Astérisque

J. PINTZ

On the sign changes of $\pi(x) - \ell ix$

Astérisque, tome 41-42 (1977), p. 255-265

<http://www.numdam.org/item?id=AST_1977__41-42__255_0>

Société Mathématique de France
Astérisque 41-42 (1977) p.255-265

ON THE SIGN CHANGES OF $\pi(x) - \ell i\, x$

by

J. PINTZ

-:-:-:-

1. - Riemann [1] stated in his famous paper from 1859 that for $x>2$

$$(1.1) \qquad \pi(x) < \ell i\, x = \int_2^x \frac{dt}{\log t}$$

holds.

More than fifty years later Littlewood [2] disproved Riemann's assertion, showing that the function

$$(1.2) \qquad \Delta_1(x) \overset{\text{def}}{=} \pi(x) - \ell i\, x$$

has infinitely many sign changes. He even proved, that with the usual notation

$$(1.3) \qquad \Delta_1(x) = \Omega \pm \left(\frac{\sqrt{x}}{\log x} \log_3 x\right) \qquad (*)$$

which means, that there are real numbers

$$x'_1 < x'_2 < x'_3 < \ldots < x'_\nu < \ldots \to \infty$$
$$x''_1 < x''_2 < x''_3 < \ldots < x''_\nu < \ldots \to \infty$$

for which

$$(1.4) \qquad \begin{aligned} \pi(x'_\nu) - \ell i\, x'_\nu &> c\, \frac{\sqrt{x'_\nu}}{\log x'_\nu} \log_3 x'_\nu \\ \pi(x''_\nu) - \ell i\, x''_\nu &< -c\, \frac{\sqrt{x''_\nu}}{\log x''_\nu} \log_3 x''_\nu \end{aligned}$$

(*) $\log_\nu x$ denotes the ν times iterated logarithm function, i. e.

$$\log_1 x = \log x \quad , \quad \log_{\nu+1} x = \log(\log_\nu x) \ ;$$

$\exp_\nu(x)$ the ν times iterated exponential function.

hold with a given absolute constant c .

However the curious problem arose that Littlewood's proof was ineffective in the following sense : it was impossible to give any upper bound for the first sign change of $\Delta_1(x)$ by the original method of Littlewood.

Only 40 years later, in 1955 Skewes [3] succeeded to give the first effective upper bound

$$(1.5) \quad \exp_4(7,705)$$

for the first sign change of $\Delta_1(x)$.

2. - But after Littlewood's paper another interesting and deeper problem remained completely open : what can we say in general about the oscillatory character of $\Delta_1(x)$.

The first essential, though conditional, result was achieved by Ingham [4].

Let us denote by Θ the upper bound of the real parts of the zeros of $\zeta(s)$.

Then Ingham's theorem asserts that if Θ is attained, i. e. if $\zeta(s)$ has a zero on the line $\sigma = \Theta$, then there exists an absolute constant c_1 such that for $Y > Y_1$, the interval

$$(2.2) \quad [c_1 Y , Y]$$

contains a sign change of $\Delta_1(x)$.

Let us denote by $V_1(Y)$ the number of sign changes of $\Delta_1(x)$ in the interval $[2, Y]$. Then Ingham's theorem asserts also, that if Θ is attained the inequality

$$(2.3) \quad V_1(Y) > c_2 \log Y$$

holds for $Y > Y_1$.

However, these results have two disadvantages. First they are conditional and the condition is very deep. Namely, as we know that $\zeta(s)$ has no zero on the line $\sigma = 1$, Ingham's condition implies the quasi Riemann-hypothesis.

Secondly, Ingham's proof similarly to Littlewood's one is also ineffective, although he didn't use the Phragmen-Lindelöf theorem. All the constants in the formulae (2.2) and (2.3), i. e. c_1 , c_2 , Y_1 are not effectively computable, even if we suppose the Riemann hypothesis instead of Ingham's condition.

3. - The first unconditional lower bound for $V_1(Y)$ was given by Knapowski [5] in 1961-62. Using Turán's method he proved the ineffective inequality

$$(3.1) \quad V_1(Y) > c_3 \log\log Y \qquad \text{for } Y > Y_2$$

(with ineffective Y_2) and the weaker effective inequality

$$(3.2) \quad V_1(Y) > c_4 \log_4 Y \qquad \text{for } Y > Y_3$$

(with effective c_4 and Y_3).

These were the best results in this direction until 1974, when Turan [6] in a joint work with Knapowski proved the unconditional but ineffective result, according to which $\Delta_1(x)$ changes his sign in every interval of the form

$$(3.3) \quad [\, Y \exp\{-\log^{3/4} Y (\log\log Y)^4\}, Y]$$

for $Y > Y_4$ (ineffective constant).

(3.3) implies for the number of sign changes of $\Delta_1(x)$ the ineffective inequality

$$(3.4) \quad V_1(Y) > c_5 \frac{(\log Y)^{1/4}}{(\log\log Y)^4}$$

for $Y > Y_5$ (ineffective constant).

The method of proving (3.3) and (3.4) led also to an improvement of Knapowski's effective inequality (3.2) ; Turan proved in 1976 [7]

$$(3.5) \quad V_1(Y) > c_6 \log_3 Y \qquad \text{for } Y > Y_6$$

(with effective c_6 and Y_6).

This result was recently improved by the author [8] to

$$(3.6) \quad V_1(Y) > c_7 (\log\log Y)^{c_8} \quad \text{for } Y > Y_7$$

(with effective constants c_7, c_8 and Y_7).

4.- Another problem raised by Littlewood's theorem was whether we can give longer intervals for which the average of $\Delta_1(x)$ is large, if possible $\frac{\sqrt{x}}{\log x} \log_3 x$, indicated by Littlewood's theorem.

The problem was dealt with by Jurkat in 1971, [9]. Let us denote by $d(x)$ the function

$$(4.1) \quad d(x) = \frac{x}{\log\log x} .$$

With this notation Jurkat's result is that, supposing the Riemann hypothesis is true, one has

$$(4.2)\quad \frac{1}{d(x)} \int_x^{x+d(x)} \Delta_1(t)\,dt = \Omega \pm \left(\frac{\sqrt{x}}{\log x} \log_3 x\right)$$

which is a conditional sharpening of Littlewood's unconditional result (1.3).

5. - In this paper we shall deal with problems of the mentioned kind. First we shall introduce some notations and expose the problems with these notations.

PROBLEM I. - To give lower bound for $V_1(Y)$ defined as the number of sign changes of $\Delta_1(x) = \pi(x) - \ell i x$ in the interval $[2, Y]$.

PROBLEM II. - To give lower bound for $V_1'(Y)$, the number of "big sign changes" of $\Delta_1(x)$ in $[2,Y]$ by which we mean, that there exist numbers

$$(5.1)\qquad x_1' < x_1'' < x_2' < x_2'' < \ldots < x'_{V_1'(Y)} < x''_{V_1'(Y)} < Y$$

for which we have

$$(5.2)\qquad \Delta_1(x_\nu') > \frac{1}{100} \frac{\sqrt{x_\nu'}}{\log x_\nu'} \log_3 x_\nu'$$

and

$$(5.3)\qquad \Delta_1(x_\nu'') < -\frac{1}{100} \frac{\sqrt{x_\nu''}}{\log x_\nu''} \log_3 x_\nu''$$

respectively.

PROBLEM III. - To give lower bound for $V_1''(Y)$, the number of "big sign changes of the average of $\Delta_1(x)$ in long intervals" in $[2, Y]$ by which we mean that with the notation

$$(5.4)\qquad d(x) = \frac{x}{\log\log x}$$

there exist numbers

$$x_1' < x_1'' < x_2' < x_2'' < \ldots < x'_{V_1''(Y)} < x''_{V_1''(Y)} < Y$$

for which the inequalities

$$(5.5)\qquad \frac{1}{d(x_\nu')} \int_{x_\nu'}^{x_\nu'+d(x_\nu')} \Delta_1(t)\,dt > \frac{1}{100} \frac{\sqrt{x_\nu'}}{\log x_\nu'} \log_3 x_\nu'$$

and

$$(5.6)\qquad \frac{1}{d(x_\nu'')} \int_{x_\nu''}^{x_\nu''+d(x_\nu'')} \Delta_1(t)\,dt < -\frac{1}{100} \frac{\sqrt{x_\nu''}}{\log x_\nu''} \log_3 x_\nu''$$

hold.

PROBLEM IV. - Localisation of the sign changes of $\Delta_1(x)$ i. e. to give a function $A_1(Y)$ for which $\Delta_1(x)$ has a sign change in every interval

$$(5.7) \quad [A_1(Y), Y] \qquad \text{for} \quad Y > Y_o .$$

PROBLEM V. - Localisation of "big sign changes" of $\Delta_1(x)$ in an interval

$$(5.8) \quad [A_1'(Y), Y] \qquad \text{for} \quad Y > Y_o .$$

PROBLEM VI. - Localisation of "big sign changes" of the average of $\Delta_1(x)$ in long intervals" in an interval

$$(5.9) \quad [A_1''(Y), Y] \qquad \text{for} \quad Y > Y_o .$$

The meaning of Problems V and VI is completely analogous to Problems II and III (with the same $d(x)$ and with the same estimates) i. e. to prove that in every interval of the form (5.8) and (5.9) resp. there exist numbers x' and x'' for which (5.2)-(5.3) or (5.5)-(5.6) resp. hold.

6. - All these problems can be considered under some conditions on the zeros of the $\zeta(s)$ function or unconditionally, i. e. we can distinguish the two cases : if we suppose the Riemann hypothesis to be true or if we don't suppose anything. In many cases the Riemann hypothesis could be substituted by the weaker condition given by Ingham, but for the sake of simplicity we shall assume always the Riemann hypothesis instead of Ingham's condition.

Another question is the very important problem of effectivisation. A theorem will be called effective, if in case of Problems I-III we give a function $f^{(i)}(Y)$, for which we assert

$$(6.1) \quad V_1^{(i)}(Y) > f^{(i)}(Y) \qquad \text{for} \quad Y > Y_o$$

where all the constants appearing in $f^{(i)}(Y)$, as well as Y_o are effectively computable. In case of Problems IV-VI the theorem will be effective, if all the constants appearing in the function $A_1^{(i)}(Y)$ and the corresponding Y_o are effectively computable. In all the other cases, i. e. if at least one constant is not effectively calculable we shall call the theorem ineffective.

In the following we shall abbreviate the Riemann hypothesis by RH , ineffective theorems by IE , effective theorems by E . Now we quote the known results with the new notations. The number behind the assertion denotes the number of the Problem (These theorems and the later ones are valid for $Y > Y_o$, where the value of Y_o is naturally different in different theorems, but for the sake of simplicity we don't repeat it in every case).

1914, Littlewood, IE : $\lim_{Y \to \infty} V_1(Y) = \infty$ (I)

1914, Littlewood, IE : $\lim_{Y \to \infty} V_1'(Y) = \infty$ (II)

1936, Ingham, RH, IE : $V_1(Y) \gg \log Y$ (I)

1936, Ingham, TH, IE : $A_1(Y) = c_1 Y$ (IV)

1961-62, Knapowski, IE : $V_1(Y) \gg \log\log Y$ (I)

1961-62, Knapowski, E : $V_1(Y) \gg \log_4 Y$ (I)

1971, Jurkat, RH, IE : $\lim_{Y \to \infty} V_1''(Y) = \infty$ (III)

1974, Turan-Knapowski, IE : $V_1(Y) \gg \frac{(\log Y)^{1/4}}{(\log\log Y)^4}$ (I)

1974, Turan-Knapowski, IE : $A_1(Y) = Y \exp\{-(\log Y)^{1/4} (\log\log Y)^4\}$ (IV)

1976, Turan-Knapowski, E : $V_1(Y) \gg \log_3 Y$ (I)

1976, Pintz, E : $V_1(Y) \gg (\log_2 Y)^{c_2}$ (I)

8. - Now we shall announce some theorems concerning Problems I-VI.

The effective improvement of previous results on Problem I is (we remember the reader that all these Theorems are valid for $Y > Y_o$).

THEOREM 1. - E : $V_1(Y) \gg \frac{\sqrt{\log Y}}{\log\log Y}$

Being valid in an essentially unchanged form for Problem II as

THEOREM 2. - E : $V_1'(Y) \gg \sqrt{\log Y} \cdot \exp(-\sqrt{\log_2 Y})$

i. e. we can guarantee on the whole the same number of "big sign changes" as the sign changes at all.

Concerning the localisation problem of number IV, we have the effective

THEOREM 3. - E : $A_1(Y) = Y^c$

Being also valid in a somewhat weaker form for Problem V

THEOREM 4. - E : $A_1'(Y) = Y^{\exp(-\sqrt{\log_2 Y})}$.

If we are satisfied with unconditional but ineffective theorems, we can say more in the following cases.

THEOREM 5. - IE : $V_1(Y) \geq V_1'(Y) \geq V_1''(Y) \gg \frac{\sqrt{\log Y}}{(\log\log Y)^2}$

and in the case of the localisation problem number IV.

THEOREM 6. - IE : $A_1(Y) = Y \exp(-\sqrt{\log Y} \log\log Y)$

further for problemx V and VI we assert

THEOREM 7. - IE : $A_1'(Y) \geq A_1''(Y) \geq Y^{\exp(-\sqrt{\log_2 Y})}$.

9. - All the mentioned unconditional theorems, and some others, mostly conditional ones could be written in the following table. We have 6 problems, and one can give 4 possible answers for each problem, namely

ineffective answer supposing the Riemann hypothesis to be true	(RH, IE)
effective answer supposing the Riemann hypothesis to be true	(RH, E)
ineffective answer without any condition	(IE)
effective answer without any condition	(E)

(For the sake of completeness we included in this table also the results of Ingham, as they are still the best results.)

10. - We can see from the table that in some cases we have far more better results if we suppose the RH, in other cases we don't gain anything by it.

It also turns out that in some cases we can't improve the best effective results even in an ineffective way, whereas in other cases ineffective methods lead to drastic improvements.

As to the connnection between the localisation problems and the corresponding lower bound of sign changes it is easy to see that in some cases the lower bound is a consequence of the result of the corresponding localisation problem, but in most cases the consequences of the localisation problem could furnish only a far weaker lower bound for $V_1^{(i)}(Y)$ (essentially an estimate with $\log_{\nu+1} Y$ instead of $\log_\nu Y$ i. e. a function with a once more iterated logarithm function).

One can also observe that in some cases, we can get as strong results if we look for a "big sign change" of $\Delta_1(x)$ or for "big sign change of $\Delta_1(x)$ in average in long intervals" as in the simple sign change problem ; in other cases far better results could be proved for the Problem I and IV than for Problems II and V , and again weaker results for Problems III and VI.

Problems	RH , IE	RH , E	IE	E
I $V_1(Y) \gg$	$\log Y$ (Ingham)	$\log Y$	$\frac{\sqrt{\log Y}}{\log\log Y}$	$\frac{\sqrt{\log Y}}{\log\log Y}$
II $V_1'(Y) \gg$	$\frac{\log Y}{\exp(\sqrt{\log_2 Y})}$	$\frac{\log Y}{\exp(\sqrt{\log_2 Y})}$	$\frac{\sqrt{\log Y}}{\log\log Y}$	$\frac{\sqrt{\log Y}}{\exp(\sqrt{\log_2 Y})}$
III $V_1''(Y) \gg$	$\frac{\log Y}{\exp(\sqrt{\log_2 Y})}$	$\frac{\log Y}{\exp(\sqrt{\log_2 Y})}$	$\frac{\sqrt{\log Y}}{(\log\log Y)^2}$	$\frac{\log\log Y}{\exp(\sqrt{\log_3 Y})}$
IV $A_1(Y) =$	$c_o Y$ (Ingham)	Y^{c_1}	$Y \exp(-\log^{\frac{1}{2}} \log_2 Y)$	Y^{c_2}
V $A_1'(Y) =$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$Y^{\exp(-\sqrt{\log_2 Y})}$
VI $A_1''(Y) =$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$Y^{\exp(-\sqrt{\log_2 Y})}$	$(\log Y)^{\exp(-\sqrt{\log_3 Y})}$

11. - The method of the proofs in case of conditional theorems is the method of Ingham with some modifications, which made possible to effectivise Ingham's theorems and some other new ideas which led to the extension of the original results on the more general Problems II, III and V, VI.

In case of unconditional Theorems we use the method of Turán-Knapowski ([6], [7]) ; again some technical refinements were necessary to improve the earlier results and to generalise it to other problems ; but all the main ideas of their proofs remained unchanged.

12. - The applied methods are also convenient to prove all the results of the table for the closely related functions

$$(12.1)\quad \Delta_2(x) = \sum_{\nu\geq 1} \frac{1}{\nu}\, \pi(x^{\frac{1}{\nu}}) - \ell i\, x \overset{\text{def}}{=} \Pi(x) - \ell i\, x$$

$$(12.2)\quad \Delta_3(x) = \sum_{n\leq x} \Lambda(n) - x = \psi(x) - x$$

$$(12.3)\quad \Delta_4(x) = \sum_{p\leq x} \log p - x$$

(where in case of $\Delta_3(x)$ and $\Delta_4(x)$ the Ω-type theorems are modified naturally to $\Omega \pm (\sqrt{x}\, \log_3 x)$).

The only change is (except that some theorems could be proved easier) that for the functions $\Delta_2(x)$ and $\Delta_3(x)$ we can give a better effective value for the localisation Problem number IV supposing the RH, i. e.

$$\text{RH , IE} : A_2(Y) = A_3(Y) = c_3 Y .$$

All the results of the table remain unchanged if we consider the function

$$(12.4)\quad \pi_1(x) - \pi_3(x) = \sum_{\substack{p\leq x \\ p\equiv 1\ (4)}} 1 - \sum_{\substack{p\leq x \\ p\equiv 3\ (4)}} 1$$

instead of $\Delta_1(x)$ or the correspondingly modified functions

$$(12.5)\quad \Pi_1(x) - \Pi_3(x)$$

$$(12.6)\quad \psi(x,4,1) - \psi(x,4,3)$$

$$(12.7)\quad \sum_{\substack{p\leq x \\ p\equiv 1\ (4)}} \log p - \sum_{\substack{p\leq x \\ p\equiv 3(4)}} \log p .$$

The method is also applicable if one considers the difference of the number of primes in two arithmetical progressions up to a given x, but only for those pairs of progressions, for which earlier results were already reached by Turan and Knapowski [10].

13. - All the results of the table would be contained in a theorem, which would assert without any condition

$$(13.1) \quad A_1''(Y) = c_4 Y \qquad \text{for} \quad Y > Y_0$$

with effectively computable constants c_4 and Y_0.

Thus, supported also by some heuristic arguments we may express using our previous notations the

CONJECTURE. - E : $A_1''(Y) = cY$ for $Y > Y_0$.

14. - Finally the author would like to express his gratitude to Professor Paul Turán for his helpful advices.

-:-:-:-

REFERENCES

[1] B. RIEMANN, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatshefte d. Press. Akad. d. Wissens. Berlin 1859-1860.

[2] J.E. LITTLEWOOD, Sur la distribution des nombres premiers, Comptes Rendus, Paris, vol. 158 (1914) 1869-1872.

[3] S. SKEWES, On the difference $(\pi(x) - \ell i\, x)$, Proc. of the Lond. Math. Soc. vol. 5 (1955) 48-70.

[4] A.E. INGHAM, A note on the distribution of primes, Acta Arith. vol. 1, fasc. 2 (1936) 201-211.

[5] S. KNAPOWSKI, On the sign changes in the remainder term in the prime number formula, Journ. of Lond. Math. Soc. vol. 36 (1961) 451-460 and On the sign changes of the difference $(\pi(x) - \ell i\, x)$, Acta Arithm. vol. VII, fasc. 2 (1962) 107-120.

[6] S. KNAPOWSKI and P. TURÁN, On the sign changes of $(\pi(x) - \ell i\, x)$ I, to appear in a volume of the Colloquia Math. Societatis Jãnos Bolyai (Proc. of the Colloquium in Debrecen Oct. 1974).

[7] S. KNAPOWSKI and P. TURÁN, On the sign changes of $(\pi(x) - li\, x)$ II, to appear in Monatshefte für Mathematik.

[8] J. PINTZ, Bemerkungen zur voranstehenden Arbeit von S. Knapowski und P. Turán, to appear in Monatshefte für Mathematik.

[9] W.B. JURKAT, On the Mertens conjecture and related general Ω-theorems, Proceedings of Symposia in Pure Mathematics, vol. 24, American Mathematical Society, 1973.

[10] S. KNAPOWSKI and P. TURÁN, Comparative prime number theory I-VIII, Acta Math. Hung. 13 (1962), 299-364, 14 (1963) 31-78 and 241-268.

-:-:-:-

J. PINTZ
Eötvös Lorand University
Dept. of Algebra and Number Theory
1088 Budapest, Muzeum krt 6-8,
Hungary