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# HOMOLOGY AND THE ZETA FUNCTION FOR DIFFEOMORPHISMS

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One of the interesting problems in smooth dynamical systems is to relate the dynamics to the geometry or topology of the manifold on which it occurs. In the case of discrete time systems, i.e. diffeomorphisms, an important invariant in this study is the zeta function of Artin and Mazur [1]. This is defined by $\zeta(t) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m)$ , where $N_m$ is the cardinality of the fixed point set of $f^m$ . If, as frequently happens, this is a rational function, then a finite set of complex numbers, the zeroes and poles of $\zeta$ determine all of the numbers $N_m$ .

We consider here diffeomorphisms of compact manifolds which satisfy Axiom A and the no-cycle property, which are described below, and survey the relation of their zeta functions and homological invariants.

We will not consider the closely related topics of the entropy conjecture or the generalized zeta function of Ruelle; however these are discussed in the remarks of Manning and Ruelle respectively, in these proceedings.

## §1. Axiom A diffeomorphisms with the no-cycle property.

We wish to study the structure of diffeomorphisms which satisfy Axiom A of Smale [12] and the no-cycle property, so we now briefly describe this class of diffeomorphisms.

Let $f : M \longrightarrow M$ be a $C^1$ diffeomorphism of a compact connected manifold $M$ . A closed f-invariant set $\Lambda \subset M$ is called <u>hyperbolic</u> if the tangent bundle of $M$ restricted to $\Lambda$ is the Whitney sum of two Df-invariant bundles, $T_\Lambda M = E^u(\Lambda) \oplus E^s(\Lambda)$ , and if there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n(v)| \leq C\lambda^n |v| \quad \text{for} \quad v \in E^s \ , \ n > 0$$

and

$$|Df^{-n}(v)| \leq C\lambda^n |v| \quad \text{for} \quad v \in E^u \ , \ n > 0 \ .$$

The diffeomorphism $f$ is said to satisfy <u>Axiom A</u> if a) the non-wandering set of $f$ , $\Omega(f) = \{x \in M : U \cap \bigcup_{n>0} f^n(U) \neq \emptyset$ for every neighbourhood $U$ of $x\}$ is a hyperbolic set, and b) $\Omega(f)$ equals the closure of the set of periodic points of f. If f satisfies Axiom A , one has the spectral decomposition theorem of Smale [12] which says $\Omega(f) = \Lambda_1 \cup \ldots \cup \Lambda_\ell$ where $\Lambda_i$ are pairwise disjoint, f-invariant closed sets and $f|_{\Lambda_i}$ is topologically transitive.

These $\Lambda_i$ are called the <u>basic sets</u> of f and because f is topologically transitive on each basic set, the restrictions of the bundles $E^s$ and $E^u$ to $\Lambda_i$ have constant dimension. The fiber dimension of $E^u(\Lambda_i)$ is called the <u>index</u> of $\Lambda_i$ and will be denoted $u_i$ .

The basic sets $\Lambda_i$ have considerable structure which we illustrate by describing the structure of zero dimensional basic sets.

If $A$ is an $n \times n$ matrix of zeroes and ones we define $\Sigma_A \subset \prod_Z \{1,2,\ldots,n\}$ by $\Sigma_A = \{(x_i)_{i=-\infty}^{\infty} | x_i \in \{1,\ldots,n\}$ and $A_{x_i x_{i+1}} = 1$ for all $i\}$ If $\{1,\ldots,n\}$ is given the discrete topology and $\Sigma_A$ a topology as a subset of the product then $\Sigma_A$ is a compact metrizable space.

The shift homomorphism $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is defined by $\sigma((x_i)) = (x_i')$ where $x_i' = x_{i+1}$ (here $(x_i)$ denotes the bi-infinite sequence whose ith element is $x_i$) .

A result of Bowen [2] shows that on any zero-dimensional basic set $\Lambda$ , $f$ is topologically conjugate to some shift $\sigma : \Sigma_A \dashrightarrow \Sigma_A$ (the matrix $A$ is not unique however).

The no-cycle property [13] implies that it is possible to find submanifolds (with boundary and of the same dimension as $M$) ,

$M = M_\ell \supset \ldots \supset M_1 \supset M_0 = \emptyset$ such that

$M_{i-1} \cup f(M_i) \subset \text{int } M_i$ , and

$$\Lambda_i = \bigcap_{m \in Z} f^m(M_i - M_{i-1}) .$$

Henceforth we will consider only diffeomorphisms which satisfy Axiom A and the no cycle property and all theorems will be assumed to include this as part of the hypothesis unless otherwise stated.

The following result is valid without the no cycle property and is the basis of our subsequent remarks.

Theorem. (Guckenheimer [6] , Manning [7]).
*If* $f : M \longrightarrow M$ *satisfies* Axiom A *then its zeta function is rational.*

In fact the proofs show that the zeta function is the quotient of two integer polynomials with constant terms 1 and that the same holds true for the zeta function of $f$ restricted to a single basic set.

## §2. Filtrations and zeta functions

Since all periodic points are contained in the basic sets $\{\Lambda_i\}$ it is useful to restrict our attention to the zeta function of $f$ restricted to a single basic set $\Lambda_i$ .

Definition : $\zeta_i = \zeta(f|_{\Lambda_i}) = \exp\,(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m)$ , where $N_m$ cardinality of $\mathrm{Fix}(f^m) \cap \Lambda_i$ .

Example : If $f : \Lambda_i \longrightarrow \Lambda_i$ is topologically conjugate to a subshift of finite type $\sigma : \Sigma_A \longrightarrow \Sigma_A$ described above, then a theorem of Bowen and Lanford [3], says $\zeta_i = \zeta(\sigma) = \frac{1}{\det(I - At)}$.

A function closely related to $\zeta_i$ is defined as follows

$$\eta_i = \exp\,(\sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m)$$

where

$$\tilde{N}_m = \sum_{x \in \mathrm{Fix}(f^m) \cap \Lambda_i}' L(f^m, x) \ ,$$

and

$$L(f^m, x)$$

is the Lefschetz index of the fixed point $x$ of $f^m$ . In our situation one can show $L(f^m; x) = \pm 1$ depending on the sign of $\det(I - Df_x^m)$ , so if this sign were always $+$ , one would have $\zeta_i = \eta_i$ . The advantage of $\eta_i$ is that by means of the Lefschetz fixed point formula it is easily computed in terms of homological invariants of $f$ . A proof of the following proposition can be found in [5].

Proposition : $\eta_i(t) = \Pi \det(I - f_{*k} t)^{(-1)^{k+1}}$ , where $f_{*k} : H_k(M_i, M_{i-1} ; R) \hookleftarrow$ is induced by $f$ .

Thus relationships between $\zeta_i$ and $\eta_i$ relate the orbit structure of $f$ on $\Lambda_i$ to homological invariants of $f$ .

We will say that $f$ satisfies the <u>orientation assumption</u> on $\Lambda_i$ if the bundle $E^u(\Lambda_i)$ is orientable and $Df$ preserves this orientation. In fact, much of the following goes through with minor changes if $Df$ reverses orientation, but there are serious difficulties if $E^u(\Lambda_i)$ is not orientable or orientation is preserved for some parts and reversed for others. The following result can be found in [12] or as (2.5) of [5].

Theorem. (Smale)

*The orientation assumption implies that*

$$\zeta_i^i = \eta_i^{(-1)^{u_i}}$$

*where* $u_i$ *is the index of* $\Lambda_i$ .

Thus in this case the zeroes and poles of $\zeta_i$ are the reciprocals of the eigenvalues of $f_* : H_*(M_i, M_{i-1} ; R) \circlearrowleft$.

In [2] Bowen shows that the radius of convergence of $\zeta_i$ is $e^{-h}$ where $h$ is the toplogical entropy of $f$ restricted to $\Lambda_i$ (see [2] for a definition). In fact using techniques of Bowen and of Manning [8] it follows that the rational function $\zeta_i$ has a pole at $e^{-h}$ and that this is the closest pole to $0$ . Thus, $e^h$ is an eigenvalue of $f_* : H_*(M_i, M_{i-1} ; R) \circlearrowleft$ by the remarks above, when the orientation assumption holds.

In [10], Ruelle and Sullivan give a very beautiful explicit construction of this eigenclass and show that it occurs in dimension $u_i$ .

Theorem. (Ruelle-Sullivan)

*The orientation assumption implies that* $e^h$ *is an eigenvalue of* $f_* : H_{u_i}(M_i, M_{i-1}; R) \circlearrowleft$ *, where* $h$ *is the topological entropy.*

This theorem was generalized by Shub and Williams in [11] to obtain an eigenclass in the homology of a relative double cover without the orientation assumption.

The following theorem from [5] gives another approach to eliminating the orientation assumption.

Theorem.

*The following are equal:*

a) *The rational function* $\zeta_i^{(-1)^{u_i}}$ *with all coefficients reduced* mod 2 .

b) *The rational function* $\eta_i$ *with all coefficients reduced* mod 2 .

c) $\prod_{k=0}^{n} \det(I - f_{*_k} t)^{(-1)^{k+1}}$ *where* $f_{*_k} : H_k(M_i, M_{i-1} ; Z_2) \circlearrowleft$ *is induced by* f.

Thus the theorem of Smale remains true modulo 2 even without the orientation assumption. This motivates the following.

Definition.: The reduced zeta function $Z_i$ of the basic set $\Lambda_i$ is the rational function $\zeta_i$ with its coefficients reduced mod 2.

This makes sense because $\zeta_i$ is of the form $P(t)/Q(t)$ where both P and Q are polynomials with integer coefficients and constant term 1 .

The theorem above can then be interpreted as saying that the zeroes and poles of $Z_i$ (in the algebraic closure of $Z_2$) are the reciprocals of eigenvalues of $f_* : H_*(M_i, M_{i-1} ; Z_2) \circlearrowleft$ .

## §3. Zeta functions and $H_*(M)$.

Thus far we have related $\zeta_i$ to homological invariants of the filtration manifolds $M_i$ . It is much more valuable to establish relationships with $f_* : H_*(M) \longrightarrow H_*(M)$ , since it is not always easy to determine the filtration manifolds or their homology, the following theorem is a combination of results of [4] and [5].

Theorem.

*If* $f : M \longrightarrow M$ *satisfies* Axiom A *and the no-cycle property then*

*a)* $\Pi\, Z_i^{(-1)^{u_i}} = \prod_{k=0}^{n} \det(I - f_{*_k} t)^{(-1)^{k+1}}$ *, where* $f_{*_k} : H_k(M ; Z_2) \circlearrowleft$ *is induced by* $f$ .

*b) If the orientation assumption holds for all basic sets then* $\Pi\, \zeta_i^{(-1)^{u_i}} = \prod_{k=0}^{n} \det(I - f_{*_k} t)^{(-1)^{k+1}}$ *, where* $f_{*_k} : H_k(M;R) \circlearrowleft$ *is induced by* $f$ .

Part a) of the above theorem implies that $\Pi\, Z_i^{(-1)^{u_i}}$ depends only on the homotopy type of $f$ , and this leads to partial answers to several interesting questions :

1) When can an isotopy remove a basic set $\Lambda_i$ of $f$ while leaving all others unchanged? A necessary condition is that $Z_i(f) = 1$ .

2) When can an isotopy "cancel" two basic sets $\Lambda_i$ and $\Lambda_j$ leaving all others unchanged? A necessary condition is $Z_i^{(-1)^{u_i}} \cdot Z_j^{(-1)^{u_j}} = 1$ .

3) When can an isotopy of $f$ to $g$ change a basic set $f : \Lambda_i \longrightarrow \Lambda_i$ to a different basic set $g : \Lambda_i' \longrightarrow \Lambda_i'$ , leaving others unaltered? A necessary condition is $Z_i(f)^{(-1)^{u_i}} = Z_i(g)^{(-1)^{u_i}}$ .

All of these problems can be seen as a generalization of the problem of simplifying a Morse function by cancelling critical points.

One can also obtain a necessary condition for a collection of abstract basic sets of a diffeomorphism of a manifold $M$ in any homotopy class.

Theorem. [5]

$\Sigma(-1)^{u_i} \deg Z_i = -X(M)$ , *where* $X(M)$ *is the Euler characteristic of* $M$ .

§4. Morse inequalities.

There are further relations between zeta functions and the homology of $M$ which are analogous to the Morse inequalities which relate the number of critical points of a Morse function on $M$ and the dimension of the homology groups of $M$, (see, for example [9]).

Recall that these inequalities say

$$C_q - C_{q-1} + \dots \pm C_o \geq B_q - B_{q-1} + \dots \pm B_o ,$$

where $C_j$ is the number of critical points of index $j$ and $B_j$ is the dimension of $H_j(M ; R)$.

To prove similar results we need dimension restrictions on the $\Lambda_i$ or in their global stable and unstable manifolds $W^u(\Lambda_i)$, (see [12] for a definition). Specifically, we will say that $f$ satisfies the dimension restrictions for $q$, if it is true that each basic $\Lambda_i$ with index $u_i \leq q$ satisfies $\dim W^u(\Lambda_i) \leq q$ and each basic set $\Lambda_j$ with index $u_j > q$ satisfies $\dim W^s(\Lambda_j) < n-q$, where $n = \dim M$. Roughly these restrictions guarantee that the basic sets can be divided into two groups, those which contribute only to homology in dimensions greater than $q$ and those which contribute to homology only in dimensions less than or equal to $q$. It is shown in [5] that the dimension restrictions are satisfied for all $q$ if $\dim \Lambda_i = 0$ for all $i$.

If we now consider an eigenvalue $\lambda$ on homology and set

$B_j(\lambda)$ = dim generalized eigenspace for $\lambda$ in $H_j(M ; R)$,

and

$C_j(\lambda) = \Sigma$ dim generalized eigenspace for $\lambda$ in $H_j(M_i, M_{i-1} ; R)$,

where the sum is over all $i$, then we have the following result.

Theorem.

*If* $C_q(\lambda)$ *and* $B_q(\lambda)$ *are as above, then*

$$C_q(\lambda) - C_{q-1}(\lambda) + \dots \pm C_o(\lambda) \geq B_q(\lambda) - B_{q-1}(\lambda) + \dots \pm B_o(\lambda) \quad .$$

We can consolidate these inequalities by considering the alternating products over $k$ of the terms $(1-\lambda t)^{C_k(\lambda)}$ and then the product over $\lambda$ of the results. The analogous product for the $B_k(\lambda)$ can be formed and one sees that it differs by a polynomial. In this way we can obtain the following result.

Theorem. [5].

*If the dimension restrictions hold for* $q$ , *then there is an integer polynomial* $P(t)$ *such that*

$$P(t)^{(-1)^q} \prod_{u_i \leq q} \eta_i = \prod_{o \leq k \leq q} \det(I - f_{*_k} t)^{(-1)^{k+1}}$$

*where* $f_{*_k} : H_k(M\ ;\ R) \circlearrowleft$ *is induced by* $f$ .

Using this result we can directly relate the $\zeta_i$ to the homology of $M$ . For example if orientation assumptions hold then by the theorem of Smale above $\eta_i = \zeta_i^{(-1)^{u_i}}$ so we have

$$P(t)^{(-1)^q} \prod_{u_i \leq q} \zeta_i^{(-1)^{u_i}} = \prod_{0 \leq k \leq q} \det(I - f_{*_k} t)^{(-1)^{k+1}} \quad .$$

Some applications of these inequalities can be found in [4] .

Inequalities relating $Z_i$ to $H_*(M)$ which do not require orientation assumptions can also be obtained from the theorem above by reducing mod 2 and using the equality $Z_i^{(-1)^{u_i}}$ = the mod 2 reduction of $\eta_i$ . The details of this can be found in [5] .

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