.Lin [108] gave new proofs of classical equivalent conditions and a new useful equivalent condition (4) as follows: Let T be a bounded linear operator on a Banach space X satisfying $\|T^n/n\| \to 0$. Then the following conditions are equivalent:

(1) There exists a bounded linear operator E such that $\|N^{-1} \sum_{n=o}^{N-1} T^n - E\| \to 0$;

(2) $(I-T)X$ is closed and $X = \{x : Tx = x\} \oplus (I-T)X$ (where I denotes the identity);

(3) $(I-T)^2X$ is closed;

(4) $(I-T)X$ is closed.

In [109] Lin derives equivalent conditions to the uniform convergence of $t^{-1} \int_o^t T_s ds$ for a semigroup $\{T_t, t\geq 0\}$: One is the closedness of the range of the infinitesimal generator, another one the uniform convergence of λR_λ as $\lambda \to 0$, where R_λ is the resolvent.

Butzer and Westphal [27] have derived the speed of convergence in the classical mean ergodic theorem in reflexive Banach spaces:

If T is power bounded, the Cesàro-averages $n^{-1} \sum_{k=o}^{n-1} T^k x$ converge to a projection Px onto $\{x : Tx = x\} = F$ and X is the direct sum of F and the closure of $G = \{x - Tx : x \in X\}$, as was shown by Lorch. In [27] it is shown that $\|n^{-1} \sum_o^{n-1} T^k x - Px\| = o(n^{-1})$ implies $x \in F$ and $\|n^{-1} \sum_o^{n-1} T^k x - Px\| = O(n^{-1})$ if and only if $x \in F \oplus G$.

Robert C.Sine [138] has proved that for a contraction T in a Banach space X the Cesàro-averages $n^{-1} \sum_o^{n-1} T^k$ converge in the strong operator topology if and only if the fixed points of T separate the fixed points of T^*. S.P.Lloyd [115] has further generalized this.

Lee K.Jones [84] has proved a refinement of the mean ergodic theorem in Banach spaces. In [83] he proves a generalization of the spectral mixing theorem of von Neumann to uniformly convex Banach spaces. Moreover, he gives several characterizations of weak mixing in uniformly convex spaces.

Lee K.Jones and M.Lin [86] give a simplified proof and refinement of an ergodic theorem of Jones [82] as follows:

Theorem 5.1.:

If T is a linear operator on a Banach space X with $\sup_n \|T^n\| < \infty$ and $x \in X$, then the following conditions are equivalent:

(1) For every $x^* \in X^*$ we have $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} | \langle x^*, T^j x\rangle | = 0$, i.e. T is weakly mixing.

(2) $\sup_{\|x^*\|\leq 1} N^{-1} \sum_{j=1}^{N} | \langle x^*, T^j x\rangle | \to 0$

(3) For every subsequence $\{k_j\}$ with positive lower density (i.e. $\sup k_j/j < \infty$) we have $\lim_{N\to\infty} \|N^{-1} \sum_{j=1}^{N} T^{k_j} x\| = 0$

(4) For every subsequence $\{k_j\}$ with positive lower density $\sup_{\|x^*\|\leq 1} N^{-1} \sum_{n=1}^{N} | \langle x^*, T^{k_j} x\rangle | \to 0$.

In [87] Jones and Lin prove for X^* separable or reflexive, and for $\{T^n x\}$ weakly conditionally compact that condition (2) holds if and only if x is orthogonal to the eigenvectors of T^* with unimodular eigenvalues. This equivalence may fail for the shift in l_∞.

Related results are contained in the papers [117] of Nagel and [114] of Lin.

The author [99] has given a characterization of continuous spectrum, i.e. weak mixing, in Hilbert space: Call a vector x weakly wandering if there exists a sequence $k_1 < k_2 < \dots$ such that the $T^{k_i}x$ are orthogonal.

Theorem 5.2.:

An isometry T in a Hilbert space X has continuous spectrum if and only if the weakly wandering vectors span X; it has discrete spectrum if and only if there are no non-zero weakly wandering vectors.

D.Graham [60] and Sund [141] have extended this to certain groups of unitary operators, but the problem if such a characterization is true for all groups or at least all Abelian groups is open.

6. Ergodic Theorems for Subsequences and Weighted Averages:

Considerable effort has been devoted during the last few years to the study of the convergence behaviour of averages

$$A_n^K f = n^{-1} \sum_{i=0}^{n-1} T^{k_i} f$$

where $K = \{k_i\}_{i=0}^{\infty}$ is a strictly increasing sequence $0 = k_0 < k_1 < \dots$ of integers and T a linear operator on a Banach space or more specially in L_p. This has been initiated by a theorem of Blum-Hanson, which, for operators

T of the form $Tf = f \circ \tau$ with τ measure preserving in a finite measure space, said that $A_n^K f$ converges in L_2 for all $f \in L_2$ and all K if and only if τ is mixing. The proper functional analytic formulation of this result is that $T^n f$ converges weakly iff $A_n^K f$ converges strongly. Short proofs for contractions in a Hilbert space habe been given in [85] and in [14]. Akcoglu, Huneke and Rost [6] have shown that such a result does not hold in all Banach spaces. M.A.Akcoglu and L.Sucheston [14] proved for arbitrary contractions in L_1 the equivalence of the convergence of T^n in the weak operator topology and the convergence of A_n^K in the strong operator topology for all K. After a partial result of Fong and Sucheston [56], Akcoglu and Sucheston [13,8,10] successfully employed the theory of dilations of positive contractions to settle the case L_p $1 < p < \infty$ $p \neq 2$. Their result in their more general formulation for weighted averages is:

<u>Theorem 6.1.</u> [8]:

Let T be a positive contraction in L_p $(1 \leq p < \infty)$. Then $\lim T^n$ exists in the weak operator topology if and only if $\lim\limits_{n\to\infty} \sum\limits_{i=o}^{\infty} a_{ni} T^i$ exists in the strong operator topology for all uniformly regular matrices (a_{ni}), i.e. all (a_{ni}) such that

(i) $\sup\limits_n \sum\limits_i |a_{ni}| < \infty$;

(ii) $\lim\limits_{n\to\infty} \sup\limits_i |a_{ni}| = 0$; and

(iii) $\lim\limits_n \sum\limits_{i=o}^{\infty} a_{ni} = 1$.

A.Ionescu-Tulcea [73] has a different approach to such convergence theorems via a new L_p-inequality. Other papers relating to these convergence problems for subsequences are Lin [105], Istratescu [78], Fong [53], and Sato [136]. Note that theorem 5.1. asserts convergence of A_n^K for K with positive lower density under a weaker condition. It does not seem to be known if the positivity assumption in theorem 6.1. is necessary, in fact, for $p = 1,2$ it is not!

N.Friedman and D.Ornstein have constructed a mixing transformation τ, a sequence K and a function f, for which the averages $A_n^K f$ with $T^k f = f \circ \tau^k$ failed to converge a.e. It was then an open problem if $A_n^K f$ did converge if τ was a Bernoulli shift. The author [98] has shown that there even exists a universal sequence K such that for all aperiodic τ there is an indicator function $f = 1_A$ for which $A_n^K f$ diverges. Different constructions have then been given by J.P.Conze [35] and Akcoglu and del Junco [7]. Conze also shows the existence of a K with positive lower density and a weakly mixing τ for which $A_n^K f$ diverges for some f. On the other hand, he proves convergence a.e. for all $f \in L_1$ if K has positive lower density and τ Lebesgue spectrum. Brunel and Keane [22], C.Ryll-Nardzweski [130], Blum and Reich [17] and Reich [125] have also proved a.e.-convergence of $A_n^K f$ under different assumptions on the sequences K or the functions f.

Making use of random ergodic theorems one can frequently assert the a.e. convergence of $A_n^K f$ for "almost all" sequences K. We refer to Eijun Kin [90] and Tsurumi [147] for recent investigations of random ergodic theorems for operators.

In all the above convergence considerations for subsequences or weighted averages the convergence of the Cesàro-averages is classic. A different point of view is that weighted averages should be used in cases when Cesàro-averages fail to converge. The author has adopted this point of view in the case of contractions T in L_1, because by the work of Chacon, $n^{-1} \sum_{k=o}^{n-1} T^k f$ in general diverges. In [95] it was pointed out that for any contraction T in L_1 there exists a matrix summation method stronger than that of Cesàro, such that the averages $\sum_{i=o}^{\infty} a_{ni} T^i f$ converge a.e. Grillenberger and the author [67] proved the existence of different summation methods with this property by showing that for any contraction T in L_1 there exists a subsequence $n_1 < n_2 < \dots$ such that $n_i^{-1} \sum_{j=o}^{n_i-1} T^j f$ converges a.e. for all $f \in L_1$. The main result in [67] asserts that there does not exist any matrix summation method compatible with the Cesàro-method which enforces a.e.-convergence for all T.

7. Vector Valued Ergodic Theorems

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, E a Banach space and L_1^E the space of E-valued Bochner-integrable random variables. In [28] Chacon has proved an ergodic theorem for operators acting in L_1^E, assuming E reflexive. Hasegawa, Sato and

Tsurumi [69] have studied a continuous time version of Chacon's theorem, a corresponding Abelian ergodic theorem and a local ergodic theorem, see also Kopp [94], Tsurumi [148], and F.Chersi and S.Invernizzi [34]. A generalization of Chacon's theorem to spaces E which need not be reflexive has been sketched by Sucheston [140]. Sucheston assumes that E has the property (W), i.e. every sequence of E-valued random variables X_n such that $\sup_n \operatorname{ess\,sup}_\Omega |X_n(\omega)| < \infty$, admits a subsequence converging weakly in L_1^E. It is known that if both E and E* have the Radon-Nikodym property, then E has property (W), there are non-reflexive spaces with this property.

8. Kingman's Subadditive Ergodic Theorem:

In [92] J.F.C.Kingman has derived a very interesting ergodic theorem for subadditive processes. Such processes (with a somewhat weaker stationarity assumption) had been introduced by Hammersley and Welsh for the study of percolation problems. A subadditive process is a realvalued stochastic process $\{X_{ik} : (i,k) \in I\}$ with parameter set $I = \{(i,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : i < k\}$ and with the following properties:

(i) Whenever $i < j < k$ $\quad X_{ik} \leq X_{ij} + X_{jk}$

(ii) The distribution of the process $(X'_{ik}, (i,k) \in I)$ with $X'_{i,k} = X_{i+1,k+1}$ is the same as the distribution of $(X_{ik}, (i,k) \in I)$.

(iii) The expectation $g_k = E(X_{ok})$ exists, and satisfies $g_t \geq -At$ for some constant A and all $t > 1$.

They arise in percolation problems, from products of random matrices, in problems on random permutations, and in many other occasions, described in a survey article by Kingman [91]. E.g., X_{ik} can be the number of different points visited by a particle in the time instances i, i+1,...,k-1.

Clearly $g_{m+n} \leq g_m + g_n$, and therefore a theorem on subadditive functions implies that the limit $\gamma = \lim n^{-1} g_n$ exists.

Theorem 8.1. [92]:

If $(X_{ik}, (i,k) \in I)$ is subadditive, then the finite limit $\xi = \lim_{n\to\infty} n^{-1} X_{on}$ exists a.e. and in the mean and $E(\xi) = \gamma$.

Kingman's proof depends on a decomposition

$$X_{i,k} = Y_{ik} + Z_{ik}$$

in which (Y_{ik}) is additive and stationary (so that Birkhoff's theorem applies) and (Z_{ik}) is nonnegative, subadditive, and $\gamma((Z_{ik})) = 0$.

As Kingman [91] has given a survey of the results obtained before 1973, I shall here restrict attention to the more recent papers on the subject.

Y.Derriennic [39] has a very nice direct proof of theorem 8.1, which does not depend on the decomposition theorem. Instead, it depends on a new maximal inequality for subadditive processes.

Andrés del Junco [88] has a simplified proof of the decomposition theorem, which uses only a martingale argument, while Kingman's proof is much longer, and Burkholder's proof, given in the appendix of [91] depends on a rather deep theorem of Komlos.

J.M.Hammersley [68] examines alternative postulates for subadditive processes and gives many examples of applications.

Hiroshi Ishitani [76] gives a central limit theorem for subadditive processes.

In a recent letter I learned that Akcoglu and Sucheston have solved the problem of extending the subadditive ergodic theorem to the operator theoretic situation.

Finally, I would like to mention a recent theorem of myself [100], even though it has no direct connections to subadditive ergodic theory. But there is some analogy which leads to an interesting open problem.

A probability measure P on $\mathbb{R}^{\mathbb{N}}$ is called stochastically smaller than a probability measure Q on $\mathbb{R}^{\mathbb{N}}$ (write $P \leq Q$), if $\int f\,dP \leq \int f\,dQ$ for all bounded measurable functions f on $\mathbb{R}^{\mathbb{N}}$, which are nondecreasing in each variable; $(x_i \leq y_i\ (i = 0,1,2,\ldots)$ implies $f(x_o,x_1,\ldots) \leq f(y_o,y_1,\ldots)$.
A process $(X_n,\ n \geq 0)$ is called superstationary if $P_o \geq P_1$, where P_i is the distribution of $(X_i,X_{i+1},X_{i+2},\ldots)$ in $\mathbb{R}^{\mathbb{N}}$.

Theorem 8.2. [100]:

If $(X_n, n \geq 0)$ is superstationary and $X_o^+ \in L_1$ the averages $n^{-1} \sum_{k=o}^{n-1} X_k$ converge a.e. The proof depends on a decomposition

$$\tilde{X}_n = \tilde{Y}_n + \tilde{Z}_n$$

where $(\tilde{X}_n)$ has the same distribution as (X_n), (but is defined on a large enough probability space), $(\tilde{Y}_n)$ is stationary, and $(\tilde{Z}_n)$ is nonnegative and $E(\tilde{Z}_n) \downarrow 0$. This decomposition is constructed with the help of a theorem of Strassen [139], which implies a characterization of the stochastic partial ordering. It would be very interesting to know if $\tilde{Z}_n$ can be constructed in such a way that it is superstationary. This would be in analogy with Kingman's decomposition.

Multiparameter subadditive processes have been discussed in [149]. Using some recent results of X.X.Nguyen and H.Zessin [152], Nguyen [153] has been able to arrive at a complete generalization of Kingman's theorem to the multiparameter case.

Supplement:

In Warszawa, M.Keane and W.Krieger directed my attention to an important new ergodic theorem of C.Lance [150]. If τ is measure preserving, $f \to f \circ \tau$ is an automorphism of the commutative von Neumann algebra L_∞. Lance has proved an extension of Birkhoffs theorem (in L_∞) to automorphisms of non-commutative von Neumann algebras. See also [36].

References

Abbreviations:

ZW	=	Zeitschrift f. Wahrscheinlichkeitstheorie u. verw. Geb.
PAMS	=	Proc.Amer.Math.Soc.
TAMS	=	Transact.Amer.Math.Soc.
AIHP	=	Ann.Inst.Henri Poincaré

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