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A GENERALIZED RUELLE PERRON-FROBENIUS THEOREM AND SOME APPLICATIONS

Peter Walters

We show how some problems on uniqueness of equilibrium states and existence of invariant measures can be deduced from a theorem about Perron-Frobenius operators.

Let (X,d) be a compact metric space. Let $T:X \longrightarrow X$ be a continuous surjection. We shall assume T satisfies the following conditions a), b), and c).

a) T is positively expansive. ie. $\exists\ \delta>0$ such that $d(T^n x, T^n y) \leq \delta\ \forall n \geq 0$ implies $x = y$. An equivalent definition is to require the existence of an open cover $\{A_1,\dots,A_k\}$ of X for which $\bigcap_{n=0}^{\infty} T^{-n} A_{i_n}$ is either empty or a one point set for all choices of the sequence $\{i_n\}$ $1 \leq i_n \leq k$. Clearly for each $x \in X$ the set $T^{-1}x$ contains at most k points.

b) T is a local homeomorphism (ie. $\forall x \in X\ \exists$ an open neighbourhood U of x so that TU is open and $T:U \longrightarrow T(U)$ is a homeomorphism.)

c) For sufficiently small δ , $d(x,t) = \delta \Rightarrow d(Tx,Ty) \geq \delta$.

Let ε_0 be chosen so that

i) ε_0 is an expansive constant for T ,

ii) $\forall x \in X$ the ball $B_{\varepsilon_o}(x)$ of radius ε_o and centre x is so that $TB_{\varepsilon_o}(x)$ is open and $T:B_{\varepsilon_o}(x) \longrightarrow TB_{\varepsilon_o}(x)$ is a homeomorphism and

iii) Condition c) holds whenever $\delta \leq \varepsilon_o$.

Examples of transformations satisfying a), b), c).

1. Subshifts of finite-type.

   Here one can take the metric $d(\{x_n\}_o^\infty, \{y_n\}_o^\infty) = \frac{1}{k+1}$ if k is the least for which $x_k \neq y_k$.

2. Expanding maps. (Shub [9]).

   Here X is a compact manifold equipped with a Riemannian metric and T is differentiable and satisfies the property : $\exists \lambda > 1$ for which

$$\|DTv\| \geq \lambda \|v\| \quad \forall v \in UT_x X \,,\ x \in X .$$

Let $C(X)$ be the Banach space of all real valued continuous functions on X, with the supremum norm. We can define for each $\phi \in C(X)$ a Perron-Frobenius operator $\mathcal{L}_\phi : C(X) \longrightarrow C(X)$ by $\mathcal{L}_\phi f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y)$. $\mathcal{L}_\phi$ is linear and positive. The members of a subclass of these are particularly useful.

Let $G(T) = \{g \in C(X) \mid g > o$ and $\sum_{y \in T^{-1}x} g(y) = 1 \quad \forall x \in X\}$.

If $\phi = \log g$ then $\mathcal{L}_{\log g} f(x) = \sum_{y \in T^{-1}x} g(y) f(y)$ and we have

$$\mathcal{L}_{\log g} U_T = \text{id}, \text{ where } U_T f = f \circ T .$$

Let $M(X)$ denote the collection of all Borel probability measures on X and let $M(T)$ consist of the T-invariant ones. In the weak*-topology the convex set $M(X)$ is compact and $M(T)$ is a compact convex subset of $M(X)$. β will denote the σ-algebra of Borel subsets of X . An interesting subset of $M(T)$ is obtained from the following

Lemma 1. (Ledrappier [5])

Let $g \in G(T)$ and we write $\mathcal{L}$ instead of $\mathcal{L}_{\log g}$. If $m \in M(X)$ the following are equivalent:

i) $\mathcal{L}^* m = m$.

ii) $m \in M(T)$ and $E_m\left(f/_{T^{-1}\beta}\right)(x) = \sum_{z \in T^{-1}Tx} g(z)\, f(z)$ a.e.m. $\forall\, f \in L'(m)$.

iii) $m \in M(T)$ and $h_m(T) + m(\log g) \geq h_\mu(T) + \mu(\log g) \quad \forall\, \mu \in M(T)$

(ie. m is an equilibrium state for $\log g$)

A measure satisfying these properties is called a g-measure. If m is a g-measure we have

$$0 = h_m(T) + m(\log g) .$$

(This says that the pressure of $\log g$ is 0).

Suppose from now on that T also satisfies the following mixing condition:

d) $\forall \varepsilon > o \ \exists\, N > o$ such that $\forall x \in X \quad T^{-N}x$ is ε-dense in X .

For $\phi \in C(X)$, $\varepsilon > o$ and $n \in Z^+$ let

$$\operatorname{var}_n(\phi,\varepsilon) = \sup \left\{ |\ \phi(x) - \phi(y)\ | \ \ d(T^i x, T^i y) \leq \varepsilon \atop o \leq i \leq n-1 \right\} .$$

We then have the following result.

Theorem 2. (Keane [3] Walters [11])

Suppose $g \in G(T)$ *and* $\sum_{n=1}^{\infty} \mathrm{var}_n(\log g, \varepsilon_1) < \infty$ *for some* $\varepsilon_1 \leq \varepsilon_0$. *Then* $\mathcal{L}^n_{\log g} f \rightrightarrows \mu(f)$ $\forall f \in C(X)$. ($\rightrightarrows$ *denotes convergence in the supremum norm*). μ *is the unique* g-*measure for* T .

Theorem 3. (Bowen [2], Ratner [7], Walters [11])

Let g *be as in theorem 2. The measure-preserving transformation* (T,μ) *has a Bernoulli natural extension.*

One can relate $\mathcal{L}_\phi$ to some $\mathcal{L}_{\log g}$ by a theorem first proved by Ruelle for the full 2-shift.

Theorem 4. (Ruelle [8], Bowen [2] for the case of subshifts of finite type, Walters [11])

Suppose $\phi \in C(X)$ *and* $\sum_{n=1}^{\infty} \mathrm{var}_n(\phi, \varepsilon_1) < \infty$ *for some* $\varepsilon_1 \leq \varepsilon_0$. *Then* $\exists$ $\lambda > 0$, $\nu \in M(X)$, $h \in C(X)$ *such that* $h > 0$, $\nu(h) = 1$,

$$\mathcal{L}_\phi h = \lambda h, \quad \mathcal{L}_\phi^* \nu = \lambda \nu \quad \textit{and} \quad \frac{\mathcal{L}_\phi^n f}{\lambda^n} \rightrightarrows h.\nu(f) \quad \forall f \in C(X) .$$

Also h *satisfies* $\dfrac{h(x)}{h(y)} \leq \exp\left(\sum_{k+1}^{\infty} \mathrm{var}_n(\phi, \varepsilon_1) \right)$

whenever $d(T^i x, T^i y) \leq \varepsilon_1$ $0 \leq i \leq k-1$.

Remarks.

1. $\lambda > 0$ and $\nu \in M(X)$ are uniquely determined by the condition $\mathcal{L}^*_\phi \nu = \lambda \nu$.
2. One can define the pressure of T to be a function $P_T : C(X) \longrightarrow R$. One has the variational principle

$$P_T(\phi) = \sup_{\mu \in M(T)} \left[h_\mu(T) + \mu(\phi) \right]$$

(Walters [10]). We say m is an equilibrium state for ϕ if

$$h_m(T) + m(\phi) = P_T(\phi) .$$

Corollary 5 .

Let ϕ be as in theorem 4. The measure μ_ϕ , defined by $\mu_\phi(f) = \nu(h.f)$, is the unique equilibrium state for ϕ . μ_ϕ is the unique g-measure for $g = \frac{e^\phi h}{\lambda . hoT}$. The natural extension of (T,μ_ϕ) is a Bernoulli shift. μ_ϕ is positive on non-empty open sets and $\nu o T^{-n} \to \mu_\phi$ in $M(X)$.

$$P_T(\phi) = \log \lambda = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_\phi 1 .$$

If $\Psi \in C(X)$ also has $\sum_{n=1}^{\infty} \operatorname{var}_n(\Psi,\varepsilon_1) < \infty$ then

$\mu_\phi = \mu_\Psi \iff \phi - \Psi = foT - f + c$ for some $f \in C(X)$ and $c \in R$.

Applications.

1. Axiom A diffeomorphisms.

These results are described fully in Bowen [2]. We just state here two results which can be deduced using corollary 5 and the Bowen-Sinai theory of Markov partitions.

Theorem 6.

Let Ω_s *be a basic set for an* Axiom A *diffeomorphism* T *and let* $\phi \in C(\Omega_s)$ *be Holder continuous* (ie. $|\phi(x)-\phi(y)| \leq a\, d(x,y)^\theta$ *for some* $a,\theta > o$) . *There is a unique equilibrium state* μ_ϕ *for* ϕ . *If* $T|_{\Omega_s}$ *is topologically mixing then* μ_ϕ *is Bernoulli.*

Theorem 7.

If $\phi,\Psi : \Omega_s \longrightarrow R$ *are both Holder continuous then*

$$\mu_\phi = \mu_\Psi \iff \phi - \Psi = uoT - u + c$$

for some Holder $u : \Omega_s \longrightarrow R$ *and some constant* c .

2. Invariant measures for expanding maps.

Here X is a compact manifold with a Riemannian metric and $T:X\longrightarrow X$ is differentiable and satisfies $\|DTv\| \geq \lambda\|v\|$ for all tangent vectors v. Here λ is a constant larger than 1. The metric d on X will be the one obtained from $\| \ \|$. T satisfies a). b). c). d). (Shub [9]).

Let m be the normalized Riemannian measure on X defined by $\| \ \|$. We are seeking an invariant measure equivalent to m. $D_xT : T_xX \to T_{Tx}X$ is linear and we can take its determinant using the Riemannian metric and so define $T'(x) = \det(D_xT)$.

Define $\phi \in C(X)$ by $\phi(x) = \log \frac{1}{|T'(x)|}$

ϕ is C^{k-1} if T is C^k. We will assume T is C^2.

Lemma 8.

Let $h \in L'(m)$ and $m(h) = 1$. Then

$$h.m \in M(T) \iff \mathcal{L}_\phi h = h \quad \text{a.e. } m.$$

(By $h.m$ we mean the measure μ defined by $\mu(f) = m(h.f)$).

Since $\phi \in C'(X)$, and since for small ε_1

$$d(x,y)<\varepsilon<\varepsilon_1 \implies d(Tx,Ty) \geq \lambda d(x,y),$$

we get

$$\sum_{n=1}^{\infty} \operatorname{var}_n(\phi,\varepsilon) < \infty .$$

Hence we can apply theorem 4 and Corollary 5. Note that $\mathcal{L}_\phi^* m=m$ so that by remark 1 $\nu=m$ and $\lambda=1$. So theorem 4 asserts the existence of $h \in C(X)$ with $m(h) = 1$, $h>o$ $\mathcal{L}_\phi h=h$ and $\mathcal{L}_\phi^n f \Rightarrow h.m(f)$ $\forall f \in C(X)$. By corollary 5 or lemma 8 we know $\mu = h.m \in M(T)$. So μ is an invariant measure equivalent to m. We list other properties of μ.

1. $mT^{-n} \to \mu$ in $M(X)$ (Corollary 5).

2. (T,μ) has a Bernoulli natural extension. (Corollary 5)

3. $h_\mu(T) = \int \log |T'(x)| \; d\mu(x)$

$$= \lim_{n\to\infty} \mathcal{L}^n_\phi(\phi) \; .$$

This is because $P_T(\phi) = \log \lambda = o$ so

$0 = h_\mu(T) + \mu(\phi)$ so that $h_\mu(T) = -\mu(\phi)$

$$= \mu(\log |T'|) \; .$$

4. $m \in M(T) \iff \sum_{y \in T^{-1}x} \frac{1}{|T'(y)|} = 1. \; \forall \, x \in X$.

5. Suppose $m \in M(T)$. Then m is the measure with maximal entropy $\iff |T'(x)| \in Z^+ \; \forall \, x \in X$.

Most of these results have been obtained by Krzyzenski [4].

3. Mappings of $[0;1]$.

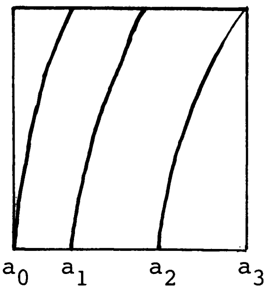
Let $T:[0,1] \to [0,1]$ be a map satisfying

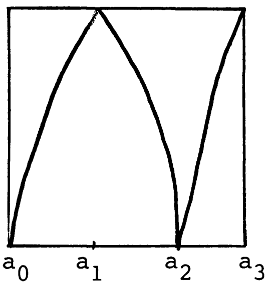
i) there is a partition $o=a_o<a_1<\ldots<a_p=1$ such that $T|_{(a_i,a_{i+1})}$ is C^2 and can be extended to a C^2 function on $[a_i,a_{i+1}]$ for each i .

ii) T maps each $[a_i,a_{i+1}]$ 1 - 1 onto $[0,1]$.

iii) $\exists \; \lambda>1$ such that $|T'(x)| \geq \lambda \;\; \forall \, x \in \bigcup_{i=o}^{p-1} (a_i;a_{i+1})$.

Examples are





Each example defines a continuous map of S^1 which is not smooth at a finite number of points. In our first example we could work as for expanding maps but in the second example conditions a), b) and c) are not satisfied. So we proceed in the usual way to use a shift system.

Let ζ denote the partition into the sets

$$\underset{A_1}{\underset{\shortparallel}{[o,a_1)}},\ \underset{A_2}{\underset{\shortparallel}{[a_1,a_2)}},\ldots\qquad,\ \underset{A_p}{\underset{\shortparallel}{[a_{p-1},1]}}\ .$$

Lemma 9.

ζ is a generator in the sense that each set of the form $\bigcap_{n=o}^{\infty} T^{-n}A_{i_n}$ contains at most one point.

Let $\Omega = \{1,2,\ldots,p\}^{Z^+}$. Define $j:[0,1]\to\Omega$ by

$$j(x) = (x_0,x_1,x_2,\ldots\ldots)\quad\text{if}\quad T^n x\in A_{x_n}\ .$$

j is 1-1 . $jT=\sigma j$ where σ is the shift on Ω .

Let $Y = [0,1]\setminus\bigcup_{n=o}^{\infty} T^{-n}\{a_o,a_1,\ldots,a_p\}$. $T^{-1}Y=Y$, $\sigma^{-1}j(Y)=j(Y)$.

Lemma 10.

j is a homeomorphism of Y with $j(Y)$.

j^{-1} extends to a continuous map $\pi:\Omega\to[0,1]$ $\pi\sigma = T\pi$.

Let m denote Lebesgue measure on $[0,1]$. Define

$$\Psi:Y\to R \quad\text{by}\quad \Psi(y) = \log\frac{1}{|\overline{T}'(y)|}\ .$$

Lift Ψ to $\phi=\Psi\circ\pi$ on $j(Y)$ and this can be extended to $\phi\in C(\Omega)$ with $\sum_{n=1}^{\infty}\mathrm{var}_n(\phi)<\infty$. m is concentrated on Y so $m\circ j^{-1}$ defines a measure on $j(Y)$ which defines a measure ν on Ω .

By definition of ϕ $\mathcal{L}^*_\phi\nu=\nu$. By theorem 4 we get

$h>o$ $h\in C(\Omega)$ with $\nu(h)=1$ and $\mathcal{L}^n_\phi f\Rightarrow h\nu(f)$ $\forall f\in C(\Omega)$.

Also $h.\nu\in M(\sigma)$.

$\mu=(h.\nu)\circ\pi^{-1}\in M(T)$

$(h.\nu)\circ\pi^{-1} = h\circ j.m$ on Y so $\mu=\ell.m\in M(T)$

where ℓ is continuous on Y and $\ell > 0$. Hence T has an invariant measure equivalent to m.

Of course (T,μ) is measure-theoretically isomorphic to $(\sigma, h\nu)$ and so has a Bernoulli natural extension. The properties listed for expanding maps also hold in this case. See Adler [1] for one of the sources of such results. Theorem 4 can be extended so that one can handle the case when ζ is a countable partition into intervals.

REFERENCES

[1] R.L. Adler, F-expansions revisited. Recent Advances in Topological Dynamics. Springer lecture notes. Vol. 318.

[2] R. Bowen, Equilibrium states and the Ergodic Theory of Anosov Diffeomorphisms. Springer lecture notes Vol. 470.

[3] M. Keane, Strongly mixing g-measures. Invent. Math. 16 (1972) 309 - 324.

[4] K. Krzyzewski, On expanding mappings. Bull. de l'académie Polonaise des sciences Vol. XIX No. 1 1971.

[5] F. Ledrappier, Principe variationnel et systèmes dynamiques symboliques. Commen.Maths.Phys. 33 (1973) pp 119 - 128.

[6] F. Ledrappier, Méchanique statistique de l'équilibre pour un revêtement. Preprint.

[7] M. Ratner, Anosov flows with Gibbs measures are also Bernoullian. Israel J. Math. To appear.

[8] D. Ruelle, Statistical mechanics of a one-dimensional lattice gas. Comm. Math. Phys. 9 (1968) 267 - 278.

[9] M. Shub, Endomorphisms of compact differentiable manifolds. A.J.M. XCI Jan. 1969.

[10] P. Walters, A variational principle for the pressure of a continuous transformation. To appear in A.J.M.

[11] P. Walters, Ruelle's operator theorem and g-measures. To appear in Trans. A.M.S.

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