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ON TOPOLOGICAL ENTROPY OF SEMIGROUPS OF COMMUTING TRANSFORMATIONS

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Table of contents.

0. Introduction.

The purpose of this paper is to study the topological entropy of semigroups of continuous commuting maps (c.c.m.) acting on a compact Hausdorff space. The semigroups considered are those isomorphic to $\mathbb{Z}_+^d$ or $\mathbb{R}_+^d$ or factors of these where d is a positive integer. Because of this isomorphism we shall speak sometimes of d-dimensional semigroups or d-dimensional semiflows. If the maps are homeomorphisms, i.e. if the semigroups are groups, the statements become sometimes sharper. Therefore results on groups of transformations isomorphic to $\mathbb{Z}^d$ or $\mathbb{R}^d$ or a factor of one of these are stated as well.

Measure-theoretic entropy of groups of invertible measure-preserving transformations acting on a Lebesgue space was investigated by several authors (Conze [5], Föllmer [8], Katznelson-Weiss [13], Pickel-Stepin [18], Thouvenot [20]). We refer to their work in sections 4, 5 and 6.

In [21], [24] and [27] the fundamental variational principle for $\mathbb{Z}_+^d$ actions is treated. Topological entropy is a special case of the notion of pressure discussed there.

Section 1 contains basic definitions and properties. It is of particular interest to know if and how the entropy of subsemigroups is related to that of the semigroup itself. Some answers to this question are given in section 2. Furthermore a first product theorem is proved there. In section 3 we state a number of results that are well-known theorems in the case $d=1$, i.e. in the theory of topological entropy of a single continuous map. Being familiar with sections 1 and 2 it is an exercise to translate the original proofs from the one-dimensional case to higher dimensions. Therefore almost all proofs are omitted. The only exception is Theorem (3.10) stating that the entropy of a semigroup equals the entropy of its restriction to the nonwandering set. Since the combinatorial part of Bowen's proof in dimension 1 (see [2]) becomes somewhat different in higher dimensions the interested reader will find a full proof in an appendix at the end of the paper. Section 4 is devoted to an extension of the notion of sequence entropy. In section 5 semigroups isomorphic to

$\mathbb{R}_+^d$ or a factor of it, i.e. d-dimensional semiflows, are considered. The first part of section 6 consists of the computation of the entropy of a d-dimensional shiftflow $S = (S_t)_{t \in \mathbb{R}^d}$ operating on a space $L^A(\mathbb{R}^d)$ of $[0,1]$-valued functions which satisfy a Lipschitz condition. The one-dimensional version of this space appears in Jacobs [12], Eberlein [7] and Denker-Eberlein [6]. We will use this flow in the second part of section 6 where we make an excursion to d-dimensional flows of measure-preserving transformations acting on a Lebesgue space $(\Omega, \mathcal{L}, m)$.

Using a d-dimensional Rokhlin theorem due to Lind [17] we state an existence theorem for generators (of the σ-algebra $\mathcal{L}$) with rather regular orbit properties. Via these generators we get the following result: Every d-dimensional flow of measure-preserving transformations acting on a non-atomic Lebesgue space can be considered - up to an isomorphism - as a d-dimensional flow of homeomorphisms operating on a compact metric space. More precisely the flow of homeomorphisms is a subflow of the shiftflow considered at the beginning of this section.

I would like to thank L. Goodwyn and P. Walters for some useful remarks and M. Misiurewicz for an improvement in section 2.

1. Definitions and basic properties.

Let X be a compact Hausdorff space. Given an open cover $\mathcal{O}$ of X we denote by $N(\mathcal{O})$ the cardinality of a minimal subcover of $\mathcal{O}$. $H(\mathcal{O}) = \log N(\mathcal{O})$ is called the <u>entropy of $\mathcal{O}$</u>. log is taken to the base 2. For two covers $\mathcal{O}, \mathcal{L}$ we write

$$\mathcal{O} \vee \mathcal{L} = \{A \cap B \mid A \in \mathcal{O}, B \in \mathcal{L}\} .$$

$\mathcal{L}$ is called <u>finer</u> than $\mathcal{O}$, in symbols $\mathcal{O} < \mathcal{L}$, iff every set of $\mathcal{L}$ is contained in a set of $\mathcal{O}$. For basic properties of the functions N and H with respect to "$\vee$" and "$\prec$" see [1]. In particular H is subadditive in the sense $H(\mathcal{A} \vee \mathcal{L}) \leq H(\mathcal{A}) + H(\mathcal{L})$.

$\mathbb{Z}^d$ denotes the d-dimensional lattice with its group structure, $\mathbb{Z}_+^d$ the subset of elements having all coordinates non-negative. $\tilde{\rho}$ stands for arbitrary subsets of $\mathbb{Z}^d$ with finite cardinality which is denoted by $|\tilde{\rho}|$. ρ stands for n-dimensional rectangles. In particular given $l = (l_1,\dots,l_d) \in \mathbb{Z}^d$ we denote by ρ_l the n-dimensional rectangle

$$\{k = (k_1,\dots,k_d) \in \mathbb{Z}^d \mid 0 \leq k_i < l_i \quad (1 \leq i \leq d)\} .$$

Given l and $\tilde{\rho}$ it is clear that $l + \tilde{\rho}$ means $\{l+k \mid k \in \tilde{\rho}\}$.
We write $l \to \infty$ if $\min_{1 \leq i \leq d} l_i \to \infty$.

Let ϕ_i $(1 \leq i \leq d)$ be continuous commuting maps (c.c.m.) of X into itself then $\phi = \langle \phi_1,\dots,\phi_d \rangle$ denotes the abelian semigroup generated by ϕ_i $(1 \leq i \leq d)$ under composition. There is a natural homomorphism λ of $\mathbb{Z}_+^d$ onto ϕ given by

$$\lambda : l = (l_1,\dots,l_d) \to \phi^l := \phi_1^{l_1} \circ \dots \circ \phi_d^{l_d} .$$

Thus ϕ is isomorphic to $\mathbb{Z}_+^d$ or a factor of it. If the transformations ϕ_i $(1 \leq i \leq d)$ are homeomorphisms then $\phi = \langle \phi_1,\dots,\phi_d \rangle$ is an abelian group isomorphic to $\mathbb{Z}^d$ or a factor of it.

Now fix for the rest of this chapter a semigroup $\phi = \langle \phi_1,\dots,\phi_d \rangle$ acting on X . Given a finite subset $\tilde{\rho} \subset \mathbb{Z}_+^d$ and an open cover $\mathcal{A}$ of X we write

$$\mathcal{A}_{\tilde{\rho}} = \bigvee_{k \in \tilde{\rho}} \phi^{-k} \mathcal{A}$$

1.1. Lemma.

Let $\mathcal{A}$ be an open cover, (l^n) a sequence in $\mathbb{Z}_+^d$, $l^n \to \infty$, then for any n-dimensional rectangle ρ_l and $\varepsilon > 0$ we have for n sufficiently large

$$|\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n}) \leq |\rho_1|^{-1} H(\mathfrak{A}\rho_1) + \varepsilon$$

Proof.

Cover ρ_{1^n} by translates $k + \rho_1$ of ρ_1 and use subadditivity of H .

1.2. Proposition.

For any open cover $\mathfrak{A}$ of X and any sequence (1^n) in $\mathbb{Z}^d_+$ such that $1^n \to \infty$

$$\lim_{n \to \infty} |\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n})$$

exists and is independent of the sequence (1^n) .

Proof.

Let (1^n) , $(\tilde{1}^n)$ be two sequences then by (1.1) for any m

$$\overline{\lim_{n\to\infty}} |\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n}) \leq |\rho_{\tilde{1}^m}|^{-1} H(\mathfrak{A}\rho_{1^m}) .$$

If m goes to infinity we get

$$\overline{\lim_{n\to\infty}} |\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n}) \leq \underline{\lim_{m\to\infty}} |\rho_{\tilde{1}^m}|^{-1} H(\mathfrak{A}\rho_{\tilde{1}^m})$$

Symmetry in (1^n) , $(\tilde{1}^n)$ yields existence and equality of the limits.

1.3. Definition.

$$h(\phi,\mathfrak{A}) := \lim_{n\to\infty} |\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n})$$

is called the entropy of $\mathfrak{A}$ with respect to the semigroup ϕ .

Using subadditivity of H we get for any n

$$|\rho_{1^n}|^{-1} H(\mathfrak{A}\rho_{1^n}) \leq H(\mathfrak{A}) .$$

Thus $h(\phi,\mathfrak{A})$ is a value in the interval $[0,H(\mathfrak{A})]$.

If ϕ is a group of homeomorphisms and we take a different system of generators for ϕ, say $\tilde{\phi}_1, \ldots, \tilde{\phi}_d$, then this corresponds to applying an automorphism A of $\mathbb{Z}^d$. A transforms n-dimensional rectangles ρ_{1^n} into parallelotopes and we see immediately using the same argument as in (1.1) that $h(\phi, \mathcal{A})$ does not depend on the generators for the group ϕ. It is clear that $\mathcal{A} < \mathcal{L}$ implies $h(\phi, \mathcal{A}) \leq h(\phi, \mathcal{L})$.

There is another way to compute $h(\phi, \mathcal{A})$ which will turn out to be useful. It is not necessary that the coordinates in (1^n) go simultaneously to infinity. Consider for some $\hat{d} \leq d$ the subsemigroup $\hat{\phi} = \langle \phi_1, \ldots, \phi_{\hat{d}} \rangle$. Denote $\hat{\phi}^c = \langle \phi_{\hat{d}+1}, \ldots, \phi_d \rangle$. Let (1^n) be a sequence in $\mathbb{Z}_+^d$, $1^n \to \infty$, and $k \in \mathbb{Z}_+^{(d-\hat{d})}$. Consider for

$$\mathcal{A}\rho_k = \bigvee_{1 \in \rho_k} (\hat{\phi}^c)^{-1} \mathcal{A}$$

$$h(\hat{\phi}, \mathcal{A}\rho_k) = \lim_{n \to \infty} |\rho_{1^n}|^{-1} H(\bigvee_{1 \in \rho_{1^n}} \hat{\phi}^{-1} \mathcal{A}\rho_k) .$$

<u>1.4. Proposition.</u>

For every sequence (k^m) in $\mathbb{Z}_+^{(d-\hat{d})}$, $k^m \to \infty$, we have

$$h(\phi, \mathcal{A}) = \lim_{m \to \infty} |\rho_{k^m}|^{-1} h(\hat{\phi}, \mathcal{A}\rho_{k^m}) .$$

<u>Proof.</u>

First observe that for any fixed k^m

$$|\rho_{k^m}|^{-1} h(\hat{\phi}, \mathcal{A}\rho_{k^m}) = \lim_{n \to \infty} |\rho_{(1^n, k^m)}|^{-1} H(\bigvee_{1 \in \rho_{1^n}} \hat{\phi}^{-1} \mathcal{A}\rho_{k^m})$$

1.1. implies that for any pair $(1^n, k^m)$

$$h(\phi, \mathcal{A}) \leq |\rho_{(1^n, k^m)}|^{-1} H(\mathcal{A}\rho_{(1^n, k^m)}) .$$

Both relations together yield

$$h(\phi, \mathfrak{A}) \leq \lim_{m\to\infty} |\rho_{k^m}|^{-1} h(\hat{\phi}, \mathfrak{A}\rho_{k^m}) .$$

To get the reverse inequality we note that again by (1.1) for every 1^n

$$h(\hat{\phi}, \mathfrak{A}\rho_{k^m}) \leq |\rho_{1^n}|^{-1} H(\bigvee_{1 \in \rho_{1^n}} \hat{\phi}^{-1} \mathfrak{A}\rho_{k^m}) .$$

Thus for any pair $(1^n, k^m) \in \mathbb{Z}_+^d$

$$|\rho_{k^m}|^{-1} h(\hat{\phi}, \mathfrak{A}\rho_{k^m}) \leq |\rho_{(1^n,k^m)}|^{-1} H(\bigvee_{1 \in \rho_{1^n}} \hat{\phi}^{-1} \mathfrak{A}\rho_{k^m}) .$$

If $(1^n, k^m) \to \infty$ we get

$$\overline{\lim_{n\to\infty}} |\rho_{k^m}|^{-1} h(\hat{\phi}, \mathfrak{A}\rho_{k^m}) \leq h(\phi, \mathfrak{A}) .$$

1.5. Definition.

$h(\phi) = \sup \{h(\phi, \mathfrak{A}) \mid \mathfrak{A} \text{ open cover}\}$ is called topological entropy of ϕ .

Let $\psi = \langle\psi_1, \ldots, \psi_d\rangle$ be another semigroup acting on a compact Hausdorff space Y . (Y,ψ) is called a homomorphic image of (X,ϕ) iff there is a continuous surjective map $\mathcal{J} : X \to Y$ such that $\mathcal{J} \circ \phi_i = \psi_i \circ \mathcal{J}$ $(1 \leq i \leq d)$. It is clear that under these conditions $h(\phi, \mathcal{J}^{-1}(\mathfrak{A})) = h(\psi, \mathfrak{A})$ for every open cover $\mathfrak{A}$ of Y . This implies

1.6. Property.

If (Y,ψ) is a homomorphic image of (X,ϕ) then $h(\psi) \leq h(\phi)$.

(Y,ψ) is isomorphic to (X,ϕ) , in symbols $(Y,\psi) \simeq (X,\phi)$, iff the map $\mathcal{J}$ above is a homeomorphism.

1.7. Property.

$$(X,\phi) \simeq (Y,\psi) \text{ implies } h(\phi) = h(\psi) .$$

A sequence $(\mathfrak{A}_n)$ of open covers of X is <u>refining</u> iff

1) $\mathfrak{A}_n < \mathfrak{A}_{n+1}$ and
2) for every open cover $\mathfrak{L}$ there exists on $\mathfrak{A}_n$ such that $\mathfrak{L} < \mathfrak{A}_n$.

The following property is an immediate consequence of the monotonicity of the function $h(\phi,\cdot)$ and simplifies the computation of $h(\phi)$.

<u>1.8. Property</u>.

If $(\mathfrak{A}_n)$ is a refining sequence of open covers then

$$h(\phi) = \lim_{n\to\infty} h(\phi, \mathfrak{A}_n) .$$

An open cover is called a <u>one-sided (topological) generator</u> for ϕ iff for every collection $(A_{i_l})_{l\in\mathbb{Z}_+^d}$ of elements of $\mathfrak{A}$ $\bigcap_{l\in\mathbb{Z}_+^d} \phi^{-l}\bar{A}_{i_l}$ consists of at most one point. In case ϕ is a group, $\mathfrak{A}$ is called a <u>(topological) generator</u> for ϕ iff the same holds with $\mathbb{Z}^d$ instead of $\mathbb{Z}_+^d$.

Rewriting Lemma 2.1., 2.3. and 2.5. in [14] in our situation of semigroups and groups of transformations yields the following result

<u>1.9. Theorem</u>.

If $\mathfrak{A}$ *is a one-sided generator or in case of* ϕ *being a group a generator then*

$$h(\phi) = h(\phi, \mathfrak{A}) .$$

The problem of existence of generators is solved by Keynes and Robertson [14]. There is a one-sided generator iff ϕ is positively expansive and in the group-case, there is a generator iff ϕ is expansive.

1.10. Example.

Let E be a finite set, $|E| = N$. $X = \{x | x: \mathbb{Z}^d \to E\}$ endowed with the product topology is a compact metric space. The shiftgroup $S = (S_l)_{l \in \mathbb{Z}^d}$ given by

$$(S_l(x))(k) = x(k+l) \quad (k,l \in \mathbb{Z}^d)$$

is a group of homeomorphisms isomorphic to $\mathbb{Z}^d$. Consider $[e]_o = \{x | x(0) = e\}$, then $\mathfrak{A}_o = \{[e]_o \mid e \in E\}$ is a generator for S. Thus $h(S) = h(S, \mathfrak{A}_o)$. An easy computation shows $h(S,\mathfrak{A}_o)=\log N$ which implies

$$h(S) = h(S, \mathfrak{A}_o) = H(\mathfrak{A}_o) = \log N .$$

Exactly the same holds if we consider $\tilde{X} = \{x | x : \mathbb{Z}_+^d \to E\}$ together with the corresponding semigroup $(\tilde{S}_l)_{l \in \mathbb{Z}_+^d}$ on it.

1.11 Example.

Let Y be a compact Hausdorff space and φ a homeomorphism of Y onto itself. Consider $X = \{x | x : \mathbb{Z}^d \to Y\}$ with the product topology and the shiftgroup $S = (S_l)_{l \in Z^d}$ acting on it. Define a homeomorphism ϕ_1 of X onto itself by

$$\phi_1(x)\ (k) = \varphi(x(k)) \quad (k \in \mathbb{Z}^d) .$$

The elements of S commute with ϕ_1. We claim the group $\phi'=\langle S,\phi_1\rangle$ satisfies

$$h(\phi') = h(\varphi) .$$

To see this define the projection $\pi_o : X \to Y$ by $\pi_o(x) = x(0)$. If $\mathcal{L}$ is an open cover of Y then $\pi_o^{-1}(\mathcal{L})$ covers X. Consider for a $q \in \mathbb{Z}_+$ the covering

$$\mathcal{L}_q = \bigvee_{j=o}^{q-1} \phi_1^{-j} \ (\pi_o^{-1} \mathcal{L})$$

then

$$H(\mathcal{L}_q) = H(\pi_o^{-1}(\bigvee_{j=o}^{q-1} \varphi^{-j}\mathcal{L})) = H(\bigvee_{j=o}^{q-1} \varphi^{-j}\mathcal{L}) \quad .$$

Observe that $h(S, \mathcal{L}_q) = H(\mathcal{L}_q)$.

Therefore

$$h(\varphi, \mathcal{L}) = \lim_{q\to\infty} q^{-1} H(\mathcal{L}_q) = \lim_{q\to\infty} q^{-1} h(S, \mathcal{L}_q) = h(\phi', \pi_o^{-1}\mathcal{L})$$

where the last equality follows by 1.4.

Taking the sup over all $\mathcal{L}$ we get

$$h(\varphi) \leq h(\phi') \quad .$$

Given any open cover $\mathcal{O}\!\mathcal{l}$ of X then by the definition of the product topology there is a n-dimensional rectangle ρ in $\mathbb{Z}^d$ and an open cover $\mathcal{L}$ of Y such that

$$\mathcal{O}\!\mathcal{l} < \bigvee_{k\in\rho} S^{-k}(\pi_o^{-1}\mathcal{L}) \quad .$$

This implies

$$h(\phi', \mathcal{O}\!\mathcal{l}) \leq h(\phi', \bigvee_{k\in\rho} S^{-k}(\pi_o^{-1}\mathcal{L})) = h(\phi', \pi_o^{-1}\mathcal{L}) = h(\varphi, \mathcal{L})$$

and we get $h(\phi') \leq h(\varphi)$.

2. Subsemigroups.

In this section we study subsemigroups $\tilde{\phi} = \langle \phi^{l^1}, \ldots, \phi^{l^q} \rangle$ of a fixed semigroup $\phi = \langle \phi_1, \ldots, \phi_d \rangle$ of c.c.m. . The vectors $l^i \in \mathbb{Z}_+^d$ $(1 \leq i \leq q)$ are assumed to be linearly independent, i.e. they cannot be written in the form

$$kl^j = \sum_{\substack{i=1 \\ i\neq j}}^{q} k^i l^i$$

for positive integers k and k^i .

Define the following equivalence relation on ϕ : $\phi^{l} \sim \phi^{l'}$ iff there exist $\phi^{k}, \phi^{k'} \in \tilde{\phi}$ such that $\phi^{l} \circ \phi^{k} = \phi^{l'} \circ \phi^{k'}$. The factorspace with respect to $\sim$ is again a semigroup.

We consider first the case of finite index. Let

$$^{p}\phi = \langle \phi^{l^1}, \dots, \phi^{l^d} \rangle \quad (l^i \in \mathbb{Z}_+^d \ , \ 1 \leq i \leq d)$$

be a subsemigroup of finite index p and Z^p its inverse image under the natural homomorphism λ . We choose a complete system of representatives in the factorsemigroup $\mathbb{Z}_+^d/Z^p$, say Q^p , by taking out of each equivalence class the element lying in the d-dimensional parallelotope $P(l^1,\dots,l^d)$ spanned by the vectors $l^1,\dots,l^d$ (In the case of elements on the boundary of $P(l^1,\dots,l^d)$ take the smaller ones).

Define $\text{comp}\,(Z^p) = Z^p + Q^p$ and $\text{comp}(^{p}\phi) = \lambda(\text{comp}(Z^p))$. $\text{comp}(^{p}\phi)$ is a semigroup which in general does not have generators. We note that it is a priori not clear how to define the entropy of the semigroup $\text{comp}(^{p}\phi)$. But since we have chosen d-dimensional rectangles ρ_{l^n} with $l^n \to \infty$ for the semigroup ϕ itself it seems natural to take d-dimensional parallelotopes $p_n \subset \mathbb{Z}_+^d$ with boundaries parallel to those of $P(l^1,\dots,l^d)$ and which cover $\text{comp}(Z^p)$ as $n \to \infty$. Then the same argument as in (1.2) shows that $h(\text{comp}(^{p}\phi))$ is independent of the sequence p_n and equals $h(\phi)$.

<u>2.1. Theorem.</u>

If $^{p}\phi = \langle \phi^{l^1}, \dots, \phi^{l^d} \rangle$ $(l^i \in \mathbb{Z}_+^d)$ *is a subsemigroup with finite index* p *then*

$$h(^{p}\phi) = p\, h(\phi) \ .$$

<u>Proof</u>.

Since $h(\phi) = h(\text{comp}(^{p}\phi))$ it is enough to consider the case

$$P_\phi = \langle \phi_1^{p_1}, \ldots, \phi_d^{p_d} \rangle \quad \text{for} \quad p_i > 0 \ (1 \le i \le d) \ .$$

Define $l(p) = (p_1, \ldots, p_d)$ and let $\rho_{l(p)}$ be the corresponding d-dimensional rectangle. Then

$$|\rho_{l(p)}| = p_1 \cdots p_d = p \ .$$

For any open cover $\mathfrak{A}$ of X we write

$$\mathfrak{A}^p = \bigvee_{l \in \rho_{l(p)}} \phi^{-l} \mathfrak{A} \ .$$

Given any $\varepsilon > 0$, $k \in \mathbb{Z}_+^d$ we get with the same argument as in (1.1) for sufficiently large n on one side

$$|\rho_{l^n}|^{-1} H(\bigvee_{l \in \rho_{l^n}} \phi^{-l} \mathfrak{A}) \le (p|\rho_k|)^{-1} H(\bigvee_{l \in \rho_k} (P_\phi)^{-l} \mathfrak{A}^p) + \varepsilon$$

which implies

$$p \, h(\phi, \mathfrak{A}) \le h(P_\phi, \mathfrak{A}^p)$$

and on the other side

$$|\rho_{l^n}|^{-1} H(\bigvee_{l \in \rho_{l^n}} (P_\phi)^{-l} \mathfrak{A}^p) \le p|\rho_k|^{-1} H(\bigvee_{l \in \rho_k} \phi^{-l} \mathfrak{A}) + \varepsilon$$

which implies

$$h(P_\phi, \mathfrak{A}^p) \le p \, h(\phi, \mathfrak{A}) \ .$$

So we have equality.

Taking the supremum over all open covers $\mathfrak{A}$ we obtain $p \, h(\phi) \le h(P_\phi)$. But given any open cover $\mathcal{L}$ we conclude

$$h(P_\phi, \mathcal{L}) \le h(P_\phi, \mathcal{L}^p) = p \, h(\phi, \mathcal{L}) \ .$$

Taking again the supremum over all open covers yields the reverse inequality.

Now we turn to subsemigroups $\hat{\phi} = \langle \phi^{l^1}, \ldots, \phi^{l^q} \rangle$ $(l^i \in \mathbb{Z}_+^d \ (1 \le i \le q))$ with infinite index, i.e. $q < d$. It is not hard to show directly that $h(\phi) \leq h(\hat{\phi})$. It should be emphasized that $h(\hat{\phi})$ of course means the entropy of the q-dimensional semigroup $\hat{\phi}$. We prove a stronger result.

2.2. Theorem.
Let $\hat{\phi} = \langle \phi^{l^1}, \ldots, \phi^{l^q} \rangle$ $(l^i \in \mathbb{Z}_+^d \ (1 \leq i \leq q))$ *be a subsemigroup with infinite index then* $h(\phi) > 0$ *implies* $h(\hat{\phi}) = \infty$.

Proof.

Consider first the particular case $\hat{\phi} = \langle \phi_1, \ldots, \phi_q \rangle$ for some $q < d$. Remember $\hat{\phi}^c$ means the semigroup $\langle \phi_{q+1}, \ldots, \phi_d \rangle$. If (k^m) is a sequence in $\mathbb{Z}_+^{(d-q)}$ such that $k^m \to \infty$ and $\mathfrak{A}$ an open cover of X then writing

$$\mathfrak{A}_{\rho_k m} = \bigvee_{k \in \rho_k m} (\hat{\phi}^c)^{-k} \mathfrak{A}$$

we get by 1.4.

$$h(\phi, \mathfrak{A}) = \lim_{m \to \infty} |\rho_k m|^{-1} h(\hat{\phi}, \mathfrak{A}_{\rho_k m}) .$$

Since $h(\phi) > 0$ we can find a cover $\mathfrak{A}$ such that $h(\phi, \mathfrak{A}) > 0$. Now $|\rho_k m| \to \infty$ as $m \to \infty$ and therefore $h(\hat{\phi}, \mathfrak{A}_{\rho_k m})$ is an unbounded sequence. We conclude $h(\hat{\phi}) = \infty$.

In the general case $\hat{\phi} = \langle \phi^{l^1}, \ldots, \phi^{l^q} \rangle$ we choose vectors $l^{q+1}, \ldots, l^d$ in $\mathbb{Z}_+^d$ such that $l^1, \ldots, l^q, l^{q+1}, \ldots, l^d$ are linearly independent vectors. Then

$$\tilde{\phi} = \langle \phi^{l^1}, \ldots, \phi^{l^d} \rangle$$

has finite index, say p , in ϕ and by 2.1 $h(\tilde{\phi}) = p\, h(\phi)$. $h(\phi) > 0$ implies therefore $h(\tilde{\phi}) > 0$. But with respect to $\tilde{\phi}$ the subsemigroup $\hat{\phi}$ has just the form required at the beginning.

2.3. Corollary

If $h(\phi_i) < \infty$ for some $i \in \{1, \ldots, d\}$ $(d \geq 2)$ then $h(\phi) = 0$.

Warning.

$h(\phi) = 0$ does not imply $h(\hat{\phi}) < \infty$ or even $h(\hat{\phi}) = 0$ for subsemigroups $\hat{\phi}$ of infinite index. See (2.5) for a counterexample.

Now we prove a first product theorem. Let $\phi = \langle\phi_1, \ldots, \phi_d\rangle$ be a semigroup acting on X and $\psi = \langle\psi_1, \ldots, \psi_{d'}\rangle$ be another one acting on Y. Given $l \in \mathbb{Z}_+^d$, $k \in \mathbb{Z}_+^{d'}$ we define the map $\phi^l \times \psi^k$ of $X \times Y$ into itself by

$$(\phi^l \times \psi^k)(x,y) = (\phi^l x, \psi^k y) \quad ((x,y) \in X \times Y) .$$

Denote by $\phi \otimes \psi$ the semigroup consisting of the family of maps $\{\phi^l \times \psi^k \mid l \in \mathbb{Z}_+^d, k \in \mathbb{Z}_+^{d'}\}$. $\phi \otimes \psi$ can be written in a different way. Namely if I_X and I_Y are the identity maps on X and Y respectively then

$$\phi \otimes \psi = \langle\phi_1 \times I_Y, \ldots, \phi_d \times I_Y, I_X \times \psi_1, \ldots, I_X \times \psi_{d'}\rangle .$$

2.4. Theorem.

$$h(\phi \otimes \psi) = 0 .$$

Proof.

Given an open cover $\mathcal{C}$ of $X \times Y$ there exist open covers $\mathcal{A}$ of X and $\mathcal{L}$ of Y such that $\mathcal{C} < \mathcal{A} \times \mathcal{L}$ (see [1] p. 312). Take sequences (l^n) in $\mathbb{Z}_+^d$, $l^n \to \infty$, (k^n) in $\mathbb{Z}_+^{d'}$, $k^n \to \infty$. Then

$$h(\phi \otimes \psi, \mathcal{C}) \leq h(\phi \otimes \psi, \mathcal{A} \times \mathcal{L})$$

$$= \lim_{n\to\infty} |\rho_{(l^n,k^n)}|^{-1} H(\bigvee_{l \in \rho_{l^n}} \phi^{-l}\mathcal{A} \times \bigvee_{k \in \rho_{k^n}} \psi^{-k}\mathcal{L})$$

$$\leq \lim_{n\to\infty} |\rho_{k^n}|^{-1}(|\rho_{l^n}|^{-1} H(\bigvee_{l \in \rho_{l^n}} \phi^{-l}\mathcal{A})) \quad +$$

$$+ \lim_{n\to\infty} |\rho_1 n|^{-1} (|\rho_k n|^{-1} \ H(\bigvee_{k \in \rho_k n} \psi^{-k} \mathcal{L}))$$

In both limits the first factor converges to 0 whereas the second expression converges to the finite values $h(\phi, \mathcal{A})$ resp. $h(\psi, \mathcal{L})$. Therefore

$$h(\phi \otimes \psi, \mathcal{C}) = 0 .$$

Since $\mathcal{C}$ was arbitrary we get

$$h(\phi \otimes \psi) = 0 .$$

2.5. Corollary.

$h(\phi) = 0$ does not imply that there is an $i \in \{1,\ldots,d\}$ for which $h(\phi_i) < \infty$ (of course $d \geq 2$) .

Proof.

Let X , Y be compact Hausdorff spaces and $\varphi : X \to X$, $\psi : Y \to Y$ be continuous maps both having infinite entropy. Consider the productsemigroup

$$\phi = \langle (\varphi \times I_Y) \ , \ (I_X \times \psi) \rangle \quad \text{on} \quad X \times Y .$$

2.4. implies $h(\phi) = 0$. On the other hand the producttheorem in [9] (Theorem 2) (see also (3.7) later) yields

$$h(\varphi \times I_Y) = h(\varphi) = \infty \text{ and}$$
$$h(I_X \times \psi) = h(\psi) = \infty .$$

3. General theorems.

The following three statements are well-known results in the case of a single continuous mapping φ of a compact Hausdorff space

X into itself [1] . Their generalization to semigroups $\phi = <\phi_1,\dots,\phi_d>$ of continuous commuting mappings of X into itself is straightforward.

3.1. Proposition.

Let X_1, X_2 be closed subsets of X such that $X = X_1 \cup X_2$ and $\phi^l x_i \in X_i$ $(x_i \in X_i, l \in \mathbb{Z}_+^d,\ i=1,2)$.

Then

$$h(\phi) = \max\{h({}^1\phi), h({}^2\phi)\}$$

where ${}^i\phi$ denote the restrictions of ϕ to X_i $(i=1,2)$.

3.2. Corollary.

Let X_1 be a closed subset of X such that

$$\phi^l x_1 \in X_1 \quad (x_1 \in X_1, l \in \mathbb{Z}_+^d)$$

then

$$h({}^1\phi) \leq h(\phi) \ .$$

3.3. Proposition.

Let $\sim$ be an equivalence relation on X compatible with ϕ in the sense $x \sim y$ implies $\phi_i x \sim \phi_i y$ $(1 \leq i \leq d)$. Define a semigroup $\tilde{\phi} = <\tilde{\phi}_1,\dots,\tilde{\phi}_d>$ on $X/\sim$ by $\tilde{\phi}_i \pi = \pi \phi_i$ $(1 \leq i \leq d)$ where π is the projection of X onto $X/\sim$. Then

$$h(\tilde{\phi}) \leq h(\phi) \ .$$

Given a directed set $\mathcal{J}$ let $(X_i)_{i \in \mathcal{J}}$ be a family of compact Hausdorff spaces on each of which acting a semigroup ${}^i\phi = <{}^i\phi_1,\dots,{}^i\phi_d>$ of continuous commuting transformations such that for $j \geq k$ $(X_k, {}^k\phi)$ is a homomorphic image of $(X_j, {}^j\phi)$ under a map λ_{jk} and the λ_{jk} are consistent.

Define

$$X = \{x = (x_i)_{i \in \mathcal{J}} \in \prod_{i \in \mathcal{J}} X_i \mid \lambda_{jk}(x_j) = x_k \quad \text{for} \quad j,k \in \mathcal{J}, j \geq k\} .$$

X is compact with the topology that is induced by the product topology. Define a semigroup ϕ of continuous commuting maps of X into itself by

$$(\phi^l(x))_i = ({}^i\phi)^l(x_i) \quad (x \in X, i \in \mathcal{J}, l \in \mathbb{Z}_+^d) .$$

(X,ϕ) is called the <u>inverse limit</u> of $(X_i, {}^i\phi)_{i \in \mathcal{J}}$.

<u>3.4. Theorem</u>.

$h(\phi) \leq \sup_{i \in \mathcal{J}} h({}^i\phi)$ *and if the maps* λ_{ij} *are surjective then*

$$h(\phi) = \lim_{i \in \mathcal{J}} h({}^i\phi) .$$

The proof is analogous to [9], Theorem 1 .

Now we turn to a particular inverse limit. Let ϕ be a semigroup acting on X . Consider the directed set $\mathbb{Z}_+^d$ where $l \geq k$ iff $l - k \in \mathbb{Z}_+^d$. Let (X^*, ϕ^*) be the inverse limit of $(X_l, {}^l\phi)_{l \in \mathbb{Z}_+^d}$ where $(X_l, {}^l\phi) = (X,\phi)$ for all $l \in \mathbb{Z}_+^d$ and the $\lambda_{lk} (l \geq k)$ are given by ϕ^{l-k}. The reason for this construction is, that ϕ^* is a group of homeomorphisms.

Define $\tilde{X} = \bigcap_{l \in \mathbb{Z}_+^d} \phi^l(X)$ and denote by $\tilde{\phi}$ the restriction of of ϕ to $\tilde{X}$. The semigroup $\tilde{\phi}$ consists of surjective maps of $\tilde{X}$ onto $\tilde{X}$. Similar to proposition 5 in [9] one proves.

<u>3.5. Property</u>.

$$h(\tilde{\phi}) = h(\phi) .$$

Since $((\tilde{X})^*, (\tilde{\phi})^*) = (X^*, \phi^*)$ we get combining the second part of (3.4) with (3.5)

3.6. Property.

$$h(\phi^*) = h(\phi) \ .$$

The following product theorem is essentially due to Goodwyn [9] in the case of a single map φ .

3.7. Theorem.

Let X *and* Y *be compact Hausdorff spaces and* $\phi = <\phi_1, \dots, \phi_d>$ *and* $\psi = <\psi_1, \dots, \psi_d>$ *semigroups of* c.c.m. *of* X *respectively* Y *into itself. Define the productsemigroup*

$$\phi \times \psi = <\phi_1 \times \psi_1, \dots, \phi_d \times \psi_d> \quad \textit{of} \quad X \times Y$$

into itself by

$$(\phi_i \times \psi_i)(x,y) = (\phi_i x, \psi_i y) \quad (x,y) \in X \times Y \ .$$

Then

$$h(\phi \times \psi) = h(\phi) + h(\psi) \ .$$

The proof runs analogously to [9]. (3.7) has the following generalization. See [9], Theorem 3 for a proof.

3.8. Theorem.

Let $({}^k\phi)_{k \in K}$ *be an arbitrary family of semigroups*

$${}^k\phi = <{}^k\phi_1, \dots, {}^k\phi_d>$$

acting on compact Hausdorff spaces $(X_k)_{k \in K}$ *respectively. Define* $X = \prod_{k \in K} X_k$ *endowed with the product topology and*

$$\phi : X \to X \quad \textit{by} \quad (\phi(x))_k = {}^k\phi(x_k) \quad (x \in X,\ k \in K)$$

then

$$h(\phi) = \sum_{k \in K} h({}^k\phi) \ .$$

For the rest of this section we assume X to be compact metric.

A group $\phi = \langle\phi_1,\ldots,\phi_d\rangle$ of commuting homeomorphisms acting on X is called an <u>expansive group</u> iff there is an $\varepsilon > 0$ such that given $x,y \in X$ either $x = y$ or there is a $l \in \mathbb{Z}^d$ and $d(\phi^l x, \phi^l y) > \varepsilon$. If ϕ is only a semigroup of continuous commuting maps then ϕ is called a <u>positively expansive semigroup</u> iff the same holds with $l \in \mathbb{Z}_+^d$. In both cases ε is called an expansive constant.

Note that ϕ is an expansive group if one of the ϕ_i is expansive in the usual sense of expansiveness for homeomorphisms. An analogous observation can be made in the semigroup case.

Let $l = (l_1,\ldots,l_d) \in \mathbb{Z}_+^d$ be given. We write

$$\mathrm{Fix}(\phi_1^{l_1},\ldots,\phi_d^{l_d}) := \bigcap_{i=1}^{d} \mathrm{Fix}(\phi_i^{l_i}) ,$$

where Fix(f) means the set of fixed points of a mapping f. Then the following result can be proved using the ideas of Proposition (2.8) [2].

<u>3.9. Theorem.</u>

If ϕ *is an expansive group of commuting homeomorphisms or a positively expansive semigroup of continuous commuting maps acting on a compact metric space* X *then*

$$h(\phi) \geq \overline{\lim_{n\to\infty}} |\rho_{l^n}|^{-1} \log |\mathrm{Fix}(\phi_1^{l_1^n},\ldots,\phi_d^{l_d^n})|$$

where (l^n) *is any sequence in* $\mathbb{Z}_+^d$ *such that* $l^n \to \infty$.

Note that for the d-dimensional shift (Example (1.10)) indeed

$$h(\phi) = \lim |\rho_{l^n}|^{-1} \log |\mathrm{Fix}(\phi_1^{l_1^n},\ldots,\phi_d^{l_d^n})| .$$

A point $x \in X$ is called a wandering point of the semigroup $\phi = \langle \phi_1, \ldots, \phi_d \rangle$ iff there is a neighbourhood U of x such that

$$U \cap \bigcup_{l \in \mathbb{Z}^d \setminus \{0\}} \phi^l U = \emptyset \; ;$$

otherwise x is called nonwandering. The set $\Omega = \Omega(\phi)$ of nonwandering points is closed and invariant $(\phi^l \Omega \subset \Omega$ for $l \in \mathbb{Z}_+^d)$. Note that the wandering points of ϕ are wandering points of ϕ_i for every $i \in \{1, \ldots, d\}$. Therefore

$$\Omega(\phi) \supset \bigcup_{i=1}^{d} \Omega(\phi_i) \; .$$

The following theorem states that as far as entropy is concerned the only interesting part of X is the nonwandering set Ω .

3.10. Theorem.

Let $\phi = \langle \phi_1, \ldots, \phi_d \rangle$ be a semigroup of continuous commuting maps of a compact metric space X into itself. If Ω denotes the set of nonwandering points of ϕ then

$$h(\phi) = h(\phi | \Omega) \; ,$$

where $\phi | \Omega$ is the restriction of ϕ to Ω .

Proof. see the appendix.

Two semigroups $\phi = \langle \phi_1, \ldots, \phi_d \rangle$ and $\psi = \langle \psi_1, \ldots, \psi_d \rangle$ acting on compact metric spaces X and Y respectively are Ω-conjugate iff there is a homeomorphism φ of $\Omega(\phi)$ onto $\Omega(\psi)$ such that

$$\varphi \circ \phi_i = \psi_i \circ \varphi \quad (1 \le i \le d) \; .$$

3.11. Corollary.

If ϕ and ψ are Ω-conjugate then

$$h(\phi) = h(\psi) .$$

3.12. Corollary.

If $\Omega(\phi)$ is a finite set then

$$h(\phi) = 0 .$$

4. Sequence entropy.

Let X be a compact Hausdorff space and $\phi = \langle\phi_1,\dots,\phi_d\rangle$ a semigroup of c.c.m. of X into itself. Throughout this section A will always denote a subset of $\mathbb{Z}_+^d$ and $L = \{l^n\}$ a sequence in $\mathbb{Z}_+^d$ such that $l^n \to \infty$. We write $A(l) = A \cap \rho_l$ and define for any open cover $\mathcal{A}$ of X

$$h_{A,L}(\phi,\mathcal{A}) = \overline{\lim_{n\to\infty}} |A(l^n)|^{-1} H(\bigvee_{j\in A(l^n)} \phi^{-j}\mathcal{A}) .$$

$$h_{A,L}(\phi) = \sup \{h_{A,L}(\phi,\mathcal{A}) \mid \mathcal{A} \text{ open cover}\}$$

is called the topological sequence entropy(t.s.e.) of ϕ with respect to A and L . Define

$$d(A,L) = \overline{\lim_{n\to\infty}} \ |A(l^n)|^{-1} |\rho_{l^n}|$$

and denote for $k > 0$ $\bar{\rho}_k = \{l \in \mathbb{Z}^d \mid |l_i| < k \ (1 \le i \le d)\}$.
A subset A of $\mathbb{Z}_+^d$ is said to have bounded gaps iff there exists a $k > 0$ such that

$$A + \bar{\rho}_k = \bigcup_{l\in A} (l+\bar{\rho}_k) \supset \mathbb{Z}_+^d .$$

$d(A,L)$ is a finite value for any subset A with bounded gaps.

4.1. Lemma

If $d(A,L)$ is finite then

$$h_{A,L}(\phi,\mathcal{A}) \le d(A,L)\ h(\phi,\mathcal{U}) .$$

Proof.

$$h_{A,L}(\phi,\mathcal{A}) = \overline{\lim_{n\to\infty}} |A(1^n)|^{-1} |\rho_{1^n}| \quad |\rho_{1^n}|^{-1} H(\bigvee_{j\in A(1n)} \phi^{-j}\mathcal{A})$$

$$\leq d(A,L) \lim_{n\to\infty} |\rho_{1^n}|^{-1} H(\mathcal{A}\rho_1 n) = d(A,L)\ h(\phi,\mathcal{A}) .$$

4.2. Theorem.

Let $\phi = \langle\phi_1,\ldots,\phi_d\rangle$ *be a semigroup of* c.c.m. *of* X *into itself and* $A\subset\mathbb{Z}_+^d$ *a subset having bounded gaps then*

$$h_{A,L}(\phi) = d(A,L)\ h(\phi) .$$

Proof.

Choose $k > 0$ such that $A + \bar{\rho}_k \supset \mathbb{Z}_+^d$ and $r\in\mathbb{Z}_+^d$ such that $r + (A + \bar{\rho}_{2k}) \subset \mathbb{Z}_+^d$ then

$$d(A,L)\ h(\phi,\mathcal{A}) = d(A,L)\ h(\phi,\phi^{-r}\mathcal{A}) =$$

$$= \overline{\lim_{n\to\infty}} |A(1^n)|^{-1} H(\bigvee_{1\in r+\rho_1 n} \phi^{-1}\mathcal{A}) \leq$$

$$\leq \overline{\lim_{n\to\infty}} |A(1^n)|^{-1} H(\bigvee_{1\in r+A(1^n)+\bar{\rho}_{2k}} \phi^{-1}\mathcal{A}) =$$

$$= h_{A,L}(\phi,\bigvee_{1\in\bar{\rho}_{2k}} \phi^{1}(\phi^{-r}\mathcal{A})) .$$

Taking the supremum over all open covers $\mathcal{A}$ we get

$$d(A,L)\ h(\phi) \leq h_{A,L}(\phi) .$$

The reverse inequality is clear by 4.1.

Whereas $h(\phi)$ does not depend on the sequence $L = \{1^n\}$ used in its definition, theorem (4.2) shows that $h_{A,L}(\phi)$ in general depends on L since there are simple examples for subsets $A\subsetneq\mathbb{Z}_+^d$ for which d(A,L) depends on L.

Furthermore we note, that theorem (4.2) partially covers theorem (2.1) since a subsemigroup $P_\phi = \langle \phi_1^{p_1}, \ldots, \phi_d^{p_d} \rangle$ $(p_i \in \mathbb{Z}_+)$ of finite index p corresponds to a subset A with bounded gaps such that $d(A,L) = p$ for any L and $h_{A,L}(\phi) = h(P_\phi)$.

Measure-theoretic sequence entropy (m.s.e.) is defined in a similar way as t.s.e.. Let $(\Omega, \mathcal{L}, m)$ be a probability measure space and $\phi = \langle \phi_1, \ldots, \phi_d \rangle$ a semigroup of measurepreserving transformations of Ω onto itself. If $\mathcal{A} = \{a_1, a_2, \ldots\}$ is an at most countable partition of Ω into measurable sets then

$$H_m(\mathcal{A}) = \sum_{i=1}^{\infty} m(a_i) \log m(a_i)$$

is called the entropy of $\mathcal{A}$. For any $\mathcal{A}$ such that $H_m(\mathcal{A}) < \infty$ we define

$$h_{A,L,m}(\phi, \mathcal{A}) = \overline{\lim_{n\to\infty}} |A(1^n)|^{-1} H_m(\bigvee_{k \in A(1^n)} \phi^{-k} \mathcal{A}) \quad .$$

$$h_{A,L,m}(\phi) = \sup \{h_{A,L,m}(\phi, \mathcal{A}) \mid \mathcal{A} \text{ partition, } H_m(\mathcal{A}) < \infty\}$$

is called the m.s.e. of ϕ with respect to A and L . The following result is an extension of Goodwyn's theorem ([10], [11]) . The proof follows that of Misiurewicz [24].

4.3. Theorem.

Let X *be a compact Hausdorff space and* $\phi = \langle \phi_1, \ldots \phi_d \rangle$ *a semigroup of* c.c.m. *of* X *into itself. Then for any* ϕ_i*-invariant* $(1 \le i \le d)$ *normalized Borelmeasure* m *on* X

$$h_{A,L,m}(\phi) \le h_{A,L}(\phi) \quad .$$

Proof.

Let $\mathcal{A} = \{a_1, \ldots, a_s\}$ be a Borel partition then for any $\varepsilon > 0$ there exist compact sets $b_i \subset a_i$ $(1 \le i \le s)$ such that

$$m\left(\bigcup_{i=1}^{s}(a_i \setminus b_i)\right) < \varepsilon \quad \text{and}$$

$$\rho_m(\mathcal{A},\mathcal{L}_o) = H_m(\mathcal{A}|\mathcal{L}_o) + H_m(\mathcal{L}_o|\mathcal{A}) < 1$$

where $\mathcal{L}_o = \{b_o, b_1, \ldots, b_s\}$ with $b_o = X \setminus \bigcup_{i=1}^{s} b_i$ and $\rho_m(\mathcal{A},\mathcal{L}_o)$ is the metric on the space of partitions as defined in [19] , 6.1 . $\mathcal{C} = \{b_o \cup b_1, \ldots, b_o \cup b_s\}$ is an open cover of X and we get

$$H_m\left(\bigvee_{k\in A(1^n)} \phi^{-k}\mathcal{L}_o\right) \le \log N\left(\bigvee_{k\in A(1^n)} \phi^{-k}\mathcal{L}_o\right) \le$$

$$\le \log N\left(\bigvee_{k\in A(1^n)} \phi^{-k}\mathcal{C}\right) + |A(1^n)| \log 2.$$

Therefore $h_{A,L,m}(\phi,\mathcal{L}_o) \le h_{A,L}(\phi,\mathcal{C}) + \log 2$. Now using the fact that

$$|h_{A,L,m}(\phi,\mathcal{A}) - h_{A,L,m}(\phi,\mathcal{L}_o)| \le \rho_m(\mathcal{A},\mathcal{L}_o)$$

which can be shown as in [19] ,8.6., we get

$$h_{A,L,m}(\phi) \le h_{A,L}(\phi) + \log 2 + 1 .$$

If $(\Omega^n, \mathcal{L}^n, m^n)$ denotes the n-fold product measure space with components $(\Omega,\mathcal{L},m)$ and ${}^n\phi = \langle {}^n\phi_1, \ldots {}^n\phi_d \rangle$ the productsemigroup on it, i.e.

$${}^n\phi_i = \phi_i \times \ldots \times \phi_i : \Omega^n \to \Omega^n ,$$

then a direct computation shows $h_{A,L,m^n}({}^n\phi) = n\, h_{A,L,m}(\phi)$. Later (4.7) we shall state that the same formula holds for the t.s.e. of the productsemigroup of a semigroup of c.c.m. . Therefore

$$n\, h_{A,L,m}(\phi) \le n\, h_{A,L}(\phi) + \log 2 + 1 .$$

Dividing by n and letting $n \to \infty$ we obtain the result.

For any subset $A \subset \mathbb{Z}_+^d$ and any sequence $L = \{l^n\}$ such that $l^n \to \infty$ we define the quantity

$$K(A,L) = \lim_{k\to\infty} \{\overline{\lim_{n\to\infty}} |A(l^n)|^{-1} |A(l^n)+\tilde{\rho}_k| \}$$

where $\tilde{\rho}_k = \{l \in \mathbb{Z}_+^d \mid 0 \leq l_i < k \ (1 \leq i \leq d)\}$. Note that the expression in brackets is always greater or equal than 1 and $K(A,L)$ itself is the limit of an increasing sequence and can be infinite.

4.4. Proposition.

Let $\phi = <\phi_1,\ldots,\phi_d>$ be a semigroup of measure-preserving transformations on a Lebesgue measure space $(\Omega, \mathcal{L}, m)$ then

$$h_{A,L,m}(\phi) \geq \begin{cases} K(A,L)\ h_m(\phi) & \text{if } K(A,L) < \infty,\ 0 < h_m(\phi) < \infty \\ 0 & \text{if } h_m(\phi) = 0 \\ \infty & \text{if } h_m(\phi) = \infty \text{ or } \\ & \quad 0 < h_m(\phi) < \infty,\ K(A,L) = \infty . \end{cases}$$

Proof.

Let $\mathcal{A}$ be a partition such that $H_m(\mathcal{A}) < \infty$ and write $\mathcal{A}^k = \bigvee_{l \in \tilde{\rho}_k} \phi^{-l} \mathcal{A}$. Then

$$h_{A,L,m}(\phi, \mathcal{A}^k) = \overline{\lim_{n\to\infty}} \ |A(l^n)|^{-1} H_m(\bigvee_{l \in A(l^n)+\tilde{\rho}_k} \phi^{-l} \mathcal{A})$$

$$\geq \overline{\lim_{n\to\infty}} \ |A(l^n)|^{-1} |A(l^n)+\tilde{\rho}_k| \ h_m(\phi, \mathcal{A})$$

where the last inequality follows from the fact (proof analogous [26] Lemma 2) that for any finite subset $S \subset \mathbb{Z}_+^d$ and any partition $\mathcal{A}$ such that $H_m(\mathcal{A}) < \infty$

$$H_m(\bigvee_{l \in S} \phi^{-l} \mathcal{A}) \geq |S| \ h_m(\phi, \mathcal{A}) .$$

Therefore

$$h_{A,L,m}(\phi) \geq \lim_{k\to\infty} \ h_{A,L,m}(\phi, \mathcal{A}^k)$$

$$\geq \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \ |A(1^n)|^{-1} |A(1^n)+\tilde{\rho}_k| \ h_m(\phi, \mathfrak{A})$$

$$= K(A,L) \ h_m(\phi, \mathfrak{A}) \quad \text{if} \quad K(A,L) \quad \text{is finite.}$$

Taking the supremum over all $\mathfrak{A}$ we get the result.

Combining (4.3) with (4.4) we obtain

4.5. Corollary.

Let X be a compact Hausdorff space and $\phi = \langle\phi_1,\dots,\phi_d\rangle$ a semigroup of c.c.m. on it then

$$h_{A,L}(\phi) \geq \begin{cases} K(A,L)\ h(\phi) & \text{if } K(A,L) < \infty \text{ , } 0 < h(\phi) < \infty \\ 0 & \text{if } h(\phi) = 0 \\ \infty & \text{if } h(\phi) = \infty \text{ or} \\ & \quad 0 < h(\phi) < \infty \text{ , } K(A,L) = \infty \text{ .} \end{cases}$$

There is another approach to t.s.e. that was first given by Bowen ([3], see also [22]) . Let X , ϕ and A be as before and $\mathfrak{A}$ an open cover of X . A set $E \subset X$ is $(A,\rho_1,\mathfrak{A})$-separated (with respect to ϕ) iff for any $x,y \in E$, $x \neq y$ there is a $k \in A(1)$ such that $\phi^k x \in U \in \mathfrak{A}$ implies $\phi^k y \notin U$. If $K \subset X$ is compact then we write $s(A,\rho_1,\mathfrak{A},K)$ for the largest cardinality of a $(A,\rho_1,\mathfrak{A})$-separated set in K . Given a sequence $L = \{1^n\}$ in $\mathbb{Z}_+^d$, $1^n \to \infty$ write

$$\bar{s}(A,L,\mathfrak{A},K) = \overline{\lim_{n\to\infty}} \ |A(1^n)|^{-1} \log s(A,\rho_1 n,\mathfrak{A},K)$$

and

$$h_{A,L}(\phi,K) = \sup\{\bar{s}(A,L,\mathfrak{A},K) \mid \mathfrak{A} \text{ open cover}\} \text{ .}$$

A set $F \subset X$ $(A,\rho_1,\mathfrak{A})$-spans a set K (with respect to ϕ) iff for any $x \in K$ there is a $y \in F$ such that given $k \in A(1)$ there is a $U \in \mathfrak{A}$ with $\phi^k x$, $\phi^k y \in U$. For any compact set $K \subset X$ we write $r(A,\rho_1,\mathfrak{A},K)$ for the smallest cardinality of any set which

$(A,\rho_1,\mathcal{A})$-spans K and

$$\overline{r}(A,L,\mathcal{A},K) = \overline{\lim_{n\to\infty}}\ |A(1^n)|^{-1} \log r(A,\rho_1 n,\mathcal{A},K) .$$

It is easy to see that taking the supremum over all open covers we get the same quantity as above, i.e.

$$h_{A,L}(\phi,K) = \sup\{\overline{r}(A,L,\mathcal{A},K) \mid \mathcal{A} \text{ open cover}\} .$$

Rewriting the proof of Proposition 2.3 in [22] in our situation yields

$$h_{A,L}(\phi,X) = h_{A,L}(\phi) .$$

If A is the full set $\mathbb{Z}_+^d$ we shall omit the A in the notation. In this case the right side is nothing but $h(\phi)$. In other words $h_L(\phi,X)$ does not depend on L. This can be shown directly too. We state

4.6. Proposition.

For any compact, invariant (i.e. $\phi^1 K \subset K (1\in \mathbb{Z}_+^d)$) subset K of X $h_L(\phi,K)$ does not depend on L.

In the following sections we shall use Bowen's definition of topological entropy. X will always be metrizable with metric d. In this case it is convenient to consider covers of ε-balls. We shall say a set $E\subset X$ is (ρ_1,ε)-separated iff for distinct $x,y\in E$ there is a $k\in\rho_1$ such that $\mathrm{d}(\phi^k x,\phi^k y) > \varepsilon$ and a set $F\subset X$ (ρ_1,ε)-spans a set $K\subset X$ iff for each $x\in K$ there is a $y\in F$ such that $\mathrm{d}(\phi^k x,\phi^k y) \le \varepsilon$ for all $k\in\rho_1$.

Finally we state a result on the t.s.e. of a productsemigroup. The proof is analogous to Proposition 2.4 [22].

4.7. Proposition.

Let X be a compact Hausdorff space, $\phi = <\phi_1,\dots,\phi_d>$ a semigroup of c.c.m. on X then the productsemigroup $\phi \times \phi = <\phi_1 \times \phi_1,\dots,\phi_d \times \phi_d>$ satisfies

$$h_{A,L}(\phi \times \phi) = 2\, h_{A,L}(\phi) \ .$$

5. Semiflows of dimension d

Let (X,d) be a compact metric space. The set of continuous mappings of X into itself is a semigroup under composition. A family $\phi = (\phi_t)_{t \in \mathbb{R}_+^d}$ contained in this set is called a continuous semiflow of dimension d (or continuous d-dimensional semiflow) iff the mapping $t \to \phi_t$ is a semigroup homomorphism and the mapping $(x,t) \to \phi_t x$ of the product space $X \times \mathbb{R}_+^d$ into X is continuous. $\phi = (\phi_t)_{t \in \mathbb{R}^d}$ is called a continuous flow of dimension d iff the same holds with $\mathbb{R}^d$ instead of $\mathbb{R}_+^d$.

For any $r \in \mathbb{R}$ we define $(r)_i \in \mathbb{R}^d$ by

$$((r)_i)_j = \begin{cases} r & i = j \\ 0 & i \neq j \end{cases} \ .$$

ϕ can be considered to be generated by the one-dimensional semiflows $\phi^i = (\phi_r^i)_{r \in \mathbb{R}_+}$ $(1 \leq i \leq d)$ where $\phi_r^i := \phi_{(r)_i}$.

Given $t = (t_1,\dots,t_d) \in \mathbb{R}_+^d$ we consider now the discrete subsemigroups

$$<\phi_{(t_1)_1},\dots,\phi_{(t_d)_d}> \ .$$

5.1. Theorem.

Let $\phi = (\phi_t)_{t \in \mathbb{R}_+^d}$ *be a continuous semiflow of dimension* d , *then for any* $t = (t_1,\dots,t_d) \in \mathbb{R}_+^d$

$$h(\langle\phi_{(t_1)_1},\ldots,\phi_{(t_d)_d}\rangle) = \prod_{i=1}^{d} t_i \; h(\langle\phi_{(1)_1},\ldots,\phi_{(1)_d}\rangle) \; .$$

If $\phi = (\phi_t)_{t\in\mathbb{R}^d}$ *is a continuous flow then the same holds for all* $t\in\mathbb{R}^d$ *with* $|t_i|$ *instead of* t_i *on the right side.*

5.2. Remark.

If $t_i = 0$ for some i and $h(\langle\phi_{(1)_1},\ldots,\phi_{(1)_d}\rangle) = \infty$ then the product on the right side is understood to be 0 since the left side is 0 in this case.

By means of (5.1) it makes sense to define the entropy of ϕ as follows.

5.3. Definition.

Let $\phi = (\phi_t)_{t\in\mathbb{R}_+^d}$ be a continuous semiflow of dimension d then

$$h(\phi) := h(\langle\phi_{(1)_1},\ldots,\phi_{(1)_d}\rangle) \; .$$

Proof of 5.1.

We shall abbreviate ${}^t\phi = \langle\phi_{(t_1)_1},\ldots,\phi_{(t_d)_d}\rangle$

Then given $s = (s_1,\ldots,s_d)$ and $t = (t_1,\ldots,t_d)$ in $\mathbb{R}_+^d$ such that

$$s_i, t_i \neq 0 \quad \text{for all} \quad 1 \leq i \leq d$$

we shall prove

$$h({}^t\phi) \leq \left(\prod_{i=1}^{d} t_i\right)\left(\prod_{i=1}^{d} s_i\right)^{-1} h({}^s\phi) \; .$$

This implies the result.

Given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$d(\phi_u(x), \phi_u(y)) < \varepsilon \quad \text{for all} \quad u\in Q_s := \prod_{i=1}^{d} [0,s_i]$$

whenever $d(x,y) < \delta$.

Let $l = (l_1, \dots, l_d) \in \mathbb{Z}_+^d$ be given and E be a set of minimal cardinality that (ρ_l, δ)-spans X with respect to ${}^s\phi$.
Then E (ρ_k, ε)-spans X with respect to ${}^t\phi$ for all
$k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$ such that

$$\prod_{i=1}^{d} [0, t_i k_i] \subset \prod_{i=1}^{d} [0, s_i l_i] \ .$$

Therefore if $L_k = (k^n)$ is the sequence in $\mathbb{Z}_+^d$ defined by $k_i^n = n$ $(1 \le i \le d)$ and $L_l = (l^n)$ the sequence defined by $l_i^n = [t_i s_i^{-1} n| + 1$ we get

$$r_{t_\phi}(\rho_{k^n}, \varepsilon, X) \le r_{s_\phi}(\rho_{l^n}, \delta, X) \quad .$$

This implies

$$\bar{r}_{t_\phi}(L_k, \varepsilon, X) \le \overline{\lim_{n\to\infty}} \ n^{-d} \prod_{i=1}^{d} ([t_i s_i^{-1} n] + 1) \ \bar{r}_{s_\phi}(L_l, \delta, X)$$

$$\le \left(\prod_{i=1}^{d} t_i\right) \left(\prod_{i=1}^{d} s_i\right)^{-1} h_{L_1}({}^s\phi, X) \quad .$$

For $\varepsilon \to 0$ we get the desired inequality.

5.4. Remark.

For flows there is a more general formulation of (5.1). ϕ can be considered as real vector space with base $\phi_{(1)_1}, \dots, \phi_{(1)_d}$.

Any real d-dimensional matrix M maps the basevectors on vectors $M(\phi_{(1)_i})$ $(1 \le i \le d)$. If ϕ_M denotes the discrete subgroup of ϕ generated by these, i.e.

$$\phi_M := \langle M(\phi_{(1)_1}), \dots, M(\phi_{(1)_d}) \rangle \quad \text{then}$$

$$h(\phi_M) = |\det M| \ h(\langle \phi_{(1)_1}, \dots, \phi_{(1)_d} \rangle) \ .$$

Knowing (5.1) the proof of this equation runs as in [5] Theorem (6.1) .

6. An example and an embedding theorem.

Given a real number $A > 0$ we consider the following space of real-valued functions on $\mathbb{R}^d$:

$$L^A(\mathbb{R}^d) = \{h : \mathbb{R}^d \to [0,1] \,|\, |h(s_1,\dots,s_d) - h(t_1,\dots,t_d)| \leq$$

$$\leq A \max_{1\leq i\leq d} |s_i - t_i| \quad \text{for all} \quad (s_1,\dots,s_d) \ , \ (t_1,\dots,t_d) \in \mathbb{R}^d\} \ .$$

Let $Q_k = [-k,k]^d$ denote the cube whose sides have length $2k$, then

$$d\!\!\mid (h,g) = \sum_{k=1}^{\infty} 2^{-k+1} \sup_{s\in Q_k} |h(s) - g(s)|$$

defines a metric such that $(L^A(\mathbb{R}^d), d\!\!\mid)$ is a compact metric space. The d-dimensional shift flow $S = (S_t)_{t\in\mathbb{R}^d}$ is given by

$$(S_t h)\ (s) = h(s+t) \quad (s,t \in \mathbb{R}^d,\ h \in L^A(\mathbb{R}^d)).$$

S is a d-dimensional flow of homeomorphisms generated by the flows $S^i = (S_{(t)_i})_{t\in\mathbb{R}}$ $(1\leq i\leq d)$ where $(t)_i \in \mathbb{R}^d$ is defined by

$$((t)_i)_j = \begin{cases} t & i=j \\ 0 & i\neq j \end{cases} \ .$$

6.1. Theorem.

$$h(S) := h(\langle S_{(1)_1},\dots,S_{(1)_d}\rangle) = \infty \ .$$

Proof.

We consider first the case $d = 2$. Given integers $k,m \geq 1$ we shall define a set $\mathcal{G}$ of $(\rho_k, 2^{-m}A)$-separated functions in $L^A(\mathbb{R}^2)$ where $\rho_k = [\,0,k[^2 \subset \mathbb{Z}_+^2$.

Divide the interval $[0,k[$ into $n = 2^m \cdot k$ intervals $(I_i)_{1\leq i\leq n}$ of equal length 2^{-m} . Consider the system F of piecewise linear, continuous functions f on $[0,k[$ s.t. $f(0)=l\cdot 2^{1-m}A$ for some $l\,(0\leq l\leq [2^{m-1}A^{-1}])$ and s.t. f has constant derivative $+A$ or $-A$ on each I_i. ($[a]$ denotes the largest integer smaller or equal than a). Clearly

$$|F| = ([2^{m-1}A^{-1}] + 1)\, 2^n .$$

Given $f\in F$ we define a function h_f on $\mathbb{R}^2$ in the following way

$$h_f(x,y) = \begin{cases} f(0) & (x,y)\in\ ]-\infty,0]\times\mathbb{R} \\ f(x) & (x,y)\in [0,k[\ \times\mathbb{R} \\ f(k) & (x,y)\in [k,\infty[\ \times\mathbb{R} \end{cases}$$

Define for $(x,y)\in[-2^{-m}, 2^{-m}]^2$ the two "disturb-functions"

$$\delta^+(x,y) = A(2^{-m} - |x|)\,2^m|y|$$

$$\delta^-(x,y) = A(|x| - 2^{-m})\,2^m|y| .$$

We say $f\in F$ has a pike at the point $i\cdot 2^{-m}\,(1\leq i\leq n-1)$ iff f has derivative $+A$ on I_i and $-A$ on I_{i+1} (positive pike) or $-A$ on I_i and $+A$ on I_{i+1} (negative pike). Then the number of pikes of any $f\in F$ is between 0 and $n-1$.

Let $f\in F$ with pikes at $i_1 2^{-m},\dots,i_t 2^{-m}$ be given. For any $i_j, j\in\{1,\dots,t\}$ and any $i\in\{1,\dots,n-1\}$ we shall define a new function $h_f((i_j,i))$ by "disturbing" h_f inside the square

$$S_q(i_j,i) = [(i_j-1)2^{-m},(i_j+1)2^{-m}]\times[(i-1)2^{-m},(i+1)2^{-m}] .$$

If the pike at $i_j\cdot 2^{-m}$ is a positive one put

$$h_f((i_j,i))(x,y) = \begin{cases} h_f((i_j-1)\cdot 2^{-m},y) + \delta^+(x-i_j\cdot 2^{-m},y-i\cdot 2^{-m}) & (x,y)\in S_q(i_j,i) \\ h_f(x,y) & \text{elsewhere}, \end{cases}$$

if it is a negative pike use δ^- in this definition. In an analogous way we can disturb h_f simultaneously on k squares $(k \leq t(n-1))$ to get functions $h_f((i_{j_1},i_1),\dots,(i_{j_k},i_k))$. If two of the k squares overlap, i.e. if we consider neighbours

$$(i_j \cdot 2^{-m}, i \cdot 2^{-m}) \quad \text{and} \quad (i_j \cdot 2^{-m}, (i+1)\, 2^{-m})$$

then let the disturbed function have constant value $h_f((i_j-1)2^{-m},y)$ on the overlapping part. The result of this procedure (including the case of 0 disturbances) is a family of functions $\mathcal{E}_f$ such that

$$|\mathcal{E}_f| = \sum_{j=0}^{t(n-1)} \binom{t(n-1)}{j} = 2^{t(n-1)} \quad .$$

If we take all $h_{\tilde{f}}$ s.t. $\tilde{f}(0)=f(0)$ and $\tilde{f}$ has t pikes we get a family of functions $\mathcal{E}_f(t)$ and

$$|\mathcal{E}_f(t)| = 2 \cdot 2^{t(n-1)} \binom{n-1}{t} \quad .$$

Considering all $h_{\tilde{f}}$ s.t. $\tilde{f}(0)=f(0)$ the procedure yields a family of functions

$$\sum_{t=0}^{n-1} \mathcal{E}_f(t) \quad \text{s.t.}$$

$$\left|\sum_{t=0}^{n-1} \mathcal{E}_f(t)\right| = 2(2^{n-1}+1)^{n-1} \quad .$$

Finally summing over all $f \in F$ we get a family of functions $\mathcal{E}$ and

$$|\mathcal{E}| = ([2^{m-1}A^{-1}]+1)\; 2(2^{n-1}+1)^{n-1} \; .$$

We want to estimate the number of functions in $\mathcal{E}$ that have values outside the interval $[0,1]$.

Let l_1, l_2 be integers, $l_1 < l_2$ and define

$$F(l_1,l_2) = \{f \in F \mid f(0) = l_1 \cdot 2^{1-m}A,\; f(k) = l_2 \cdot 2^{1-m}A\} \;,$$

i.e. $F(l_1,l_2)$ are those functions in F that have in $n_1 = n2^{-1} + (l_2 - l_1)$ intervals I_i derivative $+A$. Set $n_2 = n - n_1$.

Clearly $|F(l_1,l_2)| = \binom{n}{n_1} = \binom{n}{n_2}$. Those functions of $F(l_1,l_2)$ that have derivative $+A$ in I_1 can have at most $2n_2$ pikes (call this family $F^+(l_1,l_2)$) and those $f \in F(l_1,l_2)$ with derivative $-A$ in I_1 have at most $2n_2-1$ pikes (call this family $F^-(l_1,l_2)$) .

Disturbing functions $h_f (f \in F^+(l_1,l_2))$ by the procedure given above yields a family $\mathcal{C}^+(l_1,l_2)$ s.t.

$$|\mathcal{C}^+(l_1,l_2)| = \sum_{t=1}^{2n_2} 2^{t(n-1)} \binom{n_1-1}{\left[\frac{t}{2}\right]} \binom{n_2-1}{\left[\frac{t-1}{2}\right]} =$$

$$= \sum_{j=1}^{n_2} 2^{2j\cdot(n-1)} \binom{n_1-1}{j} \binom{n_2-1}{j-1} + \sum_{j=0}^{n_2-1} 2^{(2j+1)(n-1)} \binom{n_1-1}{j} \binom{n_2-1}{j}$$

$$< 2^{n_1-1} (n_1-1)^{-\frac{1}{2}} (2^{2(n-1)}+1)^{n_2-1} (2^{2(n-1)}+2^{n-1}) \ .$$

Here we used the following inequality

$$\binom{n_1-1}{j} \le \binom{n_1-1}{\left[\frac{n_1-1}{2}\right]} < (n_1-1)^{-\frac{1}{2}} 2^{n_1-1} \qquad (0 \le j \le n_1-1) \ .$$

Disturbing functions $h_f (f \in F^-(l_1,l_2))$ yields a family $\mathcal{C}^-(l_1,l_2)$ s.t.

$$|\mathcal{C}^-(l_1,l_2)| = \sum_{t=1}^{2n_2-1} 2^{t(n-1)} \binom{n_1-1}{\left[\frac{t-1}{2}\right]} \binom{n_2-1}{\left[\frac{t}{2}\right]}$$

$$< 2^{n_1-1} (n_1-1)^{-\frac{1}{2}} (2^{2(n-1)}+1)^{n_2-1} (2^{n-1}+1) \ .$$

If $\mathcal{C}(l_1,l_2) := \mathcal{C}^+(l_1,l_2) \cup \mathcal{C}^-(l_1,l_2)$ we get

$$|\mathcal{C}(l_1,l_2)| < 2^{n_1-1} (n_1-1)^{-\frac{1}{2}} (2^{2(n-1)}+1)^{n_2-1} (2^{n-1}+1)^2 \ .$$

Define $F(l) = \{f \in F \mid f(0) \leq l\cdot 2^{1-m}A,\ f(k) > l\cdot 2^{1-m}A\}$

then

$$|F(l)| = \sum_{\substack{l_1 \leq l \\ l_2 > l}} |F(l_1,l_2)| = \sum_{j=1}^{n\cdot 2^{-1}} \sum_{\substack{l_1 \leq l \\ l_2 > l \\ l_2 - l_1 = j}} |F(l_1,l_2)| .$$

Remember $n_1 = n.2^{-1} + (l_2-l_1) = n\cdot 2^{-1}+j$ and $n_2 = n-n_1 = n\cdot 2^{-1}-j$.

Disturbing functions $h_f (f \in F_l)$ we get a family of functions $\mathcal{E}(l)$ s.t.

$$|\mathcal{E}(l)| < \sum_{j=1}^{n.2^{-1}} j\ 2^{n\cdot 2^{-1}+j-1}(n\cdot 2^{-1}+j-1)^{-\frac{1}{2}}(2^{2(n-1)}+1)^{n\cdot 2^{-1}-j-1}\cdot$$

$$\cdot(2^{n-1}+1)^2 < (n\cdot 2^{-1})^{-\frac{1}{2}}\ 2^{n\cdot 2^{-1}-1}(2^{2(n-1)}+1)^{n\cdot 2^{-1}-1}(2^{n-1}+1)^2 .$$

$$\cdot\left(\sum_{j=1}^{\infty} j((2^{2(n-1)}+1)^{-1}2)^j\right)$$

$$= n^{-\frac{1}{2}}2^{n\cdot 2^{-1}+2^{-1}}(2^{2(n-1)}+1)^{n\cdot 2^{-1}}(2^{n-1}+1)^2\cdot(2^{2(n-1)}-1)^{-2}$$

In the last equality we used the formula

$$\sum_{j=1}^{\infty} jx^j = x(1-x)^{-2} \quad \text{for } |x| < 1 .$$

Let $F_{max}(l) = \{f \in F \mid f(0) \leq l\cdot 2^{1-m}A,\ \max_{r\in[0,k[} f(r) > l\cdot 2^{1-m}A\}$.

One easily sees that $|F_{max}(l)| \leq 2|F(l)|$.

Define

$$F_{[0,1]} = \{f \in F \mid f(0) \in [0,1],\ f(r) \notin [0,1] \text{ for some } r \in [0,k[\,\}$$

then

$$|F_{[0,1]}| \leq 2|F_{max}(l)| \leq 4|F(l)| \quad \text{for } l = [2^{m-1}A^{-1}] .$$

If $\mathcal{C}_{[0,1]}$ denotes the family of functions got by disturbing $h_f (f \in F_{[0,1]})$ one can show using the inequality above

$$|\mathcal{C}_{[0,1]}| \leq 4 \, |\mathcal{C}([2^{m-1}A^{-1}])| \quad .$$

Now consider $\mathcal{G} = \mathcal{C} \cap L^A(\mathbb{R}^2) = \mathcal{C} \setminus \mathcal{C}_{[0,1]}$ then

$$|\mathcal{G}| = |\mathcal{C}| - |\mathcal{C}_{[0,1]}| \geq ([2^{m-1}A^{-1}] + 1) 2 (2^{n-1}+1)^{n-1}$$

$$- n^{-\frac{1}{2}} 2^{(n+5)2^{-1}} (2^{2(n-1)}+1)^{n \cdot 2^{-1}} (2^{n-1}+1)^2 (2^{2(n-1)}-1)^{-2}$$

$$\geq (2^{n-1}+1)^{n-1} \left[([2^{m-1}A^{-1}]+1) 2 - n^{-\frac{1}{2}} 2^{(n+5)2^{-1}} (2^{n-1}+1)^3 \cdot \right.$$

$$\left. \cdot (2^{2(n-1)}-1)^{-2} \right]$$

$$=: (2^{n-1}+1)^{n-1} \, c(n) \; .$$

It is clear by the construction of $\mathcal{C}$ that $\mathcal{G} = \mathcal{G}(k,m)$ is a $(\rho_k, 2^{-m}A)$-separated set in $L^A(\mathbb{R}^2)$. Therefore

$$h(S) \geq \lim_{m\to\infty} \overline{\lim_{k\to\infty}} \, k^{-2} \log |\mathcal{G}| \quad .$$

Now $\log |\mathcal{G}| \geq \log (2^{n-1}+1)^{n-1} + \log c(n)$. $c(n) \geq 1$ for n sufficiently large since $[2^{m-1}A^{-1}] + 1 \geq 1$ and the second product in $c(n)$ converges to 0. Therefore $\log c(n)$ is a non-negative value for n large.

Remember $n = 2^m \cdot k$. We get

$$\log (2^{n-1}+1)^{n-1} = (2^m k-1) \log (2^{2^m k}+1) \geq (2^m k-1)^2 \quad .$$

This implies $\overline{\lim}_{k\to\infty} k^{-2} \log |\mathcal{G}| \geq 2^{2m}$ and finally

$$h(S) \geq \lim_{m\to\infty} 2^{2m} = \infty \quad .$$

For $d > 2$ we proceed by induction in the following way. Take the family $\mathcal{C}$ from the case $d-1$ as the new F, construct functions

h_f in the same way, i.e. h_f does not depend on the last coordinate and then disturb h_f using the given procedure.

Now we consider d-dimensional flows on measure spaces. Let $(\Omega, \mathcal{L}, m)$ be a Lebesgue measure space, non-atomic and of total measure one and φ an invertible measure-preserving transformation of Ω onto itself. (For details on these notions see for example [19]). The set of these transformations forms a group imp $(\Omega, \mathcal{L}, m)$ under composition. A <u>d-dimensional measure-preserving flow</u> (dim d m.p.flow) on $(\Omega, \mathcal{L}, m)$ is a family of transformations $\phi = (\phi_t)_{t \in \mathbb{R}^d}$ contained in imp $(\Omega, \mathcal{L}, m)$ such that the mapping $t \to \phi_t$ from $\mathbb{R}^d$ into imp$(\Omega, \mathcal{L}, m)$ is a group homomorphism and the mapping $(\omega, t) \to \phi_t \omega$ of the product measure space $\Omega \times \mathbb{R}^d$ into Ω is measurable.

Given two dim d m.p. flows ϕ and ψ on measure spaces $(\Omega, \mathcal{L}, m)$ and $(\Omega', \mathcal{L}', m')$ respectively we say they are <u>isomorphic</u> iff there exists a bijection $\varphi : \Omega_o \to \Omega'_o$ where Ω_o, Ω'_o are measurable subsets of Ω resp. Ω' having measure one such that φ and φ^{-1} are measurable, $\varphi m = m'$ and $\varphi \circ \phi_t = \psi_t \circ \varphi$ $(t \in \mathbb{R}^d)$.

For certain purposes (see e.g. [6]) one is interested to embed a dim d m.p. flow of arbitrary (measure-theoretic) entropy isomorphicly in a compact metric space with a flow of homeomorphisms operating on it. (Measure-theoretic) entropy is an isomorphism invariant. Therefore by theorem (4.3) a candidate for such an embedding procedure has to have infinite topological entropy. (6.1) says that the shiftflow on $L^A(\mathbb{R}^d)$ is such a candidate. We outline in the following how the embedding is done.

A dim d m.p. flow ϕ is called <u>aperiodic</u> iff there is a set of measure 0 $N \in \mathcal{L}$ such that if $\omega \notin N$ and $t \neq 0$, then $\phi_t\omega \neq \omega$. For semiopen d-dimensional rectangles $Q \subset \mathbb{R}^d$ and sets $B \in \mathcal{L}$ we consider $\phi_Q B := \bigcup_{t \in Q} \phi_t B$. $\phi_Q B$ is called <u>disjoint</u> if the sets $\phi_t B$ $(t \in Q)$ are disjoint. We recall the following theorem due to D. Lind.

<u>Theorem [17].</u>

Let ϕ be an aperiodic d-dimensional measure-preserving flow on a non-atomic Lebesgue measure space $(\Omega, \mathcal{L}, m)$ of total measure one. Then for any rectangle $Q \subset \mathbb{R}^d$ and $\varepsilon > 0$, there is a set $F \in \mathcal{L}$ such that $\phi_Q F$ is disjoint, measurable and $m(\phi_Q F) > 1-\varepsilon$. Furthermore on $\phi_Q F$ the measure m is the completed product of a measure on F with Lebesgue measure on Q.

Let $\omega \in B \subset \Omega$ then we put

$$l(B,\omega) = \sup_{Q \in R(\omega)} \min_{1 \leq i \leq d} |I_i|$$

where $R(\omega)$ is the family of all rectangles $Q = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$ (I_i intervals) such that $0 \in Q$ and $\phi_Q\omega \subset B$.

$l(B) = \inf_{\omega \in B} l(B,\omega)$ is the minimal length of a "time-interval" the flow ϕ stays inside B after having entered the set B.

Given a vector $t^0 = (t_1, \ldots, t_d) \in \mathbb{R}^d$ we write $\langle t_1, \ldots, t_d \rangle$ for the discrete subgroup of $\mathbb{R}^d$ generated by the vectors $(0, \ldots, t_i, \ldots, 0)$, i.e.

$$\langle t_1, \ldots, t_d \rangle = \{t \in \mathbb{R}^d \mid t = (k_1 t_1, \ldots, k_d t_d) \text{ for } (k_1, \ldots, k_d) \in \mathbb{Z}^d\} .$$

<u>6.2. Definition.</u>

A partition π of Ω into measurable sets is called a generator of finite type for the dim d m.p. flow ϕ iff

1) there exists a $t^0 \in \mathbb{R}^d$ such that $\bigvee_{t \in \langle t_1, \ldots t_d \rangle} \phi_t \pi = \mathcal{L}$ a.e

2) $l(B) > 0 \quad (B \in \pi)$

3) For any rectangle $Q \subset \mathbb{R}^d$ and a.e. $\omega \in \Omega$ $\phi_Q \omega \cap B \neq \emptyset$ for at most a finite number of sets $B \in \pi$.

Now we observe that Lind's theorem above yields just the d-dimensional version of the representation of ϕ that is the starting point for our iterative construction of a generator of finite type in [7] . The construction itself can be generalized to higher dimensions with some obvious modifications. We get

6.3. Theorem.

For any aperiodic d-dimensional measure-preserving flow on a non-atomic Lebesgue space $(\Omega, \mathcal{L}, m)$ *there exists a* *countable* *generator of finite type.*

6.4. Remark.

Actually we get more. Given any $t^0 = (t_1, \ldots, t_d) \in \mathbb{R}^d$ such that each $t_i \neq 0$ there exists a generator of finite type corresponding to the subgroup $\langle t_1, \ldots, t_d \rangle$,

i.e. $$\bigvee_{t \in \langle t_1, \ldots, t_d \rangle} \phi_t \pi = \mathcal{L} \quad \text{a.e.}$$.

Once we have the existence of a generator with very regular orbit properties we get

6.5. Theorem.

Every d-dimensional measure-preserving flow on a non-atomic

Lebesgue measure space is isomorphic to a d-dimensional flow of homeomorphisms operating on a compact metric space.

Proof.

Take as compact metric space the space $L^A(\mathbb{R}^d)$ with the d-dimensional shiftflow S on it and modify our embedding construction in [7] , Satz (3.1), to d dimensions.

There is probably an easier way to get theorem (6.5). In the 1-dimensional case U. Krengel [16] studied a type of generator with similar orbit-properties whose existence can be seen in a more elementary way. After [7] was written he pointed out to me that the embedding construction works as well with this type of generator. It should be not too hard to prove a d-dimensional version of this generator theorem. Also I thank U. Krengel for sending me an embedding construction for the periodic part of the flow. Such a construction was not explicitly given in [7].

Appendix.

Here we give a full proof of Theorem 3.10., compare [2] , p. 25:

Let $\mathcal{A}_\varepsilon$ be a finite open cover of Ω having diameter less than ε for some $\varepsilon > 0$. We will construct a finite open cover $\mathcal{L}_\varepsilon$ of X having diameter less than 3ε with the property

$$h(\phi, \mathcal{L}_\varepsilon) \leq h(\phi|\Omega, \mathcal{A}_\varepsilon) + \varepsilon .$$

This implies using (1.8)

$$h(\phi) = \lim_{\varepsilon\to 0} h(\phi, \mathcal{L}_\varepsilon) \leq \lim_{\varepsilon\to 0} h(\phi|\Omega, \mathcal{A}_\varepsilon) = h(\phi|\Omega) .$$

The reverse inequality is true by (3.2) .

We proceed with the construction of $\mathcal{L}_\varepsilon$. Let

$$\mathcal{A}_\varepsilon = \{A_1,\dots,A_s\} \quad \text{and} \quad c_n = N\Big(\bigvee_{1\in\rho_{1^n}} \phi^{-1}\mathcal{A}_\varepsilon\Big)$$

where $1^n = (n,\dots,n) \in \mathbb{Z}_+^d$.

Choose N large enough such that

$$|\rho_{1^N}|^{-1} \log c_N < h(\phi|\Omega, \mathcal{A}_\varepsilon) + \varepsilon$$

and $\alpha > 0$ such that $d(\phi^1 x, \phi^1 y) < \varepsilon$ for all $1 \in \rho_{1^N}$ whenever $x, y \in X$ and $d(x,y) < \alpha$. Let

$$U = \{y \in X | d(y,\Omega) < \alpha\} \text{ and } B_j = \{y \in U | d(y,A_j) < \varepsilon\} \quad (1 \leq j \leq s) .$$

Let $E = \{(i_1)_{1\in\rho_{1^N}} | i_1 \in \{1,\dots,s\}\}$ be a set of arrays such that

$$\{\bigcap_{1\in\rho_{1^N}} \phi^{-1}A_{i_1} | \ (i_1)_{1\in\rho_{1^N}} \in E\}$$

is a minimal subcover of

$$\bigvee_{1\in\rho_{1^N}} \phi^{-1}\mathcal{A}_\varepsilon, \text{ i.e.} |E| = c_N .$$

For every wandering point x let N_x be an open neighbourhood of x having diameter less than 3ε and such that

$$N_x \cap \bigcup_{1\in\mathbb{Z}^d \setminus\{0\}} \phi^1 N_x = \emptyset$$

Choose a finite subcover $\{N_{x_1},\dots,N_{x_{t'}}\}$ for $X \setminus U$ and put

$$\mathcal{L}_\varepsilon = \{B_1,\dots,B_s, N_{x_1},\dots,N_{x_{t'}}\} .$$

Let $n \geq 2t'$. In order to obtain a suitable upper bound for

$N(\bigvee_{l \in \rho_{1^n}} \phi^{-l} \mathcal{L}_\varepsilon)$ we define a subcover of $\bigvee_{l \in \rho_{1^n}} \phi^{-l} \mathcal{L}_\varepsilon$.

Let $x \in X$ be an arbitrary point. We shall define $(C_l)_{l \in \rho_{1^n}}$ with $C_l \in \mathcal{L}_\varepsilon$ such that $x \in \bigcap_{l \in \rho_{1^n}} \phi^{-l} C_l$. First observe that for $l, l' \in \rho_{1^n}$, $l \neq l'$, $\phi^l x$ and $\phi^{l'} x$ cannot occur in the same N_{x_i}. For suppose $\phi^l x$ and $\phi^{l'} x$ were both contained in the same N_{x_i}, then

$$\phi^l x \in \phi^{(l-l')} (\phi^{l'} x) \subset \phi^{l-l'} N_{x_i} .$$

This would imply $\phi^{l-l'} N_{x_i} \cap N_{x_i} \neq \emptyset$ contradicting the choice of N_{x_i}. Consequently we get $\phi^l x \in U$ except for at most t' values $l^i = (l_1^i, \ldots, l_d^i) \in \rho_{1^n}$ $(1 \leq i \leq t')$. Let $l_1^1, \ldots, l_1^t$ be the different values among $l_1^1, \ldots l_1^{t'}$.

We can assume

$$l_1^1 < l_1^2 < \ldots < l_1^t .$$

Define $l_1^0 = -1$, $l_1^{t+1} = n$ and

$$\rho_i = \{ l = (l_1, \ldots, l_d) \in \rho_{1^n} | \, l_1^{i-1} < l_1 < l_1^i \} \quad (1 \leq i \leq t+1) .$$

Then if $l \in \rho_i$ for some i we have $\phi^l x \in U$. We put

$$r_i = l_1^i - l_1^{i-1} - 1 \quad \text{then} \quad \sum_{i=1}^{t+1} r_i = n - t .$$

r_i and n can be written in a unique way $r_i = p_i N + q_i$ and $n = pN + q$ such that $0 \leq q_i, q < N$.

Now we fill the rectangle ρ_i with disjoint cubes having equal volume N^d.

Let

$$Q^i(m_1,\dots,m_d) = (l_1^i + 1 + m_1 N, m_2 N, \dots, m_d N) + \rho_{l^N}$$

where $0 \leq m_1 < p_i$, $0 \leq m_j < p$ $(2 \leq j \leq d)$.

Note that there are $p_i p^{(d-1)}$ such cubes. Since for

$$\bar{l} = (l_1^i + 1 + m_1 N, m_2 N, \dots, m_d N) \quad \phi^{\bar{l}} x \in U$$

we can choose a point $y^i_{(m_1,\dots,m_d)} \in \Omega$ such that

$$d\!l\,(y^i_{(m_1,\dots,m_d)}, \phi^{\bar{l}} x) < \alpha$$

By the choice of α we get for all $l \in \rho_{l^N}$

$$d\!l\,(\phi^l y^i_{(m_1,\dots,m_d)}, \phi^{l+\bar{l}} x) < \varepsilon .$$

Suppose

$$y^i_{(m_1,\dots,m_d)} \in \bigcap_{l \in \rho_{l^N}} \phi^{-l} A_{i_l} \quad \text{for some} \quad (i_l)_{l \in \rho_{l^N}} \in E$$

then

$$\phi^{l+\bar{l}} x \in B_{i_l} \quad \text{for all} \quad l \in \rho_{l^N} .$$

We define for these l $C_{\bar{l}+l} := B_{i_l}$. To complete the definition of

the subcover of $\bigvee_{l \in \rho_{l^n}} \phi^{-l} \mathcal{L}_\varepsilon$ we let for all

$$l \in \rho_{l^n} \setminus \sum_{1 \leq i \leq t+1} \sum_{\substack{0 \leq m_1 < p_i \\ 0 \leq m_j < p (2 \leq j \leq d)}} Q^i(m_1,\dots,m_d)$$

C_l be an arbitrary B_i containing $\phi^l x$. This construction yields the following inequality

$$N(\bigvee_{l \in \rho_{l^n}} \phi^{-l} \mathcal{L}_\varepsilon) \leq \prod_{i=1}^{t+1} c_N^{p_i p^{(d-1)}} \cdot (s+t')^{\pi(n)}$$

where $\pi(n) = t(dN + 1)n^{d-1}$. Therefore

$$h(\phi, \mathcal{L}_\varepsilon) \leq \overline{\lim_{n\to\infty}} n^{-d} \left(\sum_{i=1}^{t+1} p_i p^{(d-1)} \log c_N + \pi(n) \log (s+t')\right)$$

$$\leq \overline{\lim_{n\to\infty}} n^{-d} \left(\sum_{i=1}^{t+1} r_i N^{-1} (nN^{-1})^{d-1}\right) \log c_N$$

$$= \overline{\lim_{n\to\infty}} n^{-1} N^{-d} (n-t) \log c_N =$$

$$= N^{-d} \log c_N < h(\phi|\Omega, \mathcal{A}_\varepsilon) + \varepsilon .$$

Remark.

We do not formulate explicitly an analogue to Lemma 2.1 [2] . But it is clear that a d-dimensional version of this lemma together with its proof is implicitly given by the proof above.

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