Astérisque

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The special representation theorem for multidimensional group actions

THE SPECIAL REPRESENTATION THEOREM FOR MULTI-

DIMENSIONAL GROUP ACTIONS

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I. Introduction

A new approach to the study of measure-preserving transformations and flows based on the notion of Kakutani equivalence or monotone equivalence has been developed during the last three years independently by Feldmann, Ornstein, Weiss and Rudolph and by Satayev and myself (cf. $[1] - [6]$). A survey of these works was given in the talk of Don Ornstein. The main purpose of my talk is to establish several basic notions and results as the first step in the direction of the multi-dimensional generalization of this approach i.e. the extension of "the monotone ergodic theory" from actions of $\mathbb{Z}$ and $\mathbb{R}$ up to measure-preserving actions of the groups $\mathbb{Z}^m$ and $\mathbb{R}^m$ where m is an arbitrary positive integer. This extension is far from trivial and I am going to discuss some of the difficulties below. I have no recipes as to how all of these difficulties can be overcome. Even the formulation of the main criterion of equivalence in the positive-entropy case is vague because in the one-dimensional case it includes the notion of the LB-process which is based on the existence of the "past" in the group $\mathbb{Z}$ and this notion of the past has no canonical multi-dimensional generalization.

Indeed, there exists one special case of the LB-property, namely the property

of standardness (LB + zero-entropy, cf $[2]$, $[3]$, $[4]$) which allows several

natural generalizations. One of them is based on the consideration of the natural

partial order in the group Z^m . The corresponding notion of equivalence for

Z^m-actions was defined in ($[2]$, definition 2.3). Here we are going to prepare

some basic definitions and results for the study of a slightly more general

notion of equivalence which is based only on the integrability of an appropriated

cocycle instead of non-negativeness which would arise in considerations based

on the partial ordering. I hope that this approach admits a further generali-

zation to actions of more general groups.

Studies in "the monotone ergodic theory" depend on constructions of an

induced and special (integral) automorphism and a special flow over an automorphism

(several different terms are used for naming the last mentioned notion: an

integral flow, a flow under a function etc.). Indeed, up to the present time

appropriate multidimensional generalizations of these notions have not been

known. We begin to do away with this lack i.e. we define "special actions" of

the groups Z^m and R^m . Then we prove the multidimensional generalization of

the well known theorem of Ambrose and Kakutani and indicate another appli-

cation of our machinery. In the next work we are going to define the generali-

zation of Kakutani equivalence and the property of standardness and discuss the

prospects and difficulties of the theory.

II. Definition of special actions

The construction of a special flow over an automorphism provides a method of

definition of a flow i.e. a measure-preserving action of the group R by the

given automorphism i.e. an action of the group Z and some additional data

namely a positive integrable function. Let us recall this construction.

Let (X,μ) be a Lebesgue space, $T : X \to X$ an automorphism of X i.e. a

measurable one-to-one measure-preserving transformation and f an integrable

positive function on X . Then we can construct a new Lebesgue space $X_f \in X \times R$

$$X_f = \{(x,s), x \in X, 0 \leq s \leq f(x)\}$$

with a normalized measure induced from the space $X \times \mathbb{R}$ and define a flow $T^f = \{T_t^f\}$ on X_f in the following way: the point of X_f moves with the unit speed along the "vertical" segment and then "jumps" immediately from the point $(x,f(x))$ into the point $(Tx,0)$. Certainly, this construction is specifically one-dimensional but we can interpret it in such a way that allows natural generalization.

Let us consider the cylinder $X \times \mathbb{R}$ with the natural product measure $\mu \times \lambda$, where λ is a Lebesgue measure on $\mathbb{R}$, and define the automorphism $\hat{T}$ of this space with infinite measure in the following way

$$\hat{T}(x,s) = (Tx, s + f(x))$$

This automorphism is a principal $\mathbb{R}$ -extension of the given automorphism T , i.e. a transformation of $X \times \mathbb{R}$ which covers T and maps each fiber into another fiber by a group translation. The "vertical" flow $\{T_t\}$ acts according to the formula

$$T_t(x,s) = (x, t+s)$$

and commutes with $\hat{T}$. Thus we can formally project this flow into the factor space of $\hat{T}$-orbits. Since we assume that f is positive there exists a natural fundamental domain for the cyclic group generated by $\hat{T}$; namely the set X_f described above. Action of $\{T_t\}$ on this fundamental domain coincides with the special flow constructed over T^{-1} . The inversion of time is of no importance because we could start with T^{-1} instead of T .

Two $\mathbb{R}$ -extensions of the given automorphism T

$$T^{(1)}(x,s) = (Tx, s+f(x))$$

$$T^{(2)}(x,s) = (Tx, s+g(x))$$

are metrically isomorphic if there exists a measurable function h such that

$$f(x) = g(x) + h(Tx) - h(x) \tag{1}$$

The isomorphism is given by the following map S :

$$S(x,s) = (x,s + h(x))$$

which commutes with the vertical flow so that the factor-flows generated by f and g are also isomorphic. If for given f and g equality (1) holds for some h we will call the functions f and g R-cohomologous.

We can consider an R-extension of T generated by an arbitrary integrable (not necessary positive) function f . Ornstein, Kočergin and Weiss (see [5], [7]) proved that every integrable function with non-zero integral is R-cohomologous with a function of constant sign, so that the factor-flow generated by such a function is metrically isomorphic to the special flow over T or T^{-1} .

If $\int_X fd\mu = 0$ then the partition of the space $X \times \mathbb{R}$ into trajectories of $\hat{T}$ is nonmeasurable so that we have to consider the measurable hull of this partition and the corresponding factor-space. In this case there are two possibilities : either f is R-cohomologous with 0 and consequently the factor-space is isomorphic to R or the factor-space does not admit a good structure of a measure space compatible with the vertical flow. In both cases a good finite measure in the factor-space invariant under the factor-flow does not exist.

Now we are prepared to generalize the notion of a special flow. In the most general situation we should deal with a locally compact topological group G , a discrete subgroup H ⊂ G such that the homogenous space G/H has finite volume and a given left action $S = \{S_h\}_{h \in H}$ of H by automorphisms of a Lebesgue space (X,μ) . Then we should consider some principal G -extension $\hat{S}$ of S and project the "vertical" right action of G on $X \times G$ into the space of $\hat{S}$ -orbits. Some conditions on a G -extension are necessary to provide good properties of the factor-space and a sufficiently close connection between the given action S and the produced factor-action of the vertical action.

But we are not going to discuss the general situation in detail and we

restrict ourselves to the following two cases : $G = H = \mathbb{Z}^m$ and $G = \mathbb{R}^m$, $H = \mathbb{Z}^m$.
The former case corresponds to the multidimensional generalization of the notions
of an induced automorphism, special automorphism and their combination (cf.[2],
§ 2) , the latter corresponds to the generalization of the notion of a special
flow. Most of the arguments concerning the latter (continuous) case hold true
automatically for the first one so that we shall deal with the continuous
situation and note all the points where some special explanations for the dis-
crete case will be necessary.

Let $T = \{T_n\}_{n \in \mathbb{Z}^m}$ be an action of $\mathbb{Z}^m$ on a Lebesgue space by measure-
preserving transformations. Consider a principal $\mathbb{R}^m$ -extension $\hat{T}$ of the action
T acting on the direct product $X \times \mathbb{R}^m$. Such an extension has the following
form :

$$\hat{T}_n(x,s) = (Tx, s + h_n(x)) \tag{2}$$

where $x \in X$, $n \in \mathbb{Z}^m$, $s \in \mathbb{R}^m$. The group property of $\hat{T}$ implies the following
equation

$$h_{n_1 + n_2}(x) = h_{n_1}(x) + h_{n_2}(T_{n_1}x)$$

for every $n_1, n_2 \in \mathbb{Z}^m$.

We shall call the function $h_n(x)$ the <u>cocycle generating the given</u> $\mathbb{R}^m$ -
<u>extension</u> $\hat{T}$ of the action T .

As in the one-dimensional case we can consider the "vertical" action of $\mathbb{R}^m$:

$$T_t(x,s) = (x, s+t) , \quad t \in \mathbb{R}^m$$

and since this action commutes with $\hat{T}$ we can formally project it into the
space of $\hat{T}$ -orbits.

In general partition of $X \times \mathbb{R}^m$ into $\hat{T}$ -orbits may be nonmeasurable and
so we should consider the measurable hull of this partition and the correspon-
ding factor-space and factor-action. We are going to establish that the following
condition (J) which generalizes the condition of existence and non-vanishing
of integral $\int_X f d\mu$ in the one-dimensional case provides simultaneously good

properties of the space of $\hat{T}$ -orbits and a sufficiently close connection between the given action on X and the produced factor-action. Let us fix the standard basis in $\mathbb{Z}^m$ and $\mathbb{R}^m$: $\{e_i\}i = 1,\ldots,m$, where $e_i = (0\ldots1\ldots0)$ i-th place. Further, let $\xi = (\xi_1,\ldots,\xi_m)$ be an arbitrary basis in $\mathbb{Z}^m$. Form an $m \times m$ matrix

$$H_\xi(x) = \begin{pmatrix} h_{\xi_1}(x) \\ \vdots \\ h_{\xi_m}(x) \end{pmatrix}$$

(**J**) . Vector-functions $h_{e_i}(x), i = 1\ldots m$ and scalar functions $\det H_\xi(x)$ where ξ is an arbitrary basis in $\mathbb{Z}^m$ are absolutely integrable and vectors $h_i = \int_X h_{e_i}(x)d\mu$ are linearly independent over $\mathbb{R}$.

Theorem 1.

Let T be a measurable, measure-preserving, ergodic, free mod 0 action of the group $\mathbb{Z}^m$ on a Lebesgue space (X,μ) . Suppose that $\hat{T}$ is a principal $\mathbb{R}$ - extension of T generated by the cocycle h which satisfies the condition (**J**). Then the following construction gives the finite Lebesgue measure $\hat{\mu}$ in the space of $\hat{T}$-orbits invariant under the vertical action . Let $A \subseteq X \times \mathbb{R}^m$ be a measurable set invariant under $\hat{T}$. Denote the standard unit cell in $\mathbb{R}^m$:

$$\{(t_1,\ldots,t_m) \in \mathbb{R}^m, \ 0 \leq t_i \leq 1, \ i=1,\ldots,m\}$$

by Δ and choose measurably for every orbit $\gamma \in A$, a point $(x_\gamma,t_\gamma) \in \gamma$. Decompose $t_\gamma = s_\gamma + \tau_\gamma$ where $s_\gamma \in \Delta$, $\tau_\gamma \in \mathbb{Z}^m$ and define the function f_A on $X \times \Delta$: $f_A(x,t) = \text{card} \{\gamma \in A : x_\gamma = x, \ s_\gamma = t\}$. Then

$$\hat{\mu}(A) = \int_{X\times\Delta} f_A(x,t)d(\mu\times\lambda)$$

where λ is the standard Lebesgue measure on Δ . Measure $\hat{\mu}$ does not depend on the choice of points (x_γ,t_γ) .

The main part of the proof of this theorem is the following proposition.

Proposition 1.

   If the conditions of theorem 1 hold then there exists a measurable set
$B \subset X \times \mathbb{R}$ with the following properties.

   1. For a.e. $x \in X$ the set $B \cap (\{x\} \times \mathbb{R}^m)$ is bounded.

   2. $\int_X \lambda(B \cap (\{x\} \times \mathbb{R}^m)) d\mu < \infty$

   3. Almost every trajectory of T intersects B .

Proof of the proposition.

   Fix the point $x \in X$, denote the convex hull of vectors $(-h_n(x))$, where n
belongs to the set of all vertices of the cube Δ , i.e. all 0-1-vectors by
K_x and let $B = \bigcup_{x \in X} (\{x\} \times K_x)$. Obviously, B is a measurable subset of
$X \times \mathbb{R}^m$, so the proposition follows from the next two lemmas.

   Lemma 1. Almost every trajectory of $\hat{T}$ intersects the set B .

   Lemma 2. $\int_X \lambda(K_x) d\mu < \infty$.

Proof of lemma 1.

   Choose a triangulation Θ of the space $\mathbb{R}^m$ with the following properties.

   1. The vertexes of Θ are precisely the points of the integer lattice
$\mathbb{Z}^m \subset \mathbb{R}^m$.

   2. Θ is invariant under integer translations.

   3. Θ is a subdivision of the standard integer decomposition:$\{\Delta + n, n \in \mathbb{Z}\}$

   Using this triangulation we can extend an arbitrary given map $\phi : \mathbb{Z}^m \to \mathbb{R}^m$
in a unique way up to the map $\hat{\phi} : \mathbb{R}^m \to \mathbb{R}^m$ defining $\hat{\phi}$ as a linear map on
each simplex of Θ .

   Let $L : \mathbb{Z}^m \to \mathbb{R}^m$ denote the linear operator which is defined by its action
on the basic vectors $e_i : Le_i = h_i$. Condition (J) implies that $\det L \neq 0$.
The individual ergodic theorem is true for the cocycle h so that by the
ergodicity of T for a.e. $x \in X$ the following is true

$$\lim_{||n|| \to \infty} \frac{||h_n(x) - Ln||}{||n||} = 0 \tag{3}$$

where the norm is Euclidean.

Now fix $x \in X$ for which (3) holds and prove that for every $s \in \mathbb{R}^m$ T-orbit of the point (x,s) intersects the set B . Since

$$\hat{T}_n(\{x\} \times (K_{T_n x} - h_n(x)) = \{T_n x\} \times K_{T_n x}$$

it is enough to prove that sets

$$K_{T_n x} - h_n(x), \quad n \in \mathbb{Z}^m$$

cover the whole space $\mathbb{R}^m$.

For this purpose consider the map $\phi_x : \mathbb{Z}^m \to \mathbb{R}^m$ defined by $\phi_x(n) = -h_n(x)$ and extend it using the triangulation $\ominus$ up to the map $\hat{\phi}_x : \mathbb{R}^m \to \mathbb{R}^m$. Obviously, for every $n \in \mathbb{Z}^m$ the image $\hat{\phi}_x(\Delta + n)$ coincides with the set $K_{T_n x} - h_n(x)$ so the proof would be over if we show that $\hat{\phi}_x$ is a surjective map. But this follows from the individual ergodic theorem and standard homotopy arguments. For denote $D_z = \{t \in \mathbb{R}^m : ||t|| \leq r\}$ for $r > 0$. Fix an arbitrary $R > 0$ and choose R_1 such that $||\hat{\phi}_x(t)|| > R$ for every t with $||t|| = R_1$ and the map $\psi : \partial D_{R_1} \to S^{m-1}$ defined by $\psi(t) = \frac{\hat{\phi}_x(t)}{||\hat{\phi}_x(t)||}$ has non-zero degree. This is possible by (3) and $\det L \neq 0$. Then $\hat{\phi}_x(D_{R_1}) \supset D_R$ so that $\hat{\phi}$ is a surjective map. Lemma 1 is proved.

Proof of lemma 2.

Denote a restriction of the triangulation $\ominus$ to the unit cube Δ by $\ominus_o$ and list the simplexes of this triangulation: $\sigma_1, \ldots, \sigma_k$. Further, fix a vertex p_i for every simplex σ_i $i = 1 \ldots k$, denote $\sigma_i - p_i$ by σ_i and let $\xi_1^i, \ldots, \xi_m^i$ be the vertexes of the simplex σ_i different from the origin. Obviously

$$K_x = \bigcup_{i=1}^{k} \hat{\phi}_x(\sigma_i)$$

and

$$\lambda(\hat{\phi}_x(\sigma_i)) = \lambda(\hat{\phi}_{T_{p_i}(x)}(\bar{\sigma}_i)) \ .$$

But

$$\lambda(\hat{\phi}_{T_{p_i}(x)}(\bar{\sigma}_i)) = \frac{1}{m!} \det H_{\xi^i}(T_{p_i}(x))$$

where $\xi^i = (\xi^i_1,\ldots,\xi^i_m)$, so that the lemma follows from condition (I) .

Now we can conclude the proof of theorem 1. It is easy to see from the individual ergodic theorem that the partition of $X \times \mathbb{R}^m$ into $\hat{T}$-orbits is measurable.

Proposition 1 implies that we can choose for every trajectory $\gamma \in A$ a point (x_γ, t_γ) in the set B described above. In this case for every measurable set A , invariant under T , the function $f_A(x,s)$ is a.e. finite and $\hat{\mu}(A)$ is also finite.

Moreover, the definition of $\hat{\mu}$ does not depend on the choice of (x_γ, t_γ). For we can realize the transition from one such choice to another one by a finite or countable sequence of steps of the following kind: for each trajectory γ belonging to a subset $A' \subset A$ we change points $(x_\gamma t_\gamma)$ to

$$\hat{T}_n(x_\gamma, t_\gamma) = (T_n x_\gamma, t_\gamma + h_n(x_\gamma))$$

Furthermore we can decompose the set A' into subsets A_x for which $x_\gamma = x$ and this decomposition is evidently measurable. If we consider the unit cell $\Delta \subset \mathbb{R}^m$ as a m-dimensional torus then for every $x \in X$ the transformation from an old s_γ to a new s_γ will be a shift on this torus so that the conditional measures on these elements will be preserved. The factor-measure is also preserved because this measure is just μ and a transformation in base is just T_n which preserves the measure μ .

If we fix the choice of points (x_γ, t_γ) for ever orbit $\gamma \in X \times \mathbb{R}^m$ then the produced measure $\hat{\mu}$ is obviously non-negative and σ-additive. Since we

have proved the independence of this measure of the particular choice of $[x_\gamma, t_\gamma]$ the proof of theorem 1 is over.

Definition.

Suppose T is a measure preserving ergodic free action of the group Z^m on a Lebesgue space (X, μ), $\hat{T}$ is a principal R^m-extension of T generated by a cocycle h, which satisfies condition (I). Then we shall call the factor-action of the vertical action of the group R^m on the Lebesgue space of $\hat{T}$-orbits with the invariant measure $\hat{\mu}$ described in theorem 1 the special action over T generated by the cocycle h.

Remark.

All considerations of this section can be transfered with appropriate modifications and simplifications to the case of the special action of Z^m, i.e. a factor-action of the vertical Z^m-action generated by some principal Z^m-extension of the given measure-preserving ergodic free action of the group Z^m.

III. The special representation theorem.

Now we have established the notion of the special action and we could formulate the generalization of the Ambrose-Kakutani theorem by saying that every free ergodic action of the group R^m is metrically isomorphic to some special action over some ergodic action T of the group Z^m [*]. But we want to refine this statement by bringing some additional properties of a cocycle.

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[*] The general version of the Ambrose-Kakutani theorem deals with an arbitrary flow without fixed points. Similarly we could include more general cases in our considerations, for example arbitrary free actions, but this generalization demands some additional space and is not very essential.

Definition.

Cocycle h is called r-Lipschitz where r > 1 is a real number if for
every $n \in \mathbb{Z}^m, x \in X$

$$r^{-1} \leq \frac{||h_n(x)||}{||n||} \leq r$$

Of course, if the cocycle h is r-Lipschitz for some r then $h_n(x)$
is a bounded function for every $n \in \mathbb{Z}^m$ and h satisfies condition (I) .

Now we are completely ready to formulate the multidimensional version of
the theorem about special representation.

Theorem 2.

Let $S = \{S_t\}_{t \in \mathbb{R}^m}$ be a measurable measure-preserving ergodic free action
of the group $\mathbb{R}^m$ on a Lebesgue space (X,μ) and $\varepsilon > 0$ an arbitrary small
number. Then there exists an ergodic action of the group $\mathbb{Z}^m$ $T = \{T_n\}_{n \in \mathbb{Z}^m}$
and $(1 + \varepsilon)$ -Lipschitz cocycle h such that S is metrically isomorphic to the
special action over T generated by the cocycle h .

The first part of the proof is the following proposition of some indepen-
dent interest. It is close to some results of Feldman and Moore mentioned in
Feldman's talk and also to Rudolph's impressive result about the existence
of a measurable partition whose elements are rectangles on orbits of a given
action of the group $\mathbb{R}^m$.

Proposition 2.

Suppose S is an ergodic free action of the group $\mathbb{R}^m$, $r_n \to \infty$ is an in-
creasing sequence of positive numbers. Then there exists a decreasing sequence
of measurable sections $A_1 \supset A_2 \supset ...$ for the action S and a sequence of
measurable partitions ξ_n such that for every n = 1, 2, ... almost every
element of ξ_n contains exactly one point $x \in A_n$ and has the following form:

$$c_x^n = \{S_t x, \ t \in \Gamma_n(x)\} ,$$

where $\Gamma_n(x)$ is a convex subset of $\mathbb{R}^m$ which contains Euclidean ball of radius r_n with the center at the origin.

Proof of the proposition.

We begin with the simplified version of the proposition where only one number $r > 0$ is fixed and consequently only one section A and partition ξ have to be produced.

Represent the action S as a continuous group of homeomorphisms of a compact metric space F with Borel invariant measure ν . Without loss of generality we can suppose that $\nu(A) > 0$ for every open nonempty subset $A \subset F$. Denote the r-ball with the center at the point x in the given metric of F by $B(x,r)$ and the Euclidean r-ball on the S-orbit of the point x by $B_S(x,r)$. Fix an arbitrary point $x \in F$ and $\varepsilon > 0$ and choose $\delta > 0$ so small that

$$B_S(10r,x) \cap B(\delta,x) = B_S(\varepsilon,x) \cap B(\delta,x)$$

Further there exists $\varkappa > 0$ so that for every $y \in B(\varkappa,x)$

$$B_S(5r,y) \cap B(\delta,x) = B_S(\varepsilon,y) \cap B(\delta,x)$$

Then $y_1, y_2 \in B(\varkappa,x)$,

$$B_S(2r,y_1) \cap B_S(2r,y_2) \neq \emptyset$$

imply that

$$y_2 \in B_S(2\varepsilon,y_1) \ .$$

Thus an equivalence relation on the set $B(\varkappa,x)$ occurs : namely $y_1 \sim y_2$ if $B_S(2r,y_1) \cap B_S(2r,y_2) \neq \emptyset$. This equivalence relation generates in a natural way the measurable partition of $B(\varkappa,x)$ and we can choose a measurable section A of this partition, i.e. a set $A \subset B(\varkappa,x)$ which contains exactly one point from each element of the partition.

Sets $B_S(r,x)$, $x \in A$ form a measurable partition of some part of the space X . This fact follows from the defintion of the equivalence relation and from the measurability of the section A . Now we want to "blow up" every set $B_S(r,x)$ within its orbit and get a partition of the whole space X .

Let us fix an orbit γ of the action S and examine the intersection $\gamma \cap A$. This intersection is a discrete set in the inner topology of the orbit. Break the orbit γ into the Dirichlet regions of points of the set $\gamma \cap A$. In other words construct for each point $x \in \gamma \cap A$ the set

$$D_x = \{y \in \gamma, ||y-x|| \le ||y-z|| \quad \text{for all} \quad z \in \gamma \cap A\}$$

where $|| \cdot ||$ is the Euclidean norm in the orbit γ. The regions D_x are convex subsets of γ.

It follows from the ergodicity of the action S that for almost every orbit γ all sets D_x are bounded. Partition of X into the sets $D_x (x \in A)$ is the required partition.

We can define on every measurable section A for the given ergodic action S of the group $\mathbb{R}^m$ the natural class of equivalent measures induced from the space X. Namely, fix a sufficiently small number $\varepsilon > 0$, call subset $C \subset A$ measurable if the set $\bigcup_{x \in C} B_S(\varepsilon, x)$ is μ-measurable and suppose

$$\nu_\varepsilon (C) = \mu (\bigcup_{x \in C} B_S(\varepsilon, x))$$

Evidently, the property of C to be measurable or have measure zero does not depend on the choice of ε. We can and should construct our partition by a slightly more general way, namely replace the whole section A by its subset A' of positive measure and then do with this smaller section i.e. blow up the balls $B_S(r,x)$, $x \in A'$ to the Dirichlet regions described above. We are going to use this generalization in the proof of the general version of proposition 2 .

Recall that we deal with the topological situation so we can introduce the following definition. The point $x \in A$, where A is a measurable section of the given $\mathbb{R}^m$ -action S with the Borel invariant measure μ is called essential for the section A if for every $\delta > 0$ an intersection $A \cap B(\delta, x)$ has positive measure on A in the sense described above.

Now we proceed to the proof of the general case of the proposition by induction. Suppose that the sets $A_1, \ldots, A_n$ are already constructed with the

help of the generalized version of our procedure and denote the corresponding

points "x" and numbers "$\varkappa$" by x_i and $\varkappa_i$ i = 1...n . Then choose an essential

point $x_{n+1} \in A_n$ and fulfill all our considerations where the number $\varkappa_{n+1}$

is chosen so small that $B(\varkappa_{n+1}, x_{n+1}) \subset B(\varkappa_n, x_n)$.

Such a choice of $\varkappa_{n+1}$ provides that classes of equivalent points in

$B(\varkappa_{n+1}, x_{n+1})$ are just the intersections of such classes constructed at the

previous step (i.e. in $B(\varkappa_n, x_n)$) with the ball $B(\varkappa_{n+1}, x_{n+1})$. Hence we can

choose the set $A_n \cap B(\varkappa_{n+1}, x_{n+1})$ as a new section A_{n+1} . Non-triviality of

the intersection is provided by the choice of the point x_{n+1} (recall that x_{n+1}

is an essential point for the section A_n) . Proposition 2 is proved.

Remarks.

1. In the one-dimensional case, the assertion of theorem 2 immediately

follows from proposition 2. For, choose $r > \dfrac{100}{\varepsilon} + 2$ and consider the corres-

ponding section A and measurable partition ξ . Each element c of this

partition is a sufficiently long segment of some trajectory of the given flow,

i.e. there exists such a point $x_c \in A$ and number $t(c) \geq r$ that

$$c = \{S_t x_c\} \quad 0 \leq t < t(c) .$$

Denote $\dfrac{t(c)}{[t(c)]}$ by s(c) and consider the new measurable section

$$B = \bigcup_{c \in \xi} \bigcup_{k > 0}^{[t(c)]-1} S_{ks(c)} x_c$$

with the corresponding first return map which we shall denote by T .

Obviously for every $x \in B$ there exists $c \in \xi$ such that $T_x = S_{s(c)} x$.

Since $1 \leq s(c) \leq 1 + \dfrac{1}{[t(c)]} < 1 + \varepsilon$ we have constructed the special

representation for the flow $\{S_t\}$ with the function whose value lies between

1 and $1 + \varepsilon$.

2. The discrete version of proposition 2 also holds with the corresponding

changes in the formulation: a convex subset of $\mathbb{Z}^m$ is an intersection of a

convex set $F \subset \mathbb{R}^m$ and $\mathbb{Z}^m$ which is embedded into $\mathbb{R}^m$ as a standard integer

lattice. An easy consequence of this discrete version of the proposition is a new proof of the multi-dimensional generalization of the so-called Rochlin lemma about uniform approximation. We shall give this new proof in section IV below and now we get down to the proof of theorem 2.

The second essential step of the proof of this theorem is an extension of very "thin" sections A_n up to more regular sections B_n .

Intersection of such a section with a typical orbit consists of very large "good" pieces. Each of such pieces is a slightly distorted piece of the lattice. The following proposition provides the existence and convergence of such sections.

Proposition 3.

There exists such a sequence $r_n \to \infty$ that for every $x \in A_n$, where A_n is a set from the assertion of proposition 2 there exists a map

$$\phi_{n,x} : \Gamma_n(x) \to \mathbb{R}^m$$

with the following properties.

1. $\phi_{n,x}$ depend on x by a measurable way.

2. The Lipschitz constant of the map $\phi_{n,x}$ - id does not exceed

$$\frac{\varepsilon}{2}(1 - \frac{1}{2^n}) \, .$$

3. For every $t \in \Gamma_n(x)$ $||\phi_{n,x}(t) - t|| < m.2^n$

4. Denote the set $\displaystyle\bigcup_{x \in A_n} \bigcup_{k \in \Gamma_n(x) \cap \mathbb{Z}^m} \{S_{\phi_{n,x}(k)}x\}$ by B_n and fix the

measure $\nu = \nu_{\frac{1}{100}}$ on every of the set B_n . Then

$$\nu(B_{n+1} \Delta B_n) < \frac{1}{2^n}$$

The next simple geometrical lemma is used at the proof of this proposition as well as in the concluding part of the proof of theorem 2 and in the considerations of section IV.

Lemma 3.

Let $\Gamma \subset \mathbb{R}^m$ be a bounded convex set which contains an Euclidean ball of

radius R ; $\Gamma_r \subset \Gamma$ is an r-neighbourhood of the boundary $\partial\Gamma$ of Γ . Then the

following inequality between m-dimensional volumes $v_m(\Gamma)$ and $v_m(\Gamma_r)$ holds

$$v_m(\Gamma_r) \leq \frac{m!\,r}{R} \cdot v_m(\Gamma)$$

Proof of the lemma.

It is enough to prove the lemma for convex polyhedra because the case of

a general convex set is treated by the polyhedral approximation and limiting

process. So suppose that Γ is a convex polyhedron which contains an Euclidean

R-ball with the center in the point x . Denote the (m-1)-dimensional faces

of Γ by $F_1,\ldots,F_k$ and the cone with the vertex x and base F_i by K_i .

Cones $K_1 \ldots K_k$ form a subdivision of the polyhedron Γ . The height of each

cone K_i is greater than R so that

$$v_m(K_i) \geq \frac{R}{m!} v_{m-1}(F_i) .$$

The summing of these inequalities for $i = 1,\ldots,k$ gives

$$v_m(\Gamma) = \sum_{i=1}^{k} v_m(K_i) \geq \frac{R}{m!} \sum_{i=1}^{k} v_{m-1}(F_i) = \frac{R}{m!} v_{m-1}(\partial\Gamma)$$

On the other hand it follows from convexity of the set that

$$v_m(\Gamma_r) \leq r\, v_{m-1}(\partial\Gamma)$$

so that

$$v_m(\Gamma_r) \leq \frac{m!\,r}{R} v_m(\Gamma)$$

Lemma is proved.

Proof of proposition 3.

We choose the sequence r_n such that $r_n > \dfrac{400 \cdot 4^n \cdot (m+1)!}{\varepsilon}$. We proceed

the proof by induction. The base of the induction process is trivial because we

can choose a sufficiently large number r_1 and define $\phi_{1,x}$ as an identical

embedding of the set $\Gamma_1(x)$ into $\mathbb{R}^m$. Now let us suppose that the maps

$\phi_{i,x}$ for $i = 1 \ldots n$, $x \in A_i$ are already constructed and denote for $x \in A_n$ and $r > 0$ through $\Gamma'_{n,r}(x)$ the difference between $\Gamma_n(x)$ and the r-neighbourhood of its boundary. Condition 3 implies that for every $r > 0$

$$\phi_{n,x}(\Gamma'_{n,r}(x)) \subset \Gamma'_{n,r-m.2^n}(x) .$$

Now fix a point $x \in A_{n+1}$ and consider the element C_x^{n+1} of the partition ξ_{n+1} which contains this point. List all points $y \in A_n$ such that $C_y^n \cap C_x^{n+1} \neq \emptyset$

Each of these points has the form

$$y = S_{t(y)}x$$

Let us divide the set $\Gamma_{n+1}(x)$ into the sets

$$K_y = \Gamma_{n+1}(x) \cap (\Gamma_n(y) + t(y))$$

and let

$$K'_y = \Gamma_{n+1}(x) \cap (\Gamma'_{n,\frac{100.2^n.m}{\epsilon}}(y) + t(y))$$

We begin with the definition of the map $\phi_{n+1,x}$ on the sets K'_y , namely let for $t \in K'_y$

$$\phi_{n+1,x}(t) = [t(y)] + \phi_{n,y}(t-t(y)) \qquad (4)$$

where $[t]$ denote the entire part of the vector $t \in \mathbb{R}^m$.

Obviously, on each set K'_y condition 2 holds. Condition 3 also holds because

$$||\phi_{n+1,x}(t) - t|| = ||\phi_{n,y}(t-t(y)) - (t-t(y)) + [t(y)] -$$

$$- t(y)|| \leq \sup_{s \in \Gamma_n(y)} ||\phi_{n,y}(s) - s|| + ||[t(y)] - t(y)|| \leq$$

$$\leq m . 2^n + \sqrt{m} < m . 2^{n+1} \qquad (5)$$

Let

$$\Gamma' = \bigcup_{y : C_y^n \cap C_x^{n+1} \neq \emptyset} K'_y$$

If $t_1, t_2 \in \Gamma'$, but t_1 and t_2 lie in different sets K_y' , then the distance between t_1 and t_2 exceed $\dfrac{100.2^n.m}{\varepsilon}$ and

$$||\phi_{n+1,x}(t_1) - t_1 - (\phi_{n+1,x}(t_2) - t_2)|| \leq 2^{n+2}.m$$

so that the Lipschitz constant of the map $\phi_{n+1,x}$ - id on the set Γ' does not exceed $\dfrac{\varepsilon}{2}(1 - \dfrac{1}{2^n})$.

Now we should define the map $\phi_{n+1,x}$ on the remaining part of the set $\Gamma_{n(x)}$ with a very small increase of the Lipschitz constant of the difference $\phi_{n+1,x}$-id and without any increase of the C^0 norm of this difference. In the case of a scalar function such a prolongation (even without any increase of the Lipschitz constant) is provided by a fairly general and simple construction of Banach (see, for example, [10] , chapter 5, §5, Lemma 3) . However, I do not know any such general construction in the case of a vector function. So it is necessary to use some more special machinery.

Let $d(t)$ denote a Euclidean distance from the point $t \in K_y$ to the subset K_y' and let $s(t)$ denote the unique point of K_y' , such that the distance between t and $s(t)$ is equal to $d(t)$. Now we can define the map $\phi_{n+1,x}$ on each set K_y , namely

$$\phi_{n+1,x}(t) = t + (\max(0,1 - \frac{\varepsilon}{10.2^n.m} d(t))).([t(y)] + \phi_{n,y}(s(t) - t(y))) \quad (6)$$

Let $\lambda = \dfrac{\varepsilon}{10.2^n.m}$. Obviously $\phi_{n,x}(t) = t$ if $d(t) \geq \lambda^{-1}$, i.e. if the point t lies outside the λ^{-1} -neighbourhood of the set K_y' . But $d(t) > 2.\lambda^{-1}$ for every point

$$t \in \partial(\Gamma_n(y) + t(y))$$

so that the map $\phi_{n+1,x}$ coincides with the identity in some neighbourhood of the common boundary of every two sets K_y . Thus formula (6) gives the continuous map defined on the whole set $\Gamma_n(x)$ and it is enough to ensure conditions 2 and 3 within each set independently.

Condition 3 is obviously true because by virtue of (5) and (6)

$$\max_{t \in K_y} ||\phi_{n+1,x}(t) - t|| = \max_{t \in K_y'} ||\phi_{n+1,x}(t) - t||$$

Let us verify condition 2. Fix a number $r : 0 \leq r \leq \lambda^{-1}$ and consider the surface

$$S_{r,y} = \{t \in K_y, \; d(t) = r\}$$

It follows from the convexity of the set K_y' and from formula (6) that the Lipschitz constant of the map $\phi_{n+1,x}$ -id restricted on this surface does not exceed the corresponding constant for $\phi_{n+1,x}$ - id restricted on the surface $\partial K_y'$. The last-mentioned constant does not exceed $\frac{\varepsilon}{2}(1- \frac{1}{2^n})$ by virtue of (4). Further, the Lipschitz constant in the orthogonal direction to the surface $S_{r,y}$ does not exceed $\lambda < \frac{\varepsilon}{2^{n+2}}$ so that the Lipschitz constant of $\phi_{n+1,x}$ - id on the set K_y is less than or equal to $\frac{\varepsilon}{2}(1 - \frac{1}{2^{n+1}})$.

We omit details concerning the measurability of our construction because the construction is given by explicit formulae.

It remains to verify condition 4. Let us use the following notation

$$B_n' = \bigcup_{x \in A_{n+1}} \quad \bigcup_{y \; : \; C_y^n \cap C_x^{n+1} \neq \emptyset} \quad \bigcup_{k \in K_y' \cap Z^m} \{S_{\phi_{n,y}(k)}^y\}$$

From the definition of $\phi_{n+1,x}$ (see formula (4)) it follows that $B_n' \subset B_n \cap B_{n+1}$. Let us estimate the measure of sets $B_n \setminus B_n'$ and $B_{n+1} \setminus B_n'$. Lemma 3 and the choice of r_n imply that

$$v_m(\Gamma_n(y) \setminus \Gamma'_{n,\frac{100.2^n.m}{\varepsilon}}(y)) \leq \frac{100.2^n.m.m!}{\varepsilon.r_n} \cdot v_m(\Gamma_n(y)) \leq \frac{1}{2^{n+2}} \cdot v_m(\Gamma_n(y))$$

The conditional measure on each element C_y^n is proportional to the image of the Euclidean volume in $\Gamma_n(y)$. Consequently conditional measure of the set

$$\phi_{n,y}(\Gamma_n(y) \setminus \Gamma'_{n, \frac{100.2^n.m}{\varepsilon}})$$

is less than $\frac{1}{2^{n+2}}$. Further the number of integer points in this difference is almost proportional to its volume and consequently

$$\text{Card } \{(B_n \setminus B'_n) \cap \phi_{n,y}(\Gamma_n(y))\} \leq \frac{1}{2^{n+1}} \text{ Card } \{B_n \cap \phi_{n,y}(\Gamma_n(y))\} \qquad (7)$$

Similarly

$$\text{Card } \{(B_{n+1} \setminus B'_n) \cap \phi_{n+1,x}(t(y) + \Gamma_n(y))\} \leq \frac{1}{2^{n+1}} \text{ Card } \{B_{n+1} \cap \phi_{n,y}(\Gamma_n(y))\} \qquad (8)$$

Now suppose that $\varepsilon < \frac{1}{2}$ and consider the measure ν on the sets B_n and B_{n+1} and conditional measures induced by this measure on a fixed element C of the partition ξ_{n+1} . These measures are uniform i.e. proportional to the number of elements in the intersections $C \cap B_n$ and $C \cap B_{n+1}$ so that inequalities (7) and (8) provide that condition 4 holds.

Conclusion of the proof of theorem 2.

So we have constructed the sequence of sections B_n . Now we are going to show that this sequence converges to the section B which admits a natural action of group $\mathbb{Z}^m$. This fact allows us to represent the given $\mathbb{R}^m$-action S as a special action over this action of $\mathbb{Z}^m$ on the Lebesgue space (B, ν) .

Denote the set $\bigcup\limits_{n=1}^{\infty} \bigcap\limits_{i=n}^{\infty} B_i$ by B . From condition 4 of proposition 3 it follows that

$$\nu(B \triangle B_n) \leq \sum\limits_{i=0}^{\infty} \nu(B_{n+i+1} \triangle B_{n+i}) \leq \frac{1}{2^{n-1}}$$

For every point $y \in B$ and every sufficiently large n there exists $x_n \in A_n$ and $k_n \in \Gamma_n(x_n)$ such that

$$y = S_{\phi_{n,x_n}(k_n)} x_n$$

Now fix an element $k \in \mathbb{Z}^m$. From the definition of the maps $\phi_{n,x}$ it follows

that for a.e. $y \in B$ $k_n + k \in \Gamma_n(x_n)$ for all sufficiently large n, and the difference

$$\phi_{n,x_n}(k_n+k) - \phi_{n,x_n}(k_n)$$

does not depend on n. Denote this difference by $h_k(y)$. Obviously $S_{h_k(y)} \, y \in B$. Moreover h_k is a cocycle i.e. for every $k_1, k_2 \in \mathbb{Z}^m$

$$h_{k_1+k_2}(y) = h_{k_1}(y) + h_{k_2}(S_{h_{k_1}(y)}y)$$

So we can define an action $T = \{T_n\}$ of $\mathbb{Z}^m$ on the set B which preserves the measure ν, namely for $y \in B$, $n \in \mathbb{Z}^m$ suppose

$$T_n y = S_{h_k(y)} y$$

From condition 2 of proposition 3 we have

$$(1+\epsilon)^{-1} \leq \frac{||h_k(y)||}{||k||} \leq 1+\epsilon \tag{9}$$

Now it is easy to prove that the given $\mathbb{R}^m$-action S is metrically iso-morphic to the special action over T generated by the cocycle $-h$ which is $(1+\epsilon)$-Lipschitz by (9). For consider the space $B \times \mathbb{R}^m$ with the $\mathbb{R}^m$-extension $\hat{T}$ of the action T generated by this cocycle :

$$\hat{T}_n(y,t) = (Ty, t-h_n(y))$$

and define the map $R : B \times \mathbb{R}^m \to X$ by

$$R(y,t) = S_t y \in X$$

Suppose that $R(y_1,t_1) = R(y_2,t_2)$, i.e. $S_{t_1}y_1 = S_{t_2}y_2$ or $S_{t_1-t_2}y_1 = y_2$.

But it means that for some $k \in \mathbb{Z}^m$

$$y_2 = T_k y_1 \, , \text{ i.e. } t_1-t_2 = h_k(y_1) \, .$$

Conversely if

$$(y,t) \in B \times \mathbb{R}^m, \quad k \in \mathbb{Z}^m$$

then

$$R(T_k y, t - h_k(y)) = S_{t-h_k(y)}(T_k y) = S_t y = R(y,t)$$

So the preimage $R^{-1}(x)$ of an arbitrary point $x \in X$ is precisely one orbit of $\hat{T}$ and consequently the map R establishes the one-to-one correspondence between the space of $\hat{T}$-orbits with the factor-action generated by the vertical action and the space X with the action S. The correspondence between the invariant measure is almost obvious so that we omit details. Theorem 2 is proved.

IV. Rochlin lemma

We mentioned in the previous section about the new and simple proof of Rochlin (or Halmos-Rochlin or Kakutani-Rochlin) uniform approximation lemma for $\mathbb{Z}^m$-actions. The result itself is not new; it is known even for more general groups and under more general assumptions (see [11] for example) but our proof is quite simple and demonstrates use of our methods. Now let us pass on precise formulations.

Rochlin lemma.

Let $T = \{T_n\}$ be a measurable, measure-preserving free action of the group $\mathbb{Z}^m$ on a Lebesgue space (X,μ), $k = (k_1 \ldots k_m) \in \mathbb{Z}^m$ an arbitrary vector, $\Pi_k = \{\ell = (\ell_1 \ldots \ell_m) : \ell_i \in Z, 0 \le \ell_i < k_i, i = 1 \ldots m\}$ a rectangle in $\mathbb{Z}^m$, $\varepsilon > 0$ an arbitrary small number. Then there exists a measurable set $E \subseteq X$ such that sets $T_\ell E$, $\ell \in \Pi_k$ are pair wise disjoint and

$$\mu(\bigcup_{\ell \in \Pi_k} T_\ell E) > 1 - \varepsilon$$

Proof.

We deduce this lemma from the simplified version of proposition 2 for $\mathbb{Z}^m$-actions. Notice that ergodicity is of no importance at the proof of this proposition. For we use only the fact that every trajectory of the action T intersects the set A i.e. that A is a global measurable section for T. But it is easy to provide this property in the following way: begin with the local section as in the proof of proposition 2, then consider the minimal invariant

set F_1 containing the set A , construct the local section for the invariant

set $X \setminus F_1$, define the set F_2 by a similar way and so on.

Introduce several notations. For a fixed vector $k = (k_1 \ldots, k_m) \in \mathbb{Z}^m$

denote by $\mathbb{Z}_k^m$ the sublattice $\mathbb{Z}_k^m = \{\ell = (\ell_1, \ldots, \ell_m) \in \mathbb{Z}^m, \ell_i = t_i k_i, t_i \in \mathbb{Z}, i = 1 \ldots m\}$,

and K_k the partition of $\mathbb{Z}^m$ into the sets $\Pi_k + \ell$, $\ell \in \mathbb{Z}_k^m$.

Fix the number $r > \dfrac{100 m \cdot m! \, ||k||}{\varepsilon}$ and apply the simplified version of propo-

sition 2. For every $x \in A$ consider all elements of partition K_k which are

contained in the set $\Gamma(x)$. Obviously the union of all such elements covers the

difference between the set $\Gamma(x)$ and $(m. ||k||)$ -neighbourhood of its boundary

so that by lemma 3 the conditional measure of the set $\{T_k\}$, (where k belongs

to this difference) on the element C_x of the partition ξ exceeds

$$1 - \frac{m! m. ||k||}{r} > 1 - \frac{\varepsilon}{100}$$

Suppose $E = \bigcup_{x \in A} \; \bigcup_{\ell \in \mathbb{Z}_k^m \, , \; \Pi_k + \ell \subset \Gamma(x)} \{T_\ell x\}$

The measurability of this set follows from proposition 2. Estimation of condi-

tional measures which we have done above provides that

$$\mu \left(\bigcup_{\ell \in \Pi_k} T_\ell E \right) > 1 - \varepsilon$$

At last by definition

$$k_1, k_2 \in \Pi_k \qquad k_1 \neq k_2$$

implies that

$$T_{k_1} E \cap T_{k_2} E = \emptyset$$

Lemma is proved.

References

1. A.B.Katok, Time change, monotone equivalence and standard dyanmical systems. Dokl. Akad. Nauk SSSR, 223 (1975), N 4, 789-792 = Sov. Math. Dokl., vol. 16 (1975), N 4, 986-990.

2. А.Б.Каток,Монотонная эквивалентность в эргодической теории, Известия АН СССР,сер.матем.,41/1977/,№1, 104 - 157.

3. Е.А.Сатаев,Инвариант монотонной эквивалентности,определяющий факторн от автоморфизмов,монотонно эквивалентннх сдвигу Бернулли,Изв.АН СССР,сер.матем.,41 /1977/,№ I, 158 -181.

4. J. Feldman, New K-automorphisms and a problem of Kakutani, Isr. J. of Math., 1976

5. B. Weiss. Equivalence of measure-preserving transformations, 1-8, preprint Hebrew University Jerusalem, 1976.

6. D. Rudolph, Non-equivalence of measure-preserving transformation, 1-5, preprint Hebrew University Jerusalem, 1976.

7. А.В.Кочергин,О гомологичности функций над динамическими системами,Доклады АН СССР,231/1976/ № 4,795 - 798.

8. W. Ambrose, S. Kakutani, Structure and continuity of measurable flows, Duke Math. J., 1942, 9, p. 25.

9. P.R. Halmos, Lecture on ergodic theory, Tokyo, 1956.

10. В.В.Немыцкий,В.В.Степанов,Качественная теория дифференциальннх уравнений,Москва-Ленинград ,1949 ,издание второе.

11. D. Ornstein, B. Weiss, The Kakutani-Rochlin theorem for solvable groups, preprint, 1976.

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