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ON LOCAL MODELS OF K-TUPLES OF VECTOR FIELDS

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1 - Introduction

In this report we state some results, (with sketches of proofs only) about the local behavior of finite systems of vector fields. The full version of the paper will appear elswhere.

More precisely, we try to classify germs of generic (in C^r - Whitney topology) k-tuples of C^∞ vector fields on smooth, n-dimensional manifold. The equivalence relation is defined as follows. Consider two germs of C^∞ vector fields $\tilde{X}_p = (\tilde{X}_p^1, \ldots, \tilde{X}_p^k)$, $\tilde{Y}_q = (\tilde{Y}_q^1, \ldots, \tilde{Y}_q^k)$ at points p,q of manifolds M,N, respectively. They will be called C^r equivalent if there is a germ at p $\tilde{g}_p : (M,p) \to (N,q)$ of a C^{r+1} diffeomorphism, and there is a germ at p $\tilde{F}_p = \{\tilde{f}_p^{ij}\}_{i,j=1}^k$ of a k x k matrix valued function $F : M \to R^{k^2}$, det $F(p) \neq 0$, such that

(1) $\tilde{g}_{p*}(\tilde{X}_p\tilde{F}_p) = \tilde{Y}_p$

($\tilde{X}_p, \tilde{Y}_q$ are treated as row vectors or, when the basis in the tangent space is given, as n x k matrices).

Let $\mathcal{H}^{n,k}$ denote the set of germs at $0 \in R^n$ of k-tuples of C^∞ vector fields on R^n and let $G^{n,k,r}$ be the group of pairs

$(\tilde{g}_p, \tilde{F}_p)$, where $M = N = R^n$ and $p = q = 0$ with multiplication

$$(\tilde{g}'_o, \tilde{F}'_o)(\tilde{g}_o, \tilde{F}_o) = (\tilde{g}'_o \circ \tilde{g}_o \; , \; \tilde{F}_o \tilde{F}'_o \circ \tilde{g}_o)$$

(we shall neglect the index r, when $r = \infty$). The formula (1) defines an action of $G^{n,k}$ on $\mathcal{H}^{n,k}$ and also an action of $G^{n,k,r}$ on $\mathcal{H}^{n,k}$ in the sense that $(g_o, \tilde{F}_o)(\tilde{X}_o)$ is defined, when it belongs to $\mathcal{H}^{n,k}$. We study the structure of orbits of $G^{n,k,r}$ in $\mathcal{H}^{n,k}$.

The above classification problem has the following interpretation in the control systems theory. Consider a class of control systems on M of the form

$$(2) \quad \dot{x} = \sum_{i=1}^{k} X^i u_i = Xu \quad ,$$

where u is a column vector. Assume that we may use any control u depending on the state x : $u(t) = u(x(t))$ (feedback control). Thus the set of vector fields, which can be got on the right hand side of (2) is of the form $\mathcal{X} = \{Xu | u \in C^\infty(M ; R^k)\}$. Let

$$(3) \quad \dot{y} = \sum_{i=1}^{k} Y^i v_i = Yv$$

be a control system on the manifold N and define systems (2), (3) to be C^r equivalent if there is a C^{r+1} diffeomorphism $g : M \to N$ such that

$$(4) \quad g_*(\mathcal{X}) = \mathcal{Y} \quad , \quad \text{where} \quad \mathcal{Y} = \{Yv | v \in C^\infty(M , R^k)\} .$$

It is easy to see that the equality (4) follows from the existence of a function $F : M \to R^{k^2}$, $\det F(x) \neq 0$ for $x \in M$, such that (1) holds for any $p \in M$. The matrix F can be also interpreted as a feedback modification $u = Fv$.

Systems of the form (4) where studied by A.J. Krener [3], who used an equivalence relation like (1) with F = identity.

2. Invariant subsets in $\mathcal{H}^{n,k}$

Let $j^m : \mathcal{H}^{n,k} \to j^m(\mathcal{H}^{n,k})$ denote a projection from germs to m-jets. A subset $Q \subset \mathcal{H}^{n,k}$ will be called submanifold (algebraic set, semialgebraic set) if it is of the form $j^{m^{-1}}(P)$ for some m, where P is a submanifold (algebraic set, semialgebraic set) in $j^m(\mathcal{H}^{n,k})$. Similarly, smooth functions and foliations on $\mathcal{H}^{n,k}$ are defined as functions and foliations on $j^m(\mathcal{H}^{n,k})$ composed with j^m on the right side.

In general, our aim is to find an invariant under $G^{n,k}$, algebraic subset Q of $\mathcal{H}^{n,k}$ with codimension greater then n, such that $\mathcal{H}^{n,k} \setminus Q$ can be divided (stratified) into finite number of invariant manifolds (being also semialgebraic sets). We look for the decomposition in which invariant manifolds would be exactly orbits and model of each orbit could be found. If this is impossible, we look for foliations (of codimension as large as possible) of some of invariant manifolds. When leaves are exactly orbits and models with parameter fixed on each leaf can be found, one may regard the results to be complete.

It is understandable that such a decomposition of $\mathcal{H}^{n,k}$ would give a complete information about germs of generic k-tuples of vector fields on n-manifolds (by the transversality lemma of Thom) and structural stability of these germs.

Let us define some invariant subsets of $\mathcal{H}^{n,k}$.

Définition

Fix a couple (n,k) and denote for $i \geq 0$

$$L_i\tilde{X} = \text{span}\,\{v \in T_0R^n \mid v = ad^{i_1}_{X^{r_1}}\, ad^{i_2}_{X^{r_2}} \dots ad^{i_p}_{X^{r_p}}(X^{r_{p+1}})(0),\quad i_1 + \dots + i_p \leq i\ ,\ r_j = 1, \dots, k\}$$

and $L_{-1}\tilde{X} = R^n$.

Define $Q(ij) \subset \mathcal{H}^{n,k}$, $i \geq 0$, $j \geq 0$,

$$Q(i,j) = \{\tilde{X} \in \mathcal{H}^{n,k} | \dim L_i \tilde{X} = j\} .$$

Provided $Q((i_1,j_1)\dots(i_m,j_m))$ is defined and it is an inverse image under j^t of a difference $P = P_1 \setminus P_2$ of two algebraic sets in $j^t(\mathcal{H}^{n,k})$, we define the set $Q((i_1,j_1),\dots,(i_m,j_m)(i_{m+1},j_{m+1})$ $i_{m+1} \geq -1, j_{m+1} \geq 0$ as

$$Q((i_1,j_1)\dots(i_m,j_m)(i_{m+1},j_{m+1})) = \{\tilde{X} \in Q((i_1,j_1),\dots,(i_m,j_m)) |$$
$$\dim D(j^t\tilde{X})_o(L_{i_{m+1}}\tilde{X}) - \dim(D(j^t\tilde{X})_o(L_{i_{m+1}}\tilde{X}) \cap T_{(j^t\tilde{X})(0)} P) = j_{m+1}\}$$

Here $j^t X$ is a section of the t-jet bundle defined by X and $D(j^t X)_o$ is its differential at 0 . The symbol $T_{(j^t X)(0)}$ denotes the tangent space. The last set is also a difference of two algebraic sets.

Lemma :

The sets $Q((i_1,j_1) :\dots: (i_m,j_m))$ *are invariant under the action of* $G^{n,k}$.

3. Local models for $k \geq 2n-3$

If $k \geq 2n-3$, then the structure of orbits of $G^{n,k}$ in $\mathcal{H}^{n,k}$ is described by the following

Theorem A

If $k \geq 2n-3$ *, then there is a sequence of* $G^{n,k}$ *invariant, algebraic sets*

(5) $\mathcal{H}^{n,k} \supset Q_{i_1} \supset Q_{i_2} \supset \dots \supset Q_{i_m} \supset Q_s$,

$0 < i_1 < i_2 < \dots < i_m < s$, $s > n$, *such that codim* $Q_i = i$ *and* $Q_{i_j} \setminus Q_s$, $j = 1,\dots, m$, *are submanifolds. The orbits of the group*

$G^{n,k}$ *lying in* $\mathcal{H}^{n,k} \setminus Q_s$ *coincides with orbits of the groups* $G^{n,k,r}$, $r \geq 0$ *(exept of the set* $Q_n \setminus Q_{n+1}$ *in* d) *below).*

a) *If* $k \geq 2n$ *then the sequence* (5) *is of the form*

$$\mathcal{H}^{n,k} \supset Q_s$$

and any germ from $\mathcal{H}^{n,k} \setminus Q_s$ *is* C^∞ *equivalent to*

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, 0, \ldots, 0\right).$$

b) *If* $k = 2n-1$, *then the sequence* (5) *takes the form*

$$\mathcal{H}^{n,k} \supset Q_n \supset Q_{n+1}$$

and germs from $\mathcal{H}^{n,k} \setminus Q_n$ *and* $Q_n \setminus Q_{n+1}$ *are* C^∞ *equivalent to*

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, 0, \ldots, 0\right) \quad \textit{and}$$

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, x_1 \frac{\partial}{\partial x_n}, \ldots, x_n \frac{\partial}{\partial x_n}\right), \quad \textit{respectively} .$$

c) *If* $k = 2n-2$, *then the sequence* (5) *is of the form*

$$\mathcal{H}^{n,k} \supset Q_{n-1} \supset Q_n \supset Q_{n+1} \quad \textit{and any}$$

germ from $\mathcal{H}^{n,k} \setminus Q_{n-1}$, $Q_{n-1} \setminus Q_n$, $Q_n \setminus Q_{n+1}$ *is* C^∞ *equivalent to*

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, 0, \ldots, 0\right),$$

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, x_1 \frac{\partial}{\partial x_n}, x_2 \frac{\partial}{\partial x_n}, \ldots, x_{n-1} \frac{\partial}{\partial x_n}\right) \quad \textit{and}$$

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, (x_1^2 + x_n) \frac{\partial}{\partial x_n}, x_2 \frac{\partial}{\partial x_n}, \ldots, x_{n-1} \frac{\partial}{\partial x_n}\right),$$ *respectively.*

d) *If* $k = 2n-3$, *then the sequence* (5) *is of the form*

$$\mathcal{H}^{n,k} \supset Q_{n-2} \supset Q_n \supset Q_{n+1} \quad \textit{and}$$

germs from $\mathcal{H}^{n,k} \setminus Q_{n-2}$, $Q_{n-2} \setminus Q_n$ *are* C^∞ *equivalent to*

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, 0, \ldots, 0\right) \quad \textit{and}$$

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, x_2 \frac{\partial}{\partial x_n}, \dots, x_{n-1} \frac{\partial}{\partial x_n}\right), \quad \textit{respectively} .$$

There is a real analytic (locally rational) function ϕ *on* $Q_n \setminus Q_{n+1}$ *such that each set* P_λ *of the constant value* λ *of* ϕ *is* C^o *invariant, semialgebraic of codimension* 1 *in* $Q_n \setminus Q_{n+1}$. *All values (excluded* 0*) of this function are regular i.e.* ϕ *gives a foliation on* $Q_n \setminus (Q_{n+1} \cup P_o)$ *of codimension* 1 . *There are sets* P^i *being finite union of leaves* P_λ , $P^1 \subset P^2 \subset P^3 \subset \dots$ *such that if* $\tilde{X} \in Q_n \setminus (Q_{n+1} \sqcup P^r)$, *then* $\tilde{X}$ *is* C^r *equivalent to*

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_{n-1}}, (x_n - (x_1^2 + \lambda x_2^2)) \frac{\partial}{\partial x_n}, \right.$$
$$\left. x_3 \frac{\partial}{\partial x_n}, \dots, x_{n-1} \frac{\partial}{\partial x_n}\right),$$

$$\lambda = \frac{1}{4\phi(X)} .$$

The leaves of $Q_n \setminus (Q_{n+1} \cup \bigsqcup_{j=1}^{\infty} P^j)$ *coincide with* $G^{n,k}$ *orbits.*

<u>Idea of proof</u>

The sets Q_i may be defined as follows

a) $Q_s = \bigsqcup_{j<n} Q(0,j)$

b) $Q_{n+1} = \bigsqcup_{j<n-1} Q(0,j) \sqcup \bigsqcup_{j<n} Q(0,n-1)(-1,j)$

$Q_n = Q_{n+1} \sqcup Q(0,n-1)$

c) $Q_{n+1} = \bigsqcup_{j<n-1} Q(0,j) \sqcup \bigsqcup_{j<n-1} Q(0,n-1)(-1,j) \sqcup$
$\sqcup Q(0,n-1)(0,n-2)(-1,n-1)$

$Q_n = Q_{n+1} \sqcup Q(0,n-1)(0,n-2)$

$Q_{n-1} = Q_n \sqcup Q(0,n-1)$

d) $Q_{n+1} = \bigsqcup_{j<n-1} Q(0,j) \sqcup \bigsqcup_{j<n-2} Q(0,n-1)(-1,j) \sqcup$
$\sqcup \bigsqcup_{j=n-2,n-1} Q(0,n-1)(0,n-3)(-1,j)$

$Q_n = Q_{n+1} \cup Q(0,n-1)(0,n-3)$

$Q_{n-2} = Q_n \cup Q(0,n-1)$.

If $\tilde{X} \in Q_n \setminus Q_{n+1}$ in the case d) then it defines a germ of field of directions $\mathfrak{D}(\tilde{X})$ on the germ of the manifold $(j^t X)^{-1}(j^t(Q(1,2))) \subset R^n$ at $0 \in R^n$. The field $\mathfrak{D}(X)$ has singularity at 0 . By a field of directions we mean a class of equivalent vector fieds with the equivalence relation defined as equality up to multiplication by an invertible function. The invariant $\phi(\tilde{X})$ may be defined as $(tr(Hes\,\mathfrak{D}(\tilde{X})))^2/det(Hes\,\mathfrak{D}(\tilde{X}))$, where $Hes\,\mathfrak{D}(X)$ is the Hesjan at zero of any vector field representing $\mathfrak{D}(X)$. Note that $(tr\ A)^2/det\ A$ is a complete invariant of equivalence classes of 2×2 hyperbolic matrices under the relation of linear changes of coordinates and multiplication by nonzero numbers.

The r-exeptional leaves (which build P^r) correspond to such numbers $\phi(\tilde{X})$ for which the eigenvalues λ_1 , λ_2 of $Hes\,\mathfrak{D}(X)$ fulfil the conditions : $a\lambda_1 + b\lambda_2 = 0$ for integers a,b satisfying $|a| + |b| \leq s(r)$ - compare Hartman [2].

Let $\overset{k}{\underset{1}{\oplus}} \Gamma^\infty(TM)$ be the set of all k-tuples of C^∞ vector fields on a manifold M . Theorem A implies (by Thom's lemma) the following

Theorem A'

Assume $k \geq 2n-3$ *and let* M *be any smooth, boundaryless, n-dimensional manifold. There exists a subset* $\mathcal{U} \subset \overset{k}{\underset{1}{\oplus}} \Gamma^\infty(TM)$, C^∞ *dense and* C^i *open in the Whitney topology* $(i = 0,1,2,2$ *in the cases* a),b),c),d)), *such that for any* $X \in \mathcal{U}$ *the following is satisfied.*

a) *If* $k \geq 2n$, *then* $\tilde{X}_a$ *is* C^∞ *equivalent to the germ in a) of Theorem A for any* $a \in M$

b) If $k = 2n-1$ *, then there is a closed, 0-dimensional submanifold (family of isolated points)* $M^1(X) \subset M$ *such that* $\tilde{X}_a$ *is* C^∞ *equivalent to the first or the second germ of b) theorem A for* $a \in M \setminus M^1(X)$ *or* $a \in M^1(X)$ *, respectively*

c) If $k = 2n-2$*, then there exist closed submanifolds* $M^1(X) \supset M^2(X)$ *of dimensions* 1,0 *respectively, such that* $\tilde{X}_a$ *is* C^∞ *equivalent to the first, second or third model of c) Theorem A, if* $a \in M \setminus M^1(X)$, $a \in M^1(X) \setminus M^2(X)$ *or* $a \in M^2(X)$, *respectively.*

d) If $k = 2n-3$ *, then there exist closed submanifolds* $M^1(X) \supset M^2(X)$ *of dimensions* 2,0 *respectively, such that if* $a \in M \setminus M^1(X)$ *or* $a \in M^1(X) \setminus M^2(X)$ *, then* $\tilde{X}_a$ *is* C^∞ *equivalent to the first or second germ in d) of Theorem A. For any* $r \geq 0$ *there exists a subset* $\mathcal{U}^r \subset \mathcal{U}$, C^∞ *dense and* C^2 *open in* $\overset{k}{\underset{1}{\oplus}} \Gamma^\infty(TM)$ *in the Whitney topology such that for any* $X \in U^r$ *and* $a \in M^2(X)$ *the germ* $\tilde{X}_a$ *is* C^r *equivalent to the third germ of d) Theorem A.*

Proof

The set $\mathcal{U}$ exists by Thom's lemma. For $X \in \mathcal{U}$ we define

$M^1(X) = (j^t X)^{-1}(j^t(Q(0,n-1)))$ in the case b),c),d)

$M^2(X) = (j^t X)^{-1}(j^t(Q(0,n-1)(0,n-2)))$ in the case c)

$M^2(X) = (j^t X)^{-1}(j^t(Q(0,n-1)(0,n-3)))$ in the case d) .

The definitions do not depend on $t \geq 3$, which can be seen from the definition of sets Q(...) (they are inverse images of subsets in jets).

Observe that in the case d) the field of linear spaces span{X(.)} defines a field of directions on $M^1(X)$, which was mentioned in the proof of Theorem A. This field has singularities in $M^2(X)$.

4. The case of $n=3$, $k=2$

If $k < 2n-3$, then the situation is much more complicated. Here, we consider only the germs at 0 of couples of vector fields in R^3 .

Theorem B

There is a sequence of $G^{3,2}$ *invariant, algebraic sets* Q_i $\mathcal{H}^{3,2} \supset Q_1 \supset Q_2 \supset Q_3 \supset Q_4$ *such that codim* $Q_i = i$ *and* $Q_1 \setminus Q_4$, $Q_2 \setminus Q_3$, $Q_3 \setminus Q_4$ *are submanifolds. The germs from* $\mathcal{H}^{3,2} \setminus Q_1$, $Q_1 \setminus Q_2$, $Q_2 \setminus Q_3$ *are* C^∞ *equivalent to*

1° $(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} + x \frac{\partial}{\partial z})$,

2° $(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z})$ *and*

3° $(\frac{\partial}{\partial x} , x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z})$, *respectively .*

4° *The set* $Q_3 \setminus Q_4$ *is a disjoint union* $Q' \cup Q''$ *of two manifolds* $Q' \subset Q_2$ *and* $Q'' \cap Q_2 = \emptyset$ *and*

a) *If* $(\tilde{X} , \tilde{Y}) \in Q'$ *, then it is* C^∞ *equivalent to the germ of the couple*

$$(\frac{\partial}{\partial x} , x \frac{\partial}{\partial y} + (x^2+(z+y^2)) \psi(x,y,z)) \frac{\partial}{\partial z}) , \quad \textit{where}$$

$\psi : R^3 \to R$ *is a real function and* $\psi(0) \neq 0$ *is uniquely defined by* $(\tilde{X} , \tilde{Y})$ *. Moreover* $\psi(0)(\tilde{X} , \tilde{Y})$ *is a (locally rational) submersion and gives a foliation on* Q' *of codimension* 1*, with* $G^{3,2}$ *invariant, semialgebraic leaves*

b) *If* $(\tilde{X} , \tilde{Y}) \in Q''$ *, then it is* C^∞ *equivalent to the germ of the couple*

$$(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} + (x^3+(z+\sigma\phi(y))y^2) \psi(y,z)) \frac{\partial}{\partial z}) , \quad \textit{where}$$

ϕ *,* ψ *are functions* $\phi : R \to R$, $\psi : R^2 \to R$ *and* $\phi(0) = \psi(0) = 1$.

The number $\sigma = \pm 1$ *is uniquely defined by* $(\tilde{X}, \tilde{Y})$ *and gives a partition of* Q'' *into two semialgebraic, invariant sets.*

Idea of proof

The sets Q_i, $i = 1, \ldots, 4$ may be defined as follows :

$$Q_4 = Q(0,0) \cup Q(0,1)(0,0) \cup Q(1,1) \cup (Q(0,1)(1,1) \cup Q(2,2)) \cup Q(0,1)(-1,1) \cup Q(1,2)(-1,0) \cup \bigcup_{s=0}^{2} (Q(0,1)(1,1)(-1,s) \cup Q(2,2)(-1,s))$$

$$Q_3 = Q_4 \cup Q(0,1)(1,1) \cup (Q(0,2) \cap Q(2,2))$$

$$Q_2 = Q_3 \cup Q(0,1)$$

$$Q_1 = Q_2 \cup Q(1,2)$$

and $Q' = Q(0,1)(1,1) \setminus Q_4$, $\quad Q'' = (Q(0,2) \cap Q(2,2)) \setminus Q_4$.

Theorem B implies the following

Theorem B'

Let M *be a smooth, boundaryless, 3-dimensional manifold. There exists a* C^3 *open,* C^∞ *dense (in the Whitney topology) subset* $\mathcal{U} \subset \Gamma^\infty(TM) \oplus \Gamma^\infty(TM)$ *, which satisfies the following conditions. For any* $(X, Y) \in \mathcal{U}$ *there are closed submanifolds* M^1, M^2, M^3, M^4 *of dimensions* 2 , 1 , 0 , 0 *respectively,* $M^1 \supset M^2 \supset M^3$, $M^1 \supset M^4$, $M^2 \cap M^4 = \emptyset$ *such that if* $a \in M \setminus M^1$, $a \in M^1 \setminus (M^2 \cup M^4)$, $a \in M^2 \setminus M^3$, $a \in M^3$, $a \in M^4$, *then the germ of* (X, Y) *at "a" is* C^∞ ***equivalent*** *to the germ of the form* 1° , 2° , 3° , 4°a) , 4°b), *respectively.*

Proof

The set $\mathcal{U}$ exists by Thom's transversality lemma. For $(X, Y) \in \mathcal{U}$ we define

$$M^1 = M^1(X,Y) = (j^t(X,Y))^{-1}(j^t(Q(1,2))) ,$$
$$M^2 = M^2(X,Y) = (j^t(X,Y))^{-1}(j^t(Q(0,1))) ,$$
$$M^3 = M^3(X,Y) = (j^t(X,Y))^{-1}(j^t(Q(0,1)(1,1))) \quad \text{and}$$
$$M^4 = M^4(X,Y) = (j^t(X,Y))^{-1}(j^t(Q(0,2)\cap Q(2,2)) .$$

Observe that, similarly to the case $k = 2n-3$ of Theorem A, the couple $(X,Y) \in \mathcal{U}$ defines by span $\{X,Y, [X,Y]\}(.)$ a field of directions $\mathfrak{D}(X,Y)$ on the manifold M^1. The germs of (X,Y) out of singularities of $\mathfrak{D}(X,Y)$ have uniquely defined models. This fact was observed, first, by J. Martinet [4] for the case of a generic 1-form on 3-manifold. In his case sets like M^1 and M^4 appear, only. A study of the field of directions in the above and more genral situations can be found in [1] .

The parameter invariant $\psi(0)$ of the germ of (X , Y) at a point $a \in M^3(X,Y)$ (singular point of $\mathfrak{D}(X,Y)$) can be defined as $(\mathrm{tr}\ (\mathrm{Hes}\ \mathfrak{D}(X,Y)))^2/\det\ (\mathrm{Hes}\ \mathfrak{D}(X,Y))$ i.e. analogously as in the idea of proof of Theorem A. Since $\psi(0) \neq 0$, then the case of resonance $\mathrm{tr}\ (\mathrm{Hes}\,\mathfrak{D}(X,Y)) = 0$ is here excluded. Contrary to this, for the second kind singularities of $\mathfrak{D}(X,Y)$, those of $M^4(X,Y)$, $\mathrm{Hes}\ \mathfrak{D}(X,Y) = \begin{pmatrix} 0 , & 2\sigma \\ -6 , & 0 \end{pmatrix}$ (accounted for the model) .

Thus we have resonance singularities : hyperbolic $(\sigma = -1)$ and eliptic $(\sigma = +1)$. The problem stated by Martinet, whether these singularities have unique models, remains open .

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