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RELEVANCE OF COMPUTER EXPERIMENTS TO THE PURE MATHEMATICS OF INTEGRABLE

AND ERGODIC DYNAMICAL SYSTEMS [*]

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The computer can no longer be regarded solely as an

amusing toy for the mathematically lazy or inept. In this

paper, we describe four, simple, computer experiments which

have been, or promise to be, of major significance to purely

analytic studies of integrable and ergodic dynamical systems.

I. INTRODUCTION

Given a specified dynamical'system whose integrable or ergodic

character is in question, a computer can be used to provide an almost

definitive answer. Certainly, computer studies cannot yield a rigorous

proof, but they can exhibit overwhelming circumstantial evidence. To use

an analogy from everyday life, we easily recognize a duck by its unique

vocal call, its web feet, its oily feathers, its broad bill, its

characteristic tail, and its ability to swim; seldom do we need to

anatomically "prove" the given fowl is a duck by dissecting it.

Similarly, a computer can be programmed to "see" the distinctive

characteristics of integrable or ergodic systems. Granted the computer

can be "fooled" but not frequently, provided the programmer is careful.
In the following brief paper, we describe several specific cases in
which the computer has been used to investigate interesting problems
which had stoutly and successfully resisted analytic attack, at least
until the computer results were known.

II. THE TODA LATTICE

The Toda lattice[1] is a one-dimensional, conservative, classical
dynamical system governed by the Hamiltonian

$$H = \sum_{n=1}^{N} \left[(p_n^2/2) + e^{-(q_{n+1} - q_n)} - 1 \right] , \tag{1}$$

where we here assume periodic boundary conditions $q_{n+N} \equiv q_n$ and
where we subtract unity from each exponential in order that $H = 0$
when all the q's and p's equal zero. Without entering into the details
here, we remark only that this system has been shown to be of both
mathematical[2] and physical[3] interest, but until recently, despite the
known existance of special, particular solutions, the generic character
of the general solution (integrable or ergodic) was unknown. As a consequence,
the present author in collaboration with S. D. Stoddard and J. S. Turner
investigated this problem numerically.[4] The first non-trivial case is
$N = 3$; and it is of particular interest since for it, after neglecting the
pure translation degree of freedom, Hamiltonian (1) can be reduced to
the two degrees of freedom Hamiltonian

$$H = (1/2) (p_1^2 + p_2^2) + (1/24) \left\{ e^{(2q_2 + 2\sqrt{3}\, q_1)} + e^{(2q_2 - 2\sqrt{3}\, q_1)} \right.$$
$$\left. + e^{-4q_2} \right\} - (1/8) , \tag{2}$$

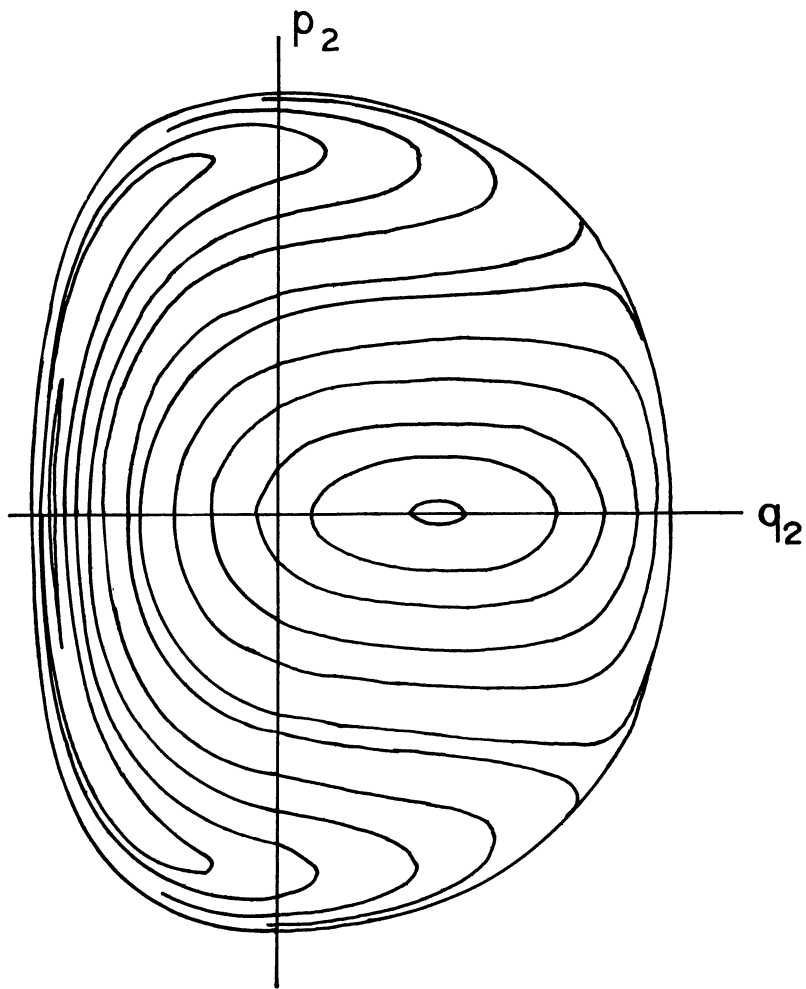which latter can be studied via the well-known Poincare surface of

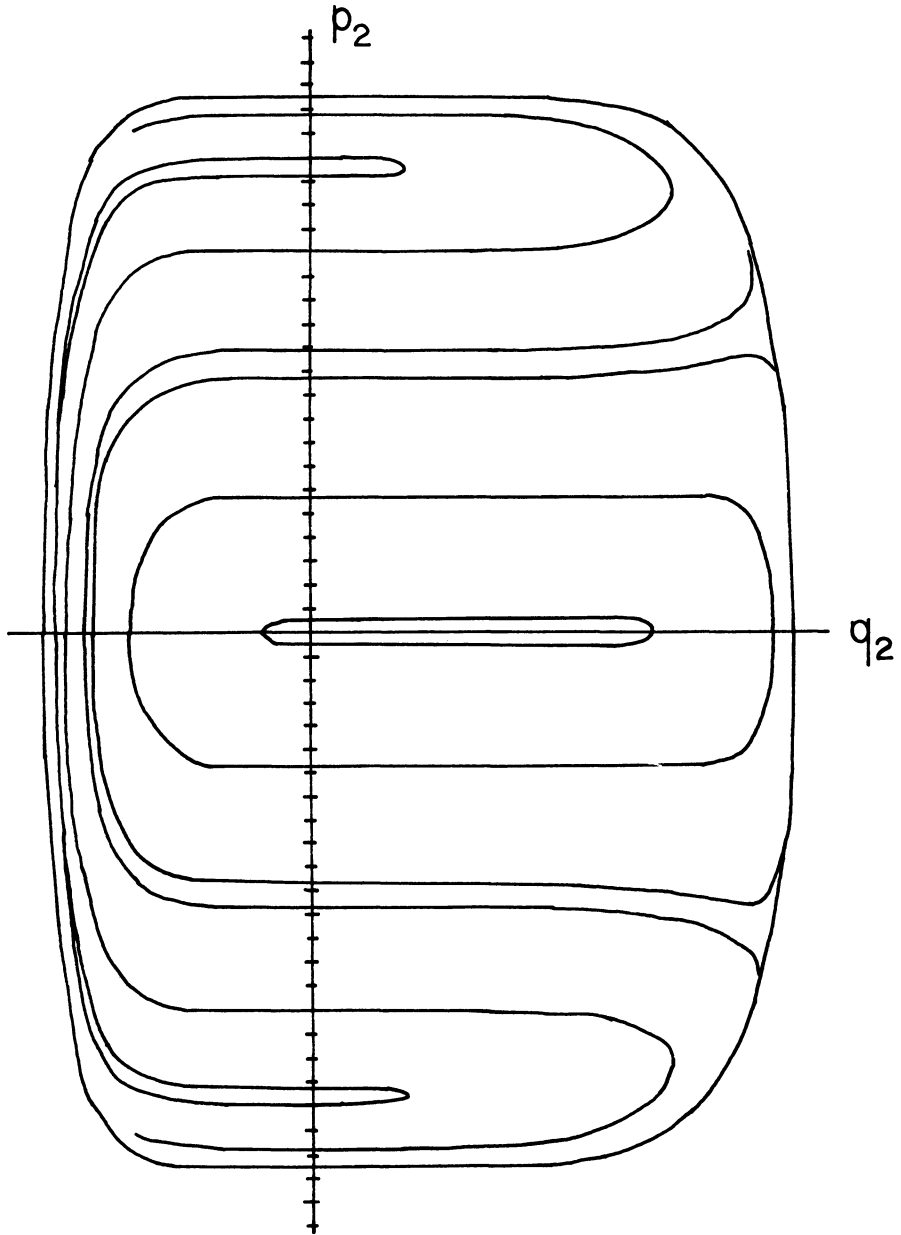Fig. 1. A surface of section for Hamiltonian (2) at E = 1.

Fig. 2. A surface of section for Hamiltonian (2) at E = 256.

section technique.[5] Since Siegal[6] has shown that generically

Hamiltonian systems are not integrable, we anticipated non-integrability

for Hamiltonian (2).

We were thus quite surprised when the computer revealed the

numerically integrated surface of section shown in Fig. 1 for energy

E = 1, and we were shocked by the surface of section shown in Fig. 2 at

energy E = 256. Our incredulity only increased when similar surfaces

of section were obtained at energies E = 1024 and 56,000. All these

surfaces of section, which exhibited curves everywhere even at extremely

large nonlinearity, clearly indicate that Hamiltonian (2) is integrable.

As an additional test, we computed a formal, truncated series expression

for a constant of the motion independent of Hamiltonian (2) using a

well-known modification of the standard Birkhoff method,[7] and we numerically

integrated the equations of motion to verify the validity of this formal

constant. We then used this constant of the motion (in conjunction with

Hamiltonian (2)) to analytically compute the numerically integrated curves

of Fig. 1, obtaining excellent agreement. We then sought and found a host

of periodic orbits for Hamiltonian (2) and established[8] that they all lay

on integral surfaces of periodic orbits, again indicating integrability.

Finally, we integrated a host of initially close orbit-pairs and observed

that the two members of all orbit-pairs separated from each other linearly

with time, a well-known characteristic of integrable systems. Lastly, we

investigated the case N = 6 for Hamiltonian (1); in particular, we

numerically integrated initially close orbit-pairs again obtaining in every

case a separation which grew linearly with time. A typical example is

shown in Fig. 3.

As we began an analytic search for Hamiltonian (1) constants of the motion, we simultaneously wrote up our computer results[4] and sent a preprint to M. Henon, among others. By return mail, Henon sent us analytic expressions[9] for N constants of the motion (in involution) thus rigorously proving integrability. Only a few days after the arrival of Henon's letter, H. Flaschka dropped by Georgia Tech, and, upon learning of Henon's results, quickly realized that he had already, without previously noticing it, derived[10] Henon's results via an application of the inverse scattering technique to the Toda lattice. Thus, independent rigorous proofs[11] of integrability for Hamiltonian (1) by Henon and Flaschka were presented in back to back Phys. Rev. B articles[9,10]. Subsequently, Kac and van Moerbeke[11] and, independently, Date and Tanaka devised an analytic technique for obtaining the general solution to the equations of motion generated by Hamiltonian (1). Hence, a computer investigation initiated a chain of remarkable achievements in pure mathematics.

III. THE HENON-HEILES HAMILTONIAN SYSTEM

If we expand the exponentials of the integrable Hamiltonian (2) in a power series and retain terms only through cubic order, we obtain the Hamiltonian

$$H = (1/2) (p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - q_2^3/3 \qquad (3)$$

studied much earlier by Henon and Heiles,[12] who sought to determine whether or not a third analytic integral (in addition to the Hamiltonian
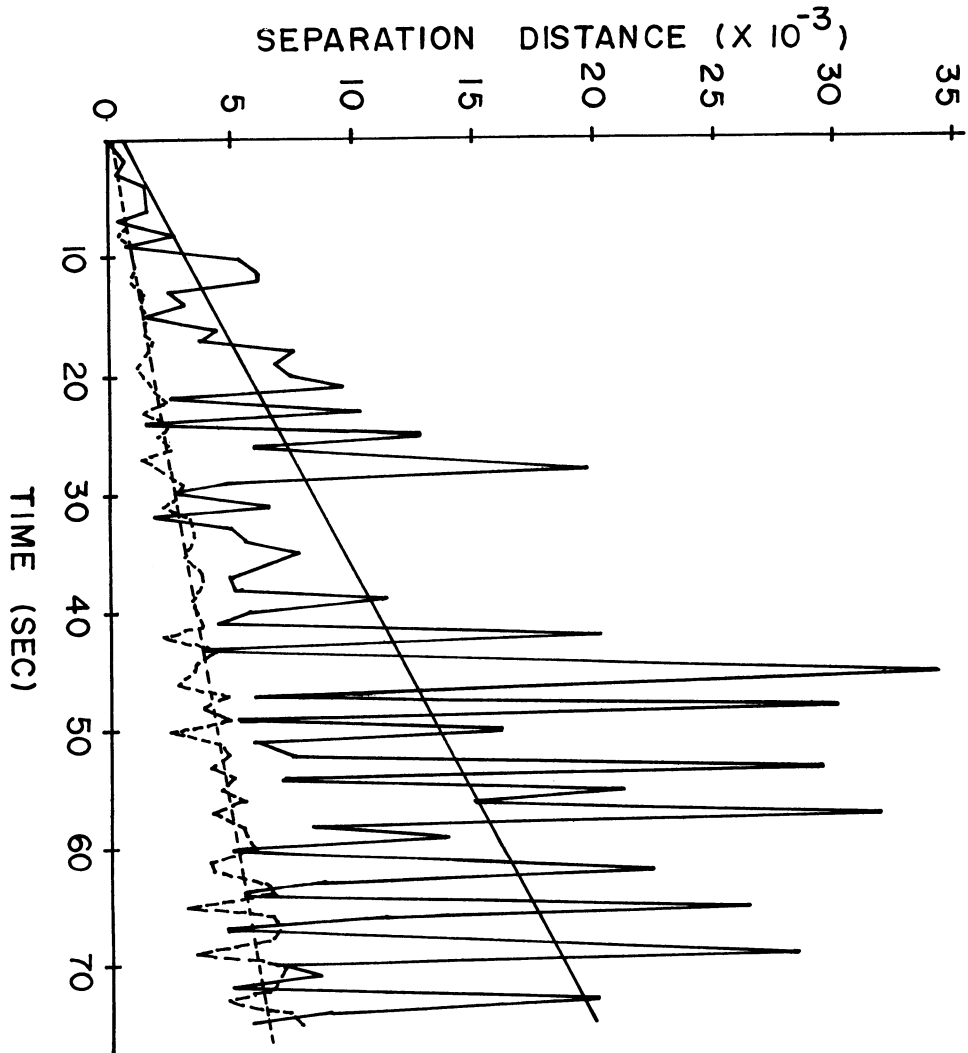
Fig. 3. A graph of separation distance versus time for an orbit-pair of Hamiltonian (1) with N = 6. Here D_q (dashed curve) is position space separation distance between two initially close orbits in phase space. D_p (solid Curve) is momentum space separation distance for the same orbit-pair. Each curve has an average linear growth (least squares straight line fit) with superimposed oscillations. Were this an ergodic, K-system, the orbits would have separated by many powers of 10 during the time interval shown.

and the z-component of angular monentum) exists for a star moving in
an axially-symmetric galactic potential. As in the previous section, one
here again seeks to determine whether or not Hamiltonian (3) is integrable.

In Fig. 4, we show a (q_2, p_2)- plane, Poincare surface of section
for Hamiltonian (3) at energy E = 1/12. To the computer accuracy used here,
curves appear to be everywhere dense, indicating integrability. However,
in Fig. 5 at E = 1/8, Hamiltonian (3) is seen to have a divided phase
space, part of which is still filled with integrable orbits but part
contains "ergodic" trajectories. The "random" splatter of dots in Fig. 5
was generated by numerical integration of a single "ergodic" trajectory.
In the integrable region, initially close orbits separate linearly with time
while in the "ergodic" region they separate exponentially as is character-
istic of C-systems[13], known to be ergodic and mixing. Finally in Fig. 6,
at energy E = 1/6, one sees that most of phase space is filled with wild
and erratic trajectories.

The Henon-Heiles work not only provides a highly unexpected answer
to the original astronomical question, but it also is now regarded as a
classic example[14] of the transition from integrable to stochastic behav-
ior. It thus serves as a non-rigorous but highly illuminating link
between the near-integrable behavior predicted by the rigorous KAM
(Kolmogorov-Arnold-Moser) theorem[15] and the rigorous K-systems[13] behavior
of Sinai's hard sphere gas.[16] The recent, as yet unpublished, work of
Benettin and Strelcyn[17] provides further numerical insight into this link
through an investigation of the transition from rigorous integrable behavior
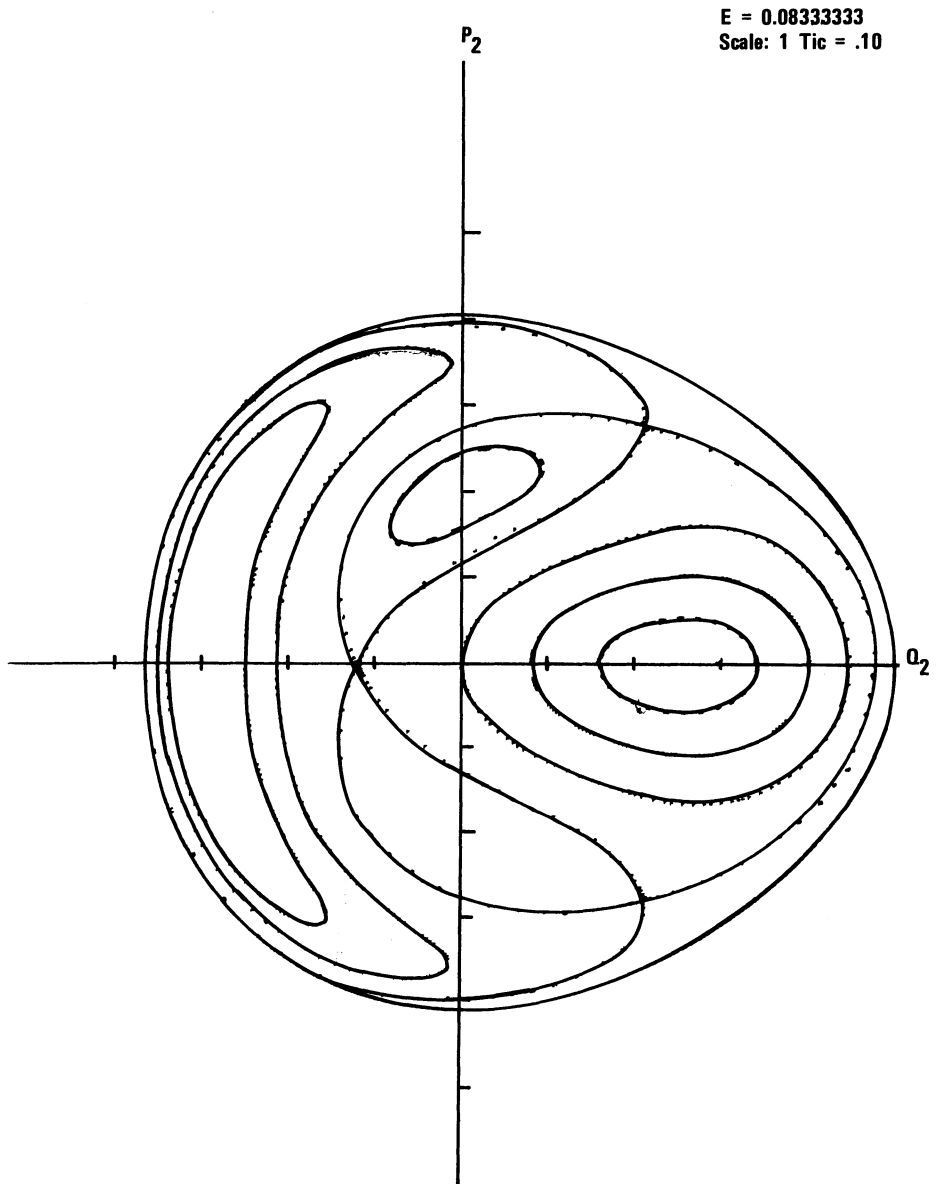
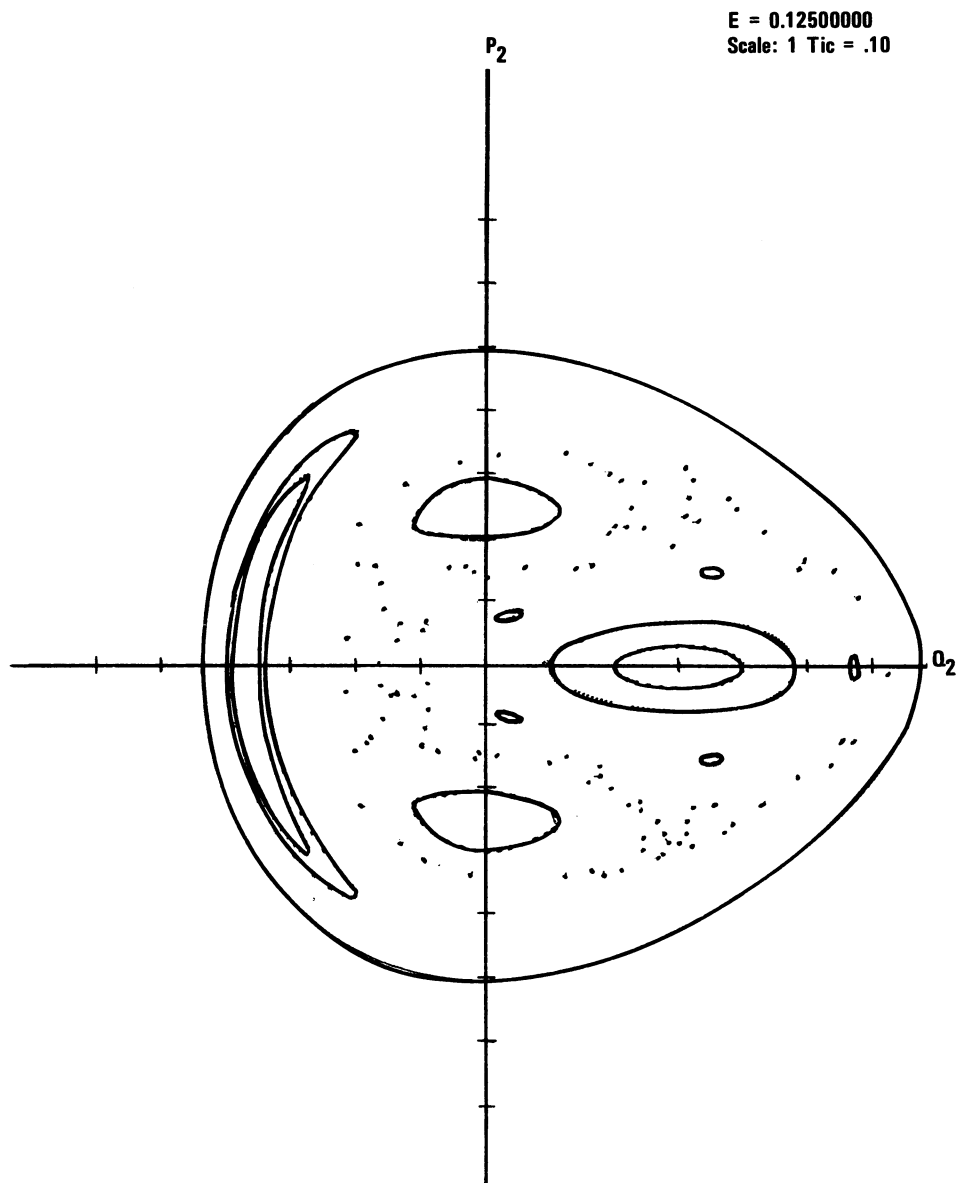Fig. 4. A surface of section for Hamiltonian (3) at E = 1/12.

E = 0.12500000
Scale: 1 Tic = .10

P_2

Q_2

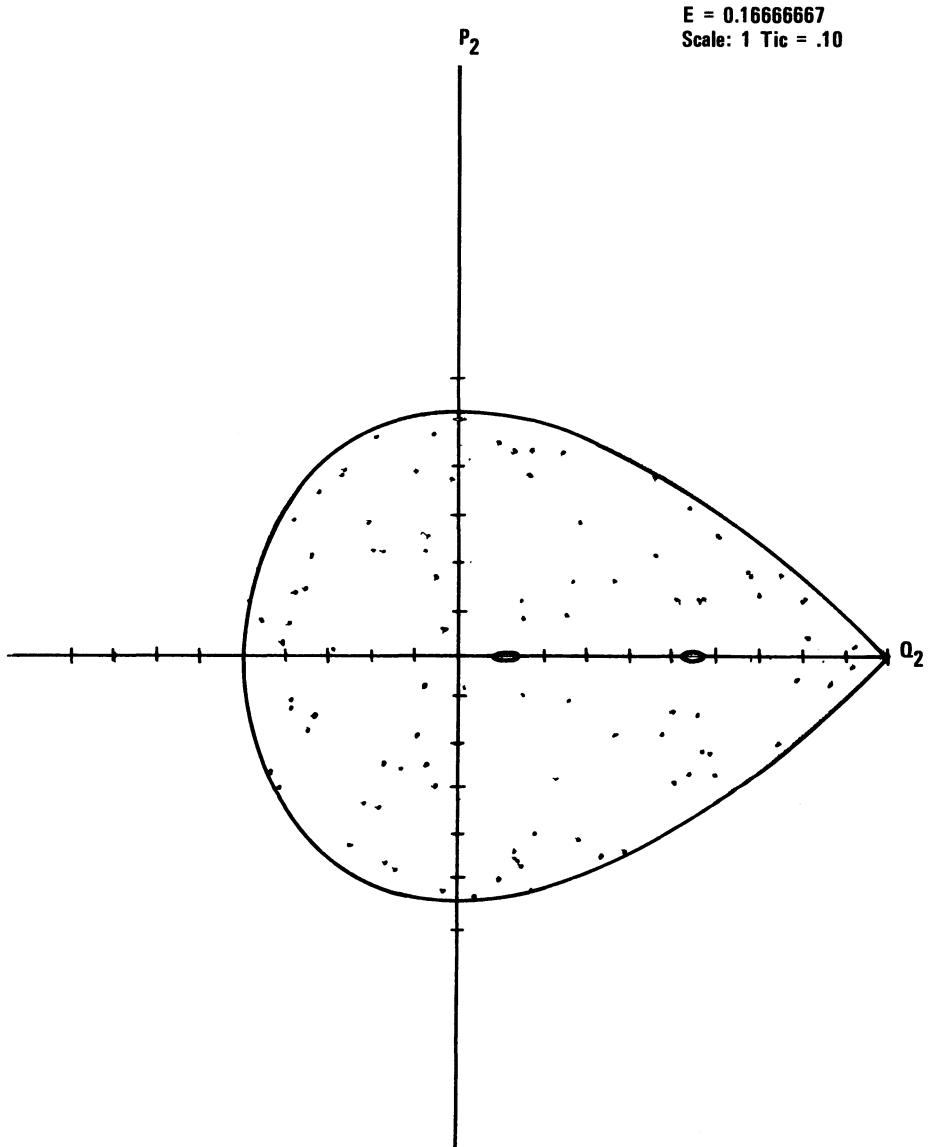Fig. 5. A surface of section for Hamiltonian (3) at E = 1/8.

Fig. 6. A surface of section for Hamiltonian (3) at E = 1/6.

to rigorous ergodic, K-system behavior for the motion of a billard confined to a plane, oval region as the boundary changes from a circle to a stadion (two parallel straight line segments joined at their ends by two semi-circles). This paper numerically computes surfaces of section as well as Lyapunov characteristic numbers for the flow. Thence it computes the Kolmogorov-Sinai entropy for any ergodic components which may exist in the flow. In the KAM, near-integrable parameter range (nearly circular boundary) for this flow, many small, disjoint ergodic components are detected. As the boundary approaches the K-system stadion shape, these disjoint components sequentially merge into the single, completely ergodic component of the stadion. This Benettin-Strelcyn work uses a remarkable blend of pure analysis and computer experiments to obtain results of interest for both physics and pure mathematics.

IV. A MAPPING WITH A STRANGE ATTRACTOR

As an example of computer experiments on strictly chaotic systems, we describe the numerical study of a mapping possessing a strange attractor which has been recently published by M. Henon.[18] This work is relevant for weather prediction,[19] turbulence theory,[20] and pure mathematics.[21]

First using "practical" and theoretical arguments, Henon sought and found a non-area-preserving, simple, quadratic mapping which appeared a likely candidate for possessing a strange attractor. The specific mapping he chose to investigate may be written

$$x_{n+1} = y_n + 1 - a x_n^2, \qquad (4a)$$

$$y_{n+1} = b x_n, \qquad (4b)$$

where a and b are constants. Henon shows that this mapping is, in fact,
the most general quadratic mapping having a constant Jacobian (here equal
to (-b)). If this mapping possesses a strange attractor, then one can
easily and accurately numerically investigate the "flow" for long "time"
intervals and one, hopefully, can develop a purely analytic theory for
this model since it would appear to be much simpler to treat than the
earlier more complicated models.

For his numerical study, Henon set a = 1.4 and b = 0.3 for reasons
which are set forth in this paper. Figure 7 shows the general appearance
of the attractor using 10^4 mapping iterates of Eq. (4a, b) starting from
an initial mapping point "on" the attractor. Aside from possible initial
transient behavior, one obtains this same figure for any initial mapping
point "trapped" by the attractor. The attractor in Fig. 7 appears to
consist of a set of parallel "curves" having a certain "thickness". Upon
magnification, the small square shown in Fig. 7 becomes Fig. 8, where the
number of mapping iterates is now 10^5. Enlarging the square in Fig. 8
yields Fig. 9 which uses 10^6 mapping iterates. These figures strongly
suggest that the attractor being graphed here is indeed a strange attractor,
i.e., a product of a one-dimensional manifold by a Cantor set. Although
these non-rigorous numerical results were initially greeted with skepticism
by the relevant mathematical community, recent additional computer results
and further (still non-rigorous) mathematical analysis have convinced all
but the most devout mathematical skeptics of the significance of Henon's
model. His computer study thus provides a significant model for future
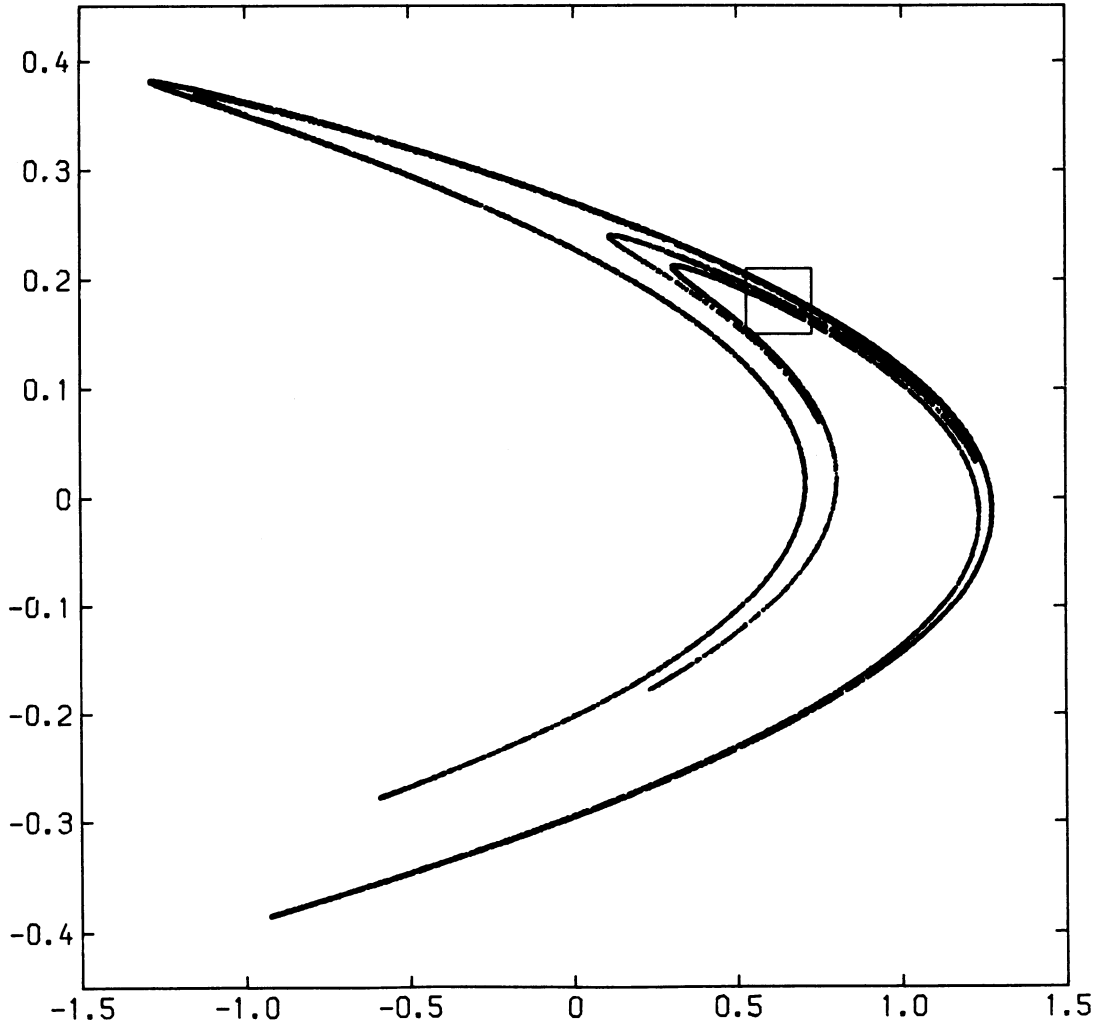purely analytic study.

Fig. 7. 10,000 iterates of the mapping in Eq. (4a, b) starting from the point $x_o = 0.63135448$ and $y_o = 0.18940634$.
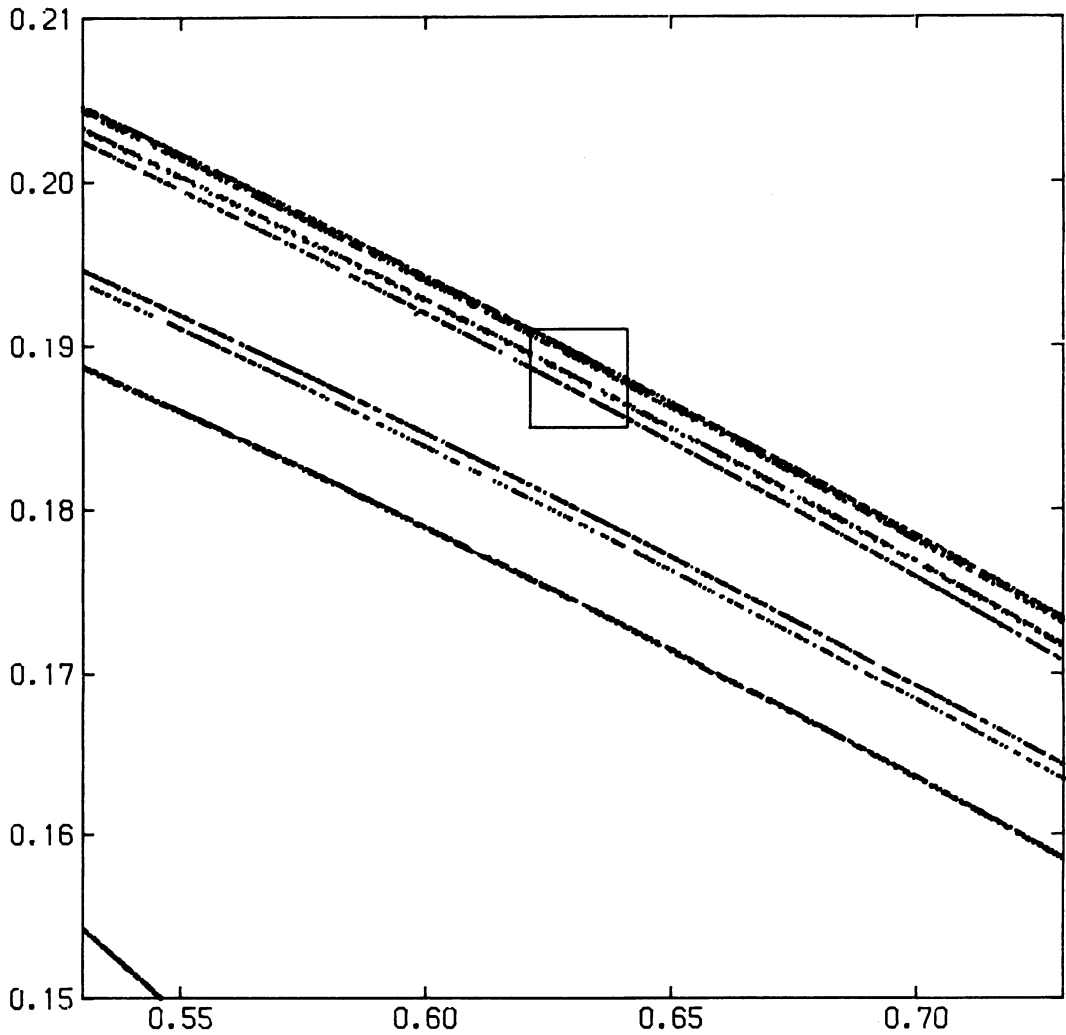
Fig. 8. A magnified view of the small square shown in Fig. 7.
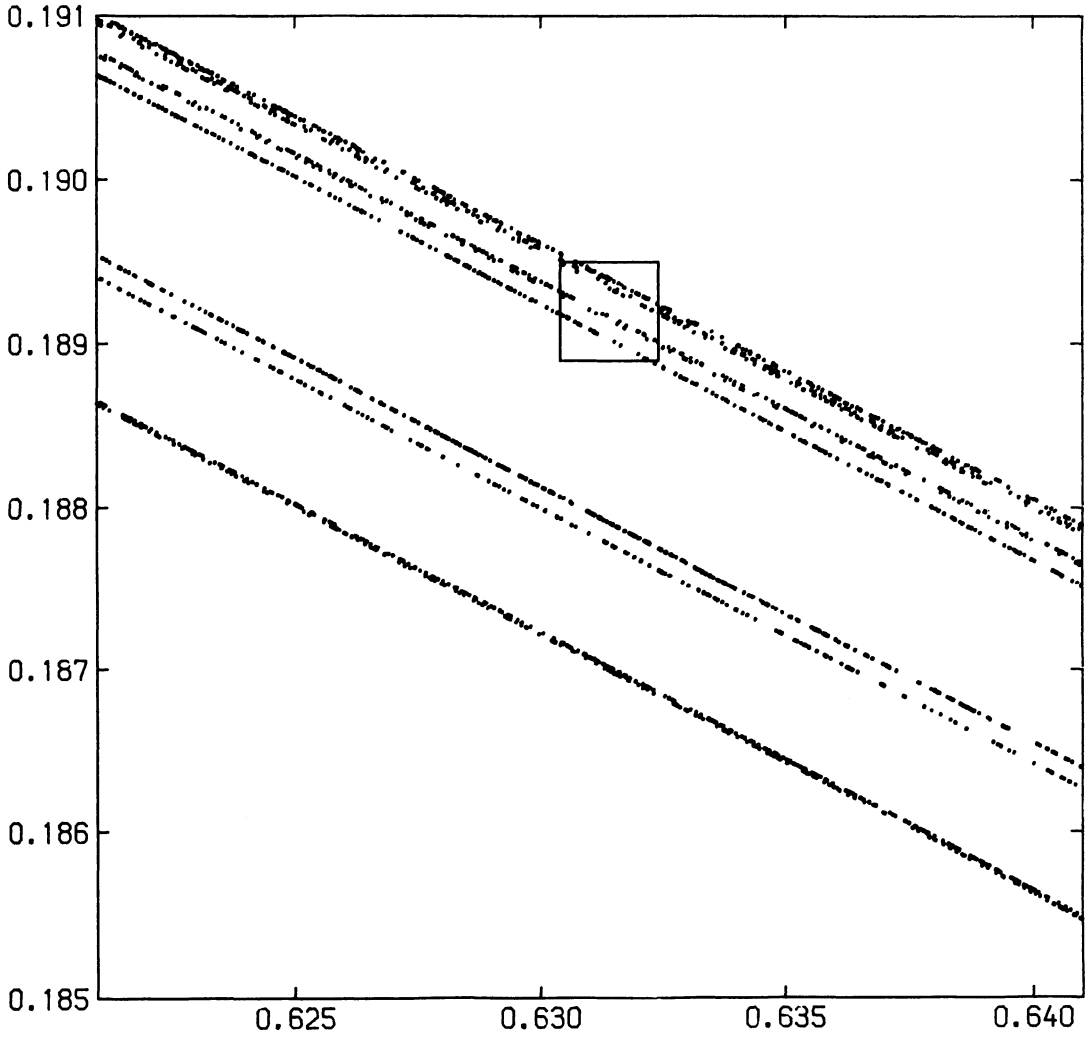
Fig. 9. An enlargement of the square shown in Fig. 8. Note the
structural similarity between Figs. 8 and 9. Further
enlargements reveal this same structure.

V. CONCLUDING REMARKS

In this paper we have discussed four mathematically significant computer experiments. By presenting this article in the proceedings of a conference dedicated primarily to pure mathematics, we hope to increase the flow of information between physical scientists performing numerical experiments and mathematicians developing rigorous theories. Finally, as a further device for increasing information flow, (to repeat an announcement made verbally at this conference), the present author has initiated a free, nonlinear science abstract service with the first, September, 1977, issue now available on request. This service collects, collates, and distributes a list of preprint and reprint abstracts covering work in all areas of nonlinear science giving titles, author's name and address, abstract, and proposed (or actual) publication journal. Please send your abstracts and/or requests to be placed on the mailing list to the present author at the by-line address.

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