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ON SEQUENTIAL ENTROPY OF MEASURE PRESERVING TRANSFORMATION

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The aim of this article is to present a simple proof of a slightly more general result than those ones obtained in [2] and [4].

Definition

Let $A = \{t_1, \dots, t_n, \dots\}$ be a subset of the set Z_+ of the set Z_+ of nonnegative integers. For a given positive integer n,k we denote

$$A_{n,k} = \bigcup_{i=1}^{n} \{t_i, t_{i+1}, \dots, t_i + k-1\}$$

$$S_{n,k}(A) = \text{card } (A_{n,k})$$

$$\overline{S}_k(A) = \overline{\lim_{n\to\infty}} S_{n,k}(A) \quad \text{and} \quad K(A) = \lim_{k\to\infty} \overline{S}_k(A)$$

Newton in [2] defined $K(A)$ in another way, but the definitions are equivalent. Let $h_A(T)$ denote the sequential entropy introduced by Kushnirenko [1].

Theorem 1.

The equation

$$h_A(T) = K(A) \cdot h(T)$$

holds for any measure transformation T except of the case $h(T) = 0$ and $K(A) = \infty$. In case $K(A) = 0$, $h_A(T) = 0$.

For the proof we recall some properties of conditional entropy. Let ξ be a finite measurable partition. We denote

$$\xi^K = \bigvee_{i=0}^{k-1} T^{-1}(\xi) \qquad \text{for any positive integer } K .$$

Lemma 1.

$$H(\xi^k) = H(\xi) + \sum_{i=1}^{k-1} H(T^{-i}(\xi) \mid \bigvee_{j=0}^{i-1} T^{-j}(\xi)) =$$

$$= H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^{i} T^{-j}(\xi))$$

Lemma 2.

For $k \leq q$ we have

$$\frac{1}{q} \cdot H(\xi^q) \leq \frac{1}{k} \cdot H(\xi^k)$$

Proof.

$$H(\xi^q) = H(\xi) + \sum_{i=1}^{q-1} H(\xi \mid \bigvee_{j=1}^{i} T^{-j}(\xi)) = H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^{i} T^{-j}(\xi)) +$$

$$+ \sum_{i=k}^{q-1} H(\xi \mid \bigvee_{j=1}^{i} T^{-j}(\xi)) \leq H(\xi^k) + (q-k) . H(\xi \mid \bigvee_{j=1}^{k-1} T^{-j}(\xi)) \leq$$

$$\leq H(\xi^k) + \frac{q-k}{k} . [H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^{i} T^{-j}(\xi))] = \frac{q}{k} . H(\xi^k)$$

Proof of the theorem.

The fact that in case $K(A) = 0$ $h_A(T) = 0$ was proved in [3].

Suppose that the product $K(A).h(T)$ is well defined. For any measurable partition ξ and any positive number k we have

$$H_A(T,\xi) \leq H_A(T,\xi^k)$$

For a given k we can write the set $A_{n,k}$ as a union of its maximal connected subsets.

$$A_{n,k} = \bigvee_{j=1}^{m} C_j$$

We denote

$$q_j = \text{card } (C_j) \quad , \quad n_j = \min (C_j)$$

From the construction of $A_{n,k}$ we have

$$q_j \geq k \quad , \quad \sum_{j=1}^{m} q_j = S_{n,k}(A)$$

According the definition of sequential entropy

$$H_A(T,\xi) \leq H_A(T,\xi^k) = \overline{\lim} \frac{1}{n} \cdot H(\bigvee_{i\in A_{n,k}} T^{-i}(\xi)) \leq$$

$$\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^{m} H(\bigvee_{i\in C} T^{-j}(\xi)) = \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^{m} H(T^{-n_j}(\xi^{q_j})) \leq$$

$$\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^{m} \frac{q_j}{k} \cdot H(\xi^k) = \frac{1}{k} \cdot H(\xi^k) \cdot \overline{\lim_{n\to\infty}} \frac{S_{n,k}(A)}{n} =$$

$$= \overline{S}_k(A) \cdot \frac{1}{k} \cdot H(\xi^k) \quad .$$

Taking limit wilh respect to k we get

$$H_A(T,\xi) \leq K(A) \cdot H(\xi)$$

Therefore the unequality

$$h_A(T) \leq K(A) \cdot h(T)$$

holds.

The converse unequality was proved in [3].

References

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