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PAUL TURÁN

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ON SOME PHENOMENA IN THE THEORY OF PARTITIONS

by

Paul TURÁN

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Three types of phenomena in the mentioned theory are discussed in this talk ; common is the fact that all have some background in group theory. Detailed proofs are given in some papers in print.

Netto conjectured in the last century that, the probability that taken two elements P_1 and P_2 from S_n , the symmetric group with n letters, they generate S_n , tends to 3/4 if $n \to \infty$. This was proved in 1967 by J. J. Dixon, combining a theorem of C . Jordan with the following lemma, due to P. Erdös and myself. If $1 \leqq b_1 < b_2 < \ldots < b_s \leqq n$ are integers then the number of $P \in S_n$ having no cycle length of b_j $(j = 1, 2, \ldots, s)$ cannot exceed $n! \left(\sum_{j=1}^{s} \frac{1}{b_j} \right)^{-1}$. (We remark that our proof for this lemma contains a mistake easy to correct). In connection with Netto's conjecture, Dr. J. Dénes raised the following interesting problem concerning partitions of n : Denoting by Π a generic partition of n ,what is the number of (Π_1, Π_2) pairs of partitions which do not have equal part sums. This problem is in its full generality unsolved yet but its investigation led already to an unexpected phenomenon. Let $p(n)$ denote the number of partitions of n ; the expression "for almost all k-tuples $(\Pi_1, \Pi_2, \ldots, \Pi_k)$ of partition" should mean "for all k-tuples with $o(p(n)^k)$ exceptions at most for $n \to \infty$". For a partition Π the set of its summands (with multiplicity) should be denoted by $\bar{\Pi}$ and the cardinality of $\bar{\Pi}$ by $|\bar{\Pi}|$. Then we assert :

THEOREM 1. - If $k \geqq 2$, $\delta > 0$ arbitrarily small but fixed then the inequality

$$|\bar{\Pi}_1 \cap \bar{\Pi}_2 \cap \ldots \cap \bar{\Pi}_k| \geqq \left(\frac{1}{k} - \delta\right) \max_{j=1,\ldots,k} |\bar{\Pi}_j|$$

holds for almost all k-tuples $(\Pi_1, \Pi_2, \ldots, \Pi_k)$.

For the sake of orientation I remark first that according to Hardy and Ramanujan (see [1])

$$p(n) = (1+o(1)) \frac{1}{4n\sqrt{3}} e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}$$

and that according to Erdös and Lehner, denoting the number of summands of the partition Π by $\ell(\Pi)$ the inequality

$$(1) \qquad \left|\ell(\Pi) - \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right| < \sqrt{n}\,\omega(n)$$

holds for almost all Π's , (see [2]) if only $\omega(x) \nearrow \infty$ for $x \to \infty$

In the case of restricted partitions Q , i. e. when each summand can occur at most once, a completely analogous theorem holds with $\frac{1}{k\,2^k \log 2}$ instead of $1/k$. We formulate even a more general theorem. Let for $n \to \infty$

$$n \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_k \leqq n(1+o(1))$$

and Q'_j an arbitrary restricted partition of n_j $(j = 1, 2, \ldots, k)$, $\overline{Q'_j}$ the set of summands of n_j , further $q(m)$ the number of unrestricted partitions of m . Then we assert the

THEOREM 2. - For an arbitrarily small fixed $\delta > 0$ and for almost all k-tuples $(Q'_1, Q'_2, \dots, Q'_k)$ (i. e. with exception of $o(q(n)^k)$ such k-tuples at most) the inequality

$$|\bar{Q}'_1 \cap \bar{Q}'_2 \cap \dots \cap \bar{Q}'_k| \geqq \left(\frac{1}{k 2^k \log 2} - \delta\right) \max_{j=1,\dots,k} |\bar{Q}'_j|$$

holds.

For the sake of orientation we remark that according to Hardy-Ramanujan [1]

$$(2) \qquad q(n) = \frac{1+o(1)}{4.\, 3^{1/4} n^{3/4}} e^{\frac{\pi}{\sqrt{3}}\sqrt{n}}$$

and denoting by $L(Q)$ the number of summands in Q the inequality of Erdös-Lehner [2]

$$(3) \qquad \left|L(Q) - \frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}\right| < n^{1/4}\,\omega(n)$$

holds for almost all Q's if only $\omega(X) \nearrow \infty$ for $x \to \infty$.

The contents of both theorems can be summarised shortly and not precisely by saying that fixing k for almost all k-tuples of partitions of n in question a positive percentage of summands occur in all the k partitions (independently of n).

The next phenomenon comes from the investigation of the asymptotical distribution of values of the expression

$$F(\Pi) = n! \frac{\prod_{1 \leqq \mu \leqq \nu \leqq m} (\lambda_\mu - \lambda_\nu + \nu - \mu)}{\prod_{j=1}^{m} (\lambda_j + m - j)!}$$

occuring in the representation theory of S_n. Here

$$\Pi : \begin{array}{l} \lambda_1 + \lambda_2 + \ldots + \lambda_m = n \\ \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m \geqq 1 \end{array}$$

stands for an arbitrary partition of n. One of the possible ways to get an "almost all" -theorem for the $F(\Pi)$ numbers is to get an orientation for almost all partitions on the distribution of its summands. The second phenomenon I want to mention is that such a theorem exists indeed. We assert namely the

THEOREM 3. - If $0 < \alpha < \frac{1}{2}$ then almost all partitions of n contain

$$(1+o(1))\, \alpha \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$$

summands which are $\leqq n^{\alpha}$.

The theorem of Erdös-Lehner in (1) gives again a useful orientation.

The third phenomenon arose in connection with our series of papers on statistical group theory with Erdös, notably with paper IV (see [3]). Here we needed an analogon of inequality (3) for the case when the summands are supposed to be only prime powers. In the paper [4] we found a general theorem in this direction. This runs as follows. The sequence

$$\Lambda : 1 \leqq a_1 < a_2 < \ldots$$

consists of integers, its counting function $\Lambda(X)$ should satisfy the restriction

$$(4) \quad \begin{array}{l} \Lambda(X) = A\, X^{\alpha}\{1 + O(\frac{1}{\log X})\} \\ A > 0 , \quad 0 < \alpha \leqq 1 . \end{array}$$

Moreover, denoting by $q(n, \Lambda)$ the number of restricted partitions of n from Λ we suppose the inequality

$$\log q(n, \Lambda) > B\, n^{\frac{\alpha}{\alpha+1}} \{ 1 - \log^{-\frac{1}{2\alpha+2}} n \} \tag{5}$$

with

$$B = (1+\alpha)\alpha^{-\frac{\alpha}{\alpha+1}} \{A\,\Gamma(\alpha+1)(1-\frac{1}{2^{\alpha}})\,\zeta(\alpha+1)\}^{\frac{1}{\alpha+1}} \tag{6}$$

holds. Then we proved in [4] that almost all restricted partition from Λ have

$$C(A,\alpha)\, n^{\frac{\alpha}{\alpha+1}} \{1+O(\log^{-\frac{1}{4\alpha+4}} n)\} \tag{7}$$

summands with a suitable explicit $C(A,\alpha)$. For the sake of orientation I remark that in the case when Λ consists of all natural numbers then (5) takes the form

$$\log q(n) > \frac{\pi}{\sqrt{3}} \sqrt{n}\, \{ 1 - \log^{-\frac{1}{4}} n \} \tag{8}$$

which can be proved by a "real Tauberian" argument. Then (7) gives back for this case Erdös-Lehner's theorem in (3) with a worse error term but deduced from a <u>general</u> theorem and without using (2). (7) can be phrased shortly (and somewhat imprecisely) that (4)-(5)-(6) imply that the "normal number" of summands in a restricted partition of n from Λ is

$$C(A,\alpha)\, n^{\frac{\alpha}{\alpha+1}} \quad . \tag{9}$$

Now it is natural to ask whether or not an analogous general theorem can be found for the normal number of summands of the <u>unrestricted</u> partitions from Λ ? We do not know such a theorem but we do know that the situation is radically different (which is the third phenomenon I want to discuss a bit this time). Let us namely consider the special case

$$\Lambda^{*} = \{ 1^2 ,\ 2^2 ,\ \ldots \} . \tag{10}$$

Our previously mentioned theorem can be applied and gives for the normal number of summands for all restricted partitions from Λ^*

$$C^* n^{1/3} \tag{11}$$

Now what can be said on the number of summands of the unrestricted partitions ? We remind the reader to the theorem of Hardy-Ramanujan (see [2]) according to which the number $r(n)$ of unrestricted partitions of n from Λ^* is

$$(1+o(1)) \frac{A_1}{n^{B_1}} e^{C_1 n^{1/3}} \qquad (A_1, B_1, C_1 \text{ positive constants}). \tag{12}$$

Denoting the number of unrestricted partitions of n from Λ^* containing at least m summands $1 = 1^2$ by $r_m^*(n)$ we have obviously

$$r_m^*(n) = r(n-m) .$$

Thus using (12) we get for

$$1 \leqq m = o(n)$$

the relation

$$\frac{r_m^*(n)}{r(n)} = \frac{r(n-m)}{r(n)} = (1+o(1))\, e^{C_1 \{(n-m)^{1/3} - n^{1/3}\}}$$

which tends to 1 as n tend to infinity when choosing

$$m = \left[\frac{n^{2/3}}{\omega(n)}\right] ,$$

$\omega(X) \nearrow \infty$ arbitrarily slowly. Hence -in great contrast to (11)- we got that for almost all unrestricted partitions of n from Λ the number of summands is

$$> \frac{n^{2/3}}{\omega(n)} , \tag{13}$$

if only $\omega(X) \nearrow \infty$ arbitrarily slowly. The result can be expressed shortly that "the lower normal number" of the summands in unrestricted partitions of n from Λ^* is $n^{2/3}/\omega(n)$. (Most probably "the upper normal number" of the summands in unrestricted partitions of n from Λ^* is

$$(14) \qquad n^{2/3}\,\omega(n)\,).$$

And considering instead of the sequence Λ^* the sequence of all ℓ^{th} powers this contrast in (11) and (13) can be made obviously much stronger.

The usefulness of "normal number theorems" is indicated by the many applications of the first such theorem, discovered by Hardy and Ramanujan (see [5]) according to which the number $U(n)$ of different prime factors of $n \leqq X$ satisfies apart from $o(X)$ such integers, the inequality

$$U(n) = (1+o(1)) \log\log X .$$

This and the above facts give interest of the following two problems. In both we suppose the counting function $\Lambda(X)$ of the sequences Λ satisfies the inequality

$$(15) \qquad \underline{\lim} \frac{\log \Lambda(X)}{\log X} \geqq C > 0 ,$$

further - denoting the number of unrestricted resp. restricted partitions of n from Λ by $F(n,\Lambda)$ resp. $G(n,\Lambda)$ that

$$(16) \qquad \lim_{n\to\infty} G(n,\Lambda) = \infty .$$

Then we ask

PROBLEM 1. - Let η be an arbitrary small fixed positive number. Does there exist a sequence Λ_o of natural numbers with (15)-(16) so that both normal numbers $f(n,\Lambda_o)$ and $g(n,\Lambda_o)$ exist for the number of summands in the unrestricted resp. restricted partitions of n from Λ_o and still we have

$$\frac{f(n, \Lambda_o)}{g(n, \Lambda_o)} > n^{1-\eta}$$

for $n > n_o(\eta)$?

In the previous example in (13)-(14) we had for the unrestricted case only lower and upper normal numbers.

Further we raise the

PROBLEM 2. - Does the existence of $f(n, \Lambda)$ imply that of $g(n, \Lambda)$ if Λ satisfies (15)-(16) ?

Omitting the explicit formulation of plausible analogous problems concerning $F(n, \Lambda)$ and $G(n, \Lambda)$ we mention finally in connection with theorem 1 the :

PROBLEM 3. - Let $k \geqq 2$ integer and real λ fixed and

$$n \leqq n_1 \leqq n_2 \leqq \dots \leqq n_k \leqq n\,(1+o(1)) .$$

Denoting by $K(n_1, n_2, \dots, n_k ; \lambda)$ the number of k-tuples of unrestricted partitions $(\Pi_1', \Pi_2', \dots, \Pi_k')$ of $n_1, n_2, \dots, n_k$ with property

$$|\overline{\Pi}_1' \cap \overline{\Pi}_2' \cap \dots \cap \overline{\Pi}_k'| \leqq \frac{\sqrt{6}}{2\pi k} \sqrt{n} \log n + \lambda \sqrt{n}$$

is it true that

$$\lim_{n \to \infty} \frac{K(n_1, n_2, \dots, n_k ; \lambda)}{\mu(n_1)\,\mu(n_2) \dots \mu(n_k)} = \Phi(\lambda)$$

exists ?

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Paul TURÁN
Mathematical Institute of
the Hungarian
Academy of Science
BUDAPEST V.
Realtanoda U. 13-15