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Jacob Feldman*

My purpose in this discussion is to give a few examples of ideas and results in the ergodic theory of single transformations which have natural and interesting generalizations to actions of continuous groups. Throughout, all groups will be assumed locally compact and second countable, and we will use left-invariant Haar measure. By an <u>action</u> of such a group G will always be meant a jointly measurable map $\phi: G \times X \to X$, where $(X,\mathcal{A},\mu)$ is a Lebesgue probability space and each ϕ_g is a homomorphism from G to 1-1 invertible μ-preserving transformations of X. Sometimes $\phi_g(x)$ will be called gx. The action will be called <u>free</u> if $g \neq e \Rightarrow gx \neq x$, all x. $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{T}$ will denote respectively the integers, real line, and circle. Many of the results on continuous groups which are described here are fairly new, and not yet published.

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I. Orbit-equivalence

Two actions ϕ, ϕ' of G, G' will be called orbit-equivalent if there is an a.e. defined and measure-preserving map $\theta: X \to X'$ carrying ϕ-orbits onto ϕ'-orbits. A striking fact, discovered by Dye [4], and later by Vershik and Belinskaya,[21] and [1], is that any two strictly ergodic actions of $\mathbb{Z}$ are orbit-equivalent. Dye called an action of a discrete group hyperfinite if it is orbit-equivalent to an action of $\mathbb{Z}$. I prefer to use the term approximately finite, abbreviated AF. Dye also showed, in [5], that any action of a discrete abelian group is AF. Recently Connes and Krieger, in [2] and [13], obtained the same conclusion for any free ergodic action of a discrete solvable group. An interesting and difficult open question is whether every action (say, free and ergodic) of a discrete amenable group is AF. Another interesting problem is whether every action (say, ergodic) of a discrete group is equivalent to a free action.

Now we permit G to be nondiscrete. It would in some ways be more natural to consider orbit-equivalences which carry over not the measure, but only the measure class, i.e. $E \subset X'$ is null $\Leftrightarrow \phi^{-1}(E)$ is null. However, for ease of exposition, I'll stay with the measure-preserving case. The main fact is that continuous groups may be reduced to discrete groups, in a manner which I will now describe. Define a lacunary section for an action to be a set $E \subset X$ such that there is a neighborhood U of e in G for which the map $(g,x) \mapsto gx$ is 1-1 on $U \times E$.

Theorem (Forrest, [9]): Every free action has a lacunary section.

In [6], Feldman, Hahn, and Moore managed to remove the freeness assumption. A consequence, using [6] and the results of [7], is the following.

Theorem. Every group action ϕ with uncountable orbits is orbit-equivalent to a product action $\phi' \times \phi''$, where ϕ' is an action of a discrete group and ϕ'' is a transitive action of $\mathbb{T}$.

Because of this, approximate finiteness has a natural generalization to continuous groups. An action ϕ will be called AF if, in the above representation as $\phi' \times \phi''$, ϕ' is AF in the discrete sense; the representation is not unique, but the property is independent of the representation. Then one has (see [6], [20]):

1. ϕ is AF $\Leftrightarrow$ ϕ is orbit-equivalent to an action of $\mathbb{R}$.
2. Any two AF strictly ergodic actions with uncountable orbits are orbit-equivalent.
3. Any action of an abelian group is AF.
4. Any free ergodic action of a solvable group is AF (Series, [20]).

(In connection with 1 and 2, I am grateful for U. Krengel for pointing out that in order to get the orbit-equivalence to be measure preserving, and not just measure class preserving, one must supplement [6] with a variant of the arguments in [12].)

The corresponding amenable group question is still there, but it has no new content, because a connected amenable group differs in only a minor way from a solvable group.

II. Tower building and tiling

A subset $A \subset G$ (which for convenience we will take to be open with compact closure) will be called a Rokhlin set if for any free action of G and any $\varepsilon > 0$ there exists $E \subset X$ with $\mu(E) > 1 - \varepsilon$ and $(g,x) \mapsto gx$ on $U \times E$. Then Rokhlin's Lemma (also proven by Kakutani) says that $\{0,\ldots,n\}$ is a Rokhlin set in $\mathbb{Z}$. The result extends to intervals in $\mathbb{Z}^n$ (Conze [3], Katznelson and Weiss [10]); and also in $\mathbb{R}^n$, a much harder result (Lind [14]). For more general groups, we must see what to substitute for intervals.

Definition: For compact $K \subset G$ and $\delta > 0$ an open set A with compact closure in G will be called (K,δ)-invariant if $\frac{|K^{-1}A \cap A|}{|A|} < \delta$ (where $|\cdot|$ denotes Haar measure). Then one of the equivalent definitions of amenability for G is the generalization of the Vǿllner condition: for any compact K and $\delta > 0$. $\exists\, A \supset K$ with A (K,δ)-invariant. Following [17] and [20], G will be called a Rokhlin group if for any compact K and δ the above A may be taken a Rokhlin set. Alternatively: define a sequence $\mathcal{J} = \{A_n\}$ to be a summing sequence if each A_n is open with compact closure, $A_1 \subset A_2 \subset \ldots,\ \bigcup_n A_n = G$, and $\frac{|KA_n \cap A_n|}{|A_n|} \to 0$ for each compact K. Then amenability of G is equivalent to the existence of a summing sequence, and G is a Rokhlin group if and only if there is a summing sequence consisting of Rokhlin sets. (At this point it should be remarked that every group has a free, strictly ergodic action; see Example 1 of V.)

It may be deduced from the work of Krieger and Connes [13], [2] that any discrete solvable G is a Rokhlin group. (Recall that any solvable group is amenable.) This is also shown, more transparently, in Ornstein-Weiss [17]. Going over to the continuous case introduces nontrivial difficulties; but it has been shown by Series [20] that solvable groups and almost connected amenable groups are Rokhlin groups.

What sort of sets in G can be Rokhlin sets? If G is discrete, then any Rokhlin set A must tile G, in the sense that there is a set $C \subset G$ for which the map $(a,s) \mapsto as$ is 1-1 on $A \times C$, and $AC = G$; and on the other hand, if G is a discrete Rokhlin group, then any A which tiles G is a Rokhlin set. This is shown in [16]. It is not difficult to get the same result for continuous G, provided we change "$AC = G$" to "$AC \sim G$ has measure 0" in the definition of tiling set. Almost any reasonable-looking definition of "tiling set" will turn out to be equivalent to this. An intriguing question is whether there is some algebraic condition on an amenable group G (such as: the existence of arbitrarily large, arbitrarily left-invariant tiling sets) which is equivalent to its being a Rokhlin group.

Definition. Say G has tiling if there is a summing sequence of tiling sets; thus any Rokhlin group has tiling.

III. Entropy

The entropy of a $\mathbb{Z}$ action, introduced by Kolmogorov and Sinai, has proven to be an extremely important tool in ergodic theory. It extends in a straightforward way to $\mathbb{Z}^n$ actions; this is carried out in [3] and [17]. By use of "large, almost invariant" sets in the role of intervals, it may be generalized to actions of arbitrary discrete amenable groups; this is carried out in Kieffer's paper [11]. For actions of $\mathbb{R}$, the definition of the entropy $h(\phi,\mathcal{P})$ for a finite partition $\mathcal{P}$ is usually given as $h(\phi,\mathcal{P}) = H(\phi_1|\bar{\mathcal{P}})$, where $H(\phi,|\bar{\mathcal{P}})$ is the usual entropy of the transformation $\phi_1|\bar{\mathcal{P}}$, and $\bar{\mathcal{P}}$ is the smallest ϕ-invariant σ-field containing $\mathcal{P}$. This has the weakness that it depends strongly on the existence of discrete subgroups, and I want to propose a way around this.

For a $\mathbb{Z}$-action, entropy may be alternately defined by the formula

$$h(\phi, \ ) = \inf\{e > 0 \ \text{ for every } \ \varepsilon > 0 \ \exists N \text{ s.t. for } \ n > N \ \text{ there is a set } \mathcal{B} \text{ of } 2^{ne} \text{ atoms of } \bigvee_{j=0}^{n} \phi_{-j}\mathcal{P} \text{ with } \mu(\bigcup_{B\in\mathcal{B}}) > 1 - \varepsilon\}.$$

For an action of a discrete amenable group, one may replace "... $\exists N$ s.t. if $n > N$..." by "... $\exists$ finite $K \subset G$ and $\delta > 0$ s.t. if $A \supset K$ and is (K,δ)-invariant...", and "...2^{ne} atoms of $\bigvee_{j=0}^{n} \phi_j^{-1}\mathcal{P}$..." by "...$2^{|A|e}$ atoms of $\mathcal{P}_A$...", where $\mathcal{P}_A$ denotes $\bigvee_{g\in A} \phi_g^{-1}\mathcal{P}$. At this point it should be remarked that I do <u>not</u> want to assume ergodicity; but if ϕ is decomposed into its ergodic components ϕ^γ, then

$$h(\phi,\mathcal{P}) = \text{ess sup}_\gamma\ h(\phi^\gamma,\mathcal{P}^\gamma),$$

where $\mathcal{P}^\gamma$ is the trace of $\mathcal{P}$ on the γth ergodic component. The above definition may be shown to be the same as Kieffer's, modulo the "ess sup" remark.

Now, instead of covering X by atoms, let us cover it by $\mathcal{P}_A$-measurable sets of small $d_A^{\mathcal{P}}$ diameter, where

$$d_A^{\mathcal{P}}(x,y) = \frac{|\{g \in A\colon\ gx \text{ and } gy \text{ have different } \mathcal{P}\text{-names}\}|}{|A|}:$$

the generalization of the Hamming distance. If a $\mathcal{P}_A$-measurable subset of X has $d_A^{\mathcal{P}}$-diameter $< \alpha$, then it contains no more than $\binom{|A|}{|A|\alpha}p^{|A|\alpha}$ atoms, where p is the cardinality of $\mathcal{P}$. Define $h_\alpha(\phi,\mathcal{P})$ as before, except that instead of covering X up to measure ε by $\mathcal{P}_A$-atoms, we cover by $\mathcal{P}_A$-measurable sets of $d_A^{\mathcal{P}}$ - radius $< \alpha$. Then

$$h_\alpha(\phi,\mathcal{P}) < h(\phi,\mathcal{P}) < h_\alpha(\phi,\mathcal{P}) + \rho(\alpha),$$

where $\rho(\alpha) = -\ \alpha \log \alpha - (1-\alpha)\log\ (1 - \alpha)$, and Stirling's formula is used to get the right inequality. It is now perfectly clear how to generalize to continuous amenable groups: simply use Haar measure to define $d_A^{\mathcal{P}}$. The quantity $h_\alpha(\phi,\mathcal{P})$ thus defined increases as $\mathcal{P}$ increases or as α decreases. We now set $h(\phi,\mathcal{P}) = \sup_{\alpha} h_{\alpha}(\phi,\mathcal{P})$ and $h(\phi) = \sup_{\mathcal{P}} h(\phi,\mathcal{P})$.

Entropy may also be defined with respect to a particular summing sequence: instead of looking at coverings by $\mathcal{P}_A$-measurable sets of small $d_A^{\mathcal{P}}$-diameter, for large and almost invariant A, we may look at coverings by $\mathcal{P}_{A_n}$-measurable sets of small $d_{A_n}^{\mathcal{P}}$-diameter, for large n. Thus $h_{\alpha}^{\mathcal{J}}(\phi,\mathcal{P})$ and consequently $h^{\mathcal{J}}(\phi,\mathcal{P})$, and $h^{\mathcal{J}}(\phi)$ may be defined for a summing sequence $\mathcal{J}$ in G. Then $h_{\alpha}^{\mathcal{J}}(\phi,\mathcal{P}) \leqslant h_{\alpha}(\phi,\mathcal{P})$, etc. Conceivably there are cases where the inequality is strict, but it may be shown without difficulty that if $\mathcal{J}$ consists of <u>tiling</u> sets then we have equality. It is also the case that for <u>discrete</u> G equality holds for <u>all</u> $\mathcal{J}$; this is a consequence of the main result of [11]. It would be interesting to know if equality <u>always</u> holds.

How shall we see that the definition is correct for flows? This involves examining the relation between the entropies for groups and closed cocompact subgroups. Here is a fact which takes care of the case at hand:

<u>Theorem</u>. If H is discrete and cocompact in G, then $h_{\alpha}^{\mathcal{J}}(\phi,\mathcal{P}) = h_{\alpha}(\phi,\mathcal{P})$ for all summing sequences $\mathcal{J}$, and

$$h_{\alpha}(\phi|H) \geqslant |G/H| h_{\alpha}(\phi,\mathcal{P}) \geqslant h_{\alpha}(\phi|H,\mathcal{P})$$

for all finite partitions $\mathcal{P}$.

IV. Equipartition

The "asymptotic equipartition theorem" of Shannon and Macmillan may be stated as follows: for any action of $\mathbb{Z}$, finite partition $\mathcal{P}$ of X, and $\varepsilon > 0$, there exists N so that if $n > N$ then a set $\mathcal{B}$ of atoms of $\mathcal{P}_0^n$ may be chosen with $\mu(\cup\mathcal{B}) > 1 - \varepsilon$ and each member of $\mathcal{B}$ having measure lying in the interval $2^{-n(h(\phi,\mathcal{P})\pm\varepsilon)}$. This theorem has been generalized to other discrete groups; see for example Pickel and Stëpin [18]; and eventually Kieffer [11] got the same sort of statement for arbitrary amenable groups.

For a continuous group, even $\mathbb{R}$, it is not a priori clear what an asymptotic equipartition theorem should say. However, the idea of counting sets of small $d_A^{\mathcal{P}}$ diameter, rather than atoms of $\mathcal{P}_A$, leads to a natural statement.

Asymptotic Equipartition property (for an action ϕ of G, summing sequence $\mathcal{A} = \{A_n\}$ in G, and finite partition $\mathcal{P}$). Given $\varepsilon > 0 \, \exists \, N$ such that if $n > N$ then there is a collection $\mathcal{B}$ of disjoint $\mathcal{P}_{A_n}$-messurable sets with $\mu(\cup\mathcal{B}) > 1 - \varepsilon$, and each member of $\mathcal{B}$ having $d_{A_n}^{\mathcal{P}}$ diameter $< \varepsilon$ and measure in the interval $2^{-|A_n|(h^{\mathcal{A}}(\phi,\mathcal{P})\pm\varepsilon)}$

Theorem. The asymptotic equipartition property holds for any amenable G having a discrete cocompact subgroup.

There are two improvements to make. The first is, to get an asymptotic equipartition property for any collection $\mathcal{B}$ of $\mathcal{P}_A$-measurable of small $d_A^{\mathcal{P}}$-diameter, when A is large and invariant enough and $\mathcal{B}$ has close to $2^{|A|h(\phi,\mathcal{P})}$ members (rather than simply asserting the existence of such $\mathcal{B}$, as we have done). The second is to get a "second order" refinement, by fixing the upper bound on the $d_A^{\mathcal{P}}$ diameter, and then getting members of $\mathcal{B}$ to have measure about $2^{-|A|h_\alpha(\phi,\mathcal{P})}$. The latter is new even for $\mathbb{Z}$-actions. Here is a formulation which subsumes both.

Strong asymptotic α-equipartition property (for an action ϕ of G, summing sequence $\mathcal{S} = \{A_n\}$ in G, finite partition $\mathcal{P}$, and $\alpha > 0$): Given $\varepsilon > 0$, $\exists\, N$ and $\delta > 0$ such that if $n > N$ and $\mathcal{B}$ is a collection of disjoint $\mathcal{P}_{A_n}$-measurable sets of $\bar{d}_{A_n}$-diameter $< \alpha$ with, cardinality of $\mathcal{B}$ $< 2^{|A_n|(h_\alpha^{\mathcal{S}}(\phi,\mathcal{P})+\delta)}$, and with $\mu(\cup\mathcal{B}) > 1 - \delta$, then $\exists$ subcollection $\mathcal{B}_0$ with $\mu(\cup\mathcal{B}_0) > 1 - \varepsilon$ and each member of $\mathcal{B}_0$ having measure in the interval $2^{-|A_n|(h_\alpha^{\mathcal{S}}(\phi,\mathcal{P})\pm\varepsilon)}$.

Theorem. If the amenable group G has a discrete cocompact subgroup with tiling, then it has the strong asymptotic α-equipartition property.

Bernoulli actions.

For a discrete G, there is a notion of Bernoulli action: choose a probability measure space (S,π) - typically S will be finite or countable - and let $X = S^G$, μ the product measure which is π on each factor, and $\phi_g(x)(h) = gh$. If G is also amenable, then $h(\phi)$ turns out to be the usual entropy of the probability distribution π. Ornstein [16] proved the deep and important fact that when $G = \mathbb{Z}$ then the entropy is a complete isomorphism invariant of the action. This result and part of the theory surrounding it have been extended to $\mathbb{Z}^n$ by Katznelson and Weiss [10]. The ingredients of the arguments, the

Rokhlin-Kakutani theorem, entropy and asymptotic equipartition, are present in a <u>discrete solvable</u> group, as we have seen. Thus it is to be expected that the theory will carry over to this case without essential difficulty.

The situation for continuous groups is more delicate. For $G = \mathbb{R}$, Ornstein [16] showed that there exists, for each $a > 0$, an action whose restriction to $\mathbb{Z}$ is Bernoulli with entropy a, and that this action of $\mathbb{R}$ is unique up to isomorphism. Similar results hold in $\mathbb{R}^n$, as shown by Lind [15].

<u>Question</u>: What should be meant by a Bernoulli action of a general continuous G, and what sort of isomorphism theorems are there? Here are two examples.

<u>Example 1</u>. This is a candidate for the title "Bernoulli action of infinite entropy." Let G be noncompact. There is a stochastic process $\{\xi_A;\ A \subset G\}$, where A ranges over sets of finite measure, and with the ξ_A satisfying

(a) ξ_A is a Poisson random variable with mean $|A|$.

(b) If A_i are disjoint and $A = \bigcup_i A_i$ then $\{\xi_{A_i}\}$ are independent and $\xi_A = \sum_i \xi_{A_i}$.

The process is unique. Thus for each g there is a measure-preserving transformation ϕ_g satisfying $\xi_A \circ \phi_g = \xi_{g^{-1}A}$. If g is a generator for an infinite discrete subgroup H of G, then $\phi|H$ is isomorphic to a Bernoulli action of infinite entropy.

Example 2. This is deeper, and is contained in unpublished work of D. Rudolph and me, [19], [8]. Let K be a compact group, α any automorphism of K, and G the semidirect product $\mathbb{Z} \overset{\alpha}{\times} K$. Thus, $\mathbb{Z}$ is a subgroup, K a normal subgroup, and $\mathbb{Z}$ acts on K by inner automorphisms of G, the action of 1 being precisely α.

Theorem. For each $a > 0$ there is a free action of G whose restriction to $\mathbb{Z}$ is a Bernoulli action with entropy a. This action is unique up to isomorphism.

All proofs of Bernoulli isomorphism to date have used "discrete subgroup" arguments. It would be of great interest to find a proof, say for $G = \mathbb{R}$, in which the discrete case is not used: presumably by using continuous versions of entropy, the Rokhlin lemma, finitely determined, asymptotic equipartition, etc. This was one of the purposes of the entropy definition in III, but it is not yet clear whether such a program can work.

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