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# A THEOREM ON THE MONODROMY OF ISOLATED SINGULARITIES

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INTRODUCTION . – Let $\pi : (X, x) \to (T, t)$ be a flat morphism between germs of smooth complex spaces, $(\Delta, t)$ its discriminant. Suppose that the fibre $(X_t, t) = \pi^{-1}(t)$ is a hypersurface with an isolated singularity at $x$. Then $\pi$ induces a fibre bundle on $T - \Delta$ whose fibre $M$ has the homotopy type of a bouquet of spheres of dimension $r = \dim X - \dim T$ (see [4]) ; the associated representation $\pi_1(T - \Delta, t) \to \mathrm{Aut}(H_r(M, \mathbb{Z}))$ is called the monodromy of $\pi$. By looking at the properties of a representation of $\pi_1(T - \Delta, t)$ in the case that $\pi$ is semiuniversal, we show an irreducibility property of such a representation. As a consequence we get a no-splitting principle for a hypersurface isolated singularity, that extends a known result for curves [3] .

1 . THE MONODROMY OF $\pi$ . – Let $\pi : (\mathbb{C}^N, 0) \to (\mathbb{C}^n, 0)$ be a flat morphism whose fibre $(X_0, 0) = \pi^{-1}(0)$ is a hypersurface with an isolated singularity at $0$. Denote by $(\Delta, 0)$ the discriminant of $\pi$. Choose a small ball $B$ around $0$ in $\mathbb{C}^N$. Then there exists a contractible open neighborhood $U$ of $0$ in $\mathbb{C}^n$ such that $\pi : B \cap \pi^{-1}(U) \to U$ is a smooth proper map and $\pi$ is of maximal rank on $\pi^{-1}(U - \Delta) \cap \mathring{B}$ and on $\pi^{-1}(U) \cap \partial B$. It follows that $\pi : B \cap \pi^{-1}(U - \Delta) \to U - \Delta$ is a differentiable fibre bundle, whose fibre $M$ is a compact differentiable $2r$-dimensional manifold with boundary, where $r = N - n$ ; moreover since $\pi$ is of maximal rank on $\pi^{-1}(U) \cap \partial B$ we may suppose that the group of the bundle is made with diffeomorphisms of $M$ which are the identity on $\partial M$.
One knows (see [4]) that $M$ is a parallelizable manifold which is homotopically equivalent to a bouquet of $\mu$ spheres of dimension $r$. In particular $H_r(M, \mathbb{Z})$ is a free module of rank $\mu$ over $\mathbb{Z}$. Let $p \in U - \Delta$ and identify $M$ with $\pi^{-1}(p) \cap B$. Then there is a homomorphism $\pi_1(U - \Delta, p) \to \mathrm{Aut}\, H_r(M, \mathbb{Z})$. Letting $U$ vary, one deduces a homomorphism $\sigma : \pi_1(\mathbb{C}^n - \Delta, 0) \to \mathrm{Aut}\, H_r(M, \mathbb{Z})$, where $\pi_1(\mathbb{C}^n - \Delta, 0)$ denotes the local fundamental group of $\mathbb{C}^n - \Delta$ at $0$. (*)

(*) Remark that $\pi_1(\mathbb{C}^n - \Delta, 0)$ is defined up to an inner automorphism ant that the indeterminacy of $\sigma$ is just that of an inner automorphism of $\pi_1(\mathbb{C}^n-\Delta,0)$.

2 . <u>THE PRESENTATION OF</u> $\pi_1(\mathbb{C}^n - \Delta, 0)$ . – Let $(\Delta, 0) \subset (\mathbb{C}^n, 0)$ be defined by an equation

$$w^m + a_1(z)w^{m-1} + \ldots + a_m(z) = 0$$

where $(w, z_1, \ldots, z_{n-1})$ are local coordinates on $(\mathbb{C}^n, 0)$ and $a_i(0) = 0$ for $i = 1, \ldots, m$ .

Consider a nice stratification of $\Delta$ , say a stratification verifying Whitney's conditions. By the curve selection lemma one can see that there exists $\varepsilon_0 > 0$ s.th. for $0 < \varepsilon < \varepsilon_0$ the hypersurface $\|z\| = \varepsilon$ is transversal to each stratum. It follows that, if $U_\varepsilon = \{(w, z) \mid \|z\| < \varepsilon\}$ then for $0 < \varepsilon' < \varepsilon < \varepsilon_0$ the inclusion $U_{\varepsilon'} - \Delta \to U_\varepsilon - \Delta$ is a homotopy equivalence. Moreover let $\eta(\varepsilon) = \sup\limits_{U_\varepsilon \cap \Delta} |w|$ , $U_{\varepsilon,\eta(\varepsilon)} = \{(w, z) \in U_\varepsilon \mid |w| < \eta(\varepsilon)\}$. Then the inclusion $U_{\varepsilon,\eta(\varepsilon)} - \Delta \to U_\varepsilon - \Delta$ is a homotopy equivalence. Since $a_i(0) = 0$ for $i = 1, .., m$ one has that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ , so that $U_{\varepsilon,\eta(\varepsilon)}$ is an arbitrary small neighborhood of $0$ in $\mathbb{C}^n$ . In particular if $0 < \varepsilon < \varepsilon_0$ and $p \in U_\varepsilon - \Delta$ , then $\pi_1(\mathbb{C}^n - \Delta, 0) \to \pi_1(U_\varepsilon - \Delta, p)$ is an isomorphism.

<u>NOTATIONS</u>. – $U = U_\varepsilon$ ; $V = U \cap \{w = 0\}$; $\varphi : U \to V$ the projection. Fix $|w_0| >> \varepsilon$ . For $z \in V$ , $L_z$ is the straight line $\varphi^{-1}(z)$ and $p_z = (w_0, z) \in L_{z_0}$ .

Finally let $\Gamma$ denote the discriminant of $\varphi : U \cap \Delta \to V$ , $\tilde{\Gamma} = \varphi^{-1}(\Gamma)$ . Suppose $z_0 \in V - \Gamma$ . then one has a diagram :

$$\begin{array}{ccccccc} 0 \to \pi_1(L_{z_0} - \Delta) & \xrightarrow{j} & \pi_1(U - \Delta \cup \tilde{\Gamma}) & \underset{\gamma}{\overset{\beta}{\rightleftarrows}} & \pi_1(V - \Gamma) \to 0 \\ & & \downarrow \alpha & & \\ & & \pi_1(U - \Delta) & & \end{array}$$

where the base point is always $p_{z_0}$ , $j$ and $\alpha$ are induced by inclusions, $\beta$ by $\varphi$ and $\gamma$ by the map $z \mapsto (w_0, z)$ .

Remark that $\varphi : U - \Delta \cup \tilde{T} \to V - T$ is a fibre bundle with fibre $L_{z_0} - \Delta$ and that $z \to (w_0, z)$ induces a cross section of such a bundle. So from the homotopy sequence of a fibre bundle we get

1) the horizontal line is exact and $\beta \circ \gamma$ is the identity on $\pi_1(V - \Gamma)$ .

Moreover one has obviously

2) $\alpha$ is surjective

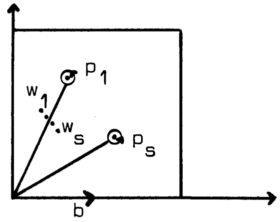
3) $\alpha \circ \gamma$ is the null homomorphism

Consider the sequence $0 \to \pi_1(V - \Gamma) \xrightarrow{\gamma} \pi_1(U - \Delta \cup \tilde{\Gamma}) \xrightarrow{\alpha} \pi_1(U - \Delta) \to 0$ .

This must not be exact at $\pi_1(U - \Delta \cup \tilde{\Gamma})$ . Nevertheless one has

4) $\ker \alpha$ is generated by the conjugated of $\operatorname{Im} \gamma$ .

<u>Proof</u>. – Obviously for $v \in \operatorname{Im} \gamma$ and $b \in \pi_1(U - \Delta \cup \tilde{\Gamma})$ one has $b^{-1}vb \in \ker \alpha$ . Let $b \in \ker \alpha$ . Then $b = \partial c$ with $c \in \pi_2(U - \Delta\ ,\ U - \Delta \cup \tilde{\Gamma})$ . Let us represent $c$ by a map $\hat{c} : [0, 1] \times [0, 1] \to U - \Delta$ which is transversal to $\tilde{\Gamma}$ . Then $\hat{c}^{-1}(\tilde{\Gamma})$ is a finite set of points, let say $p_1, \ldots, p_s$ . The following picture shows that $b$ is equivalent in $\pi_1(U - \Delta \cup \tilde{\Gamma})$ to a product $w_1 \ldots w_s$ of simple loops around $\tilde{\Gamma}$




$w_i$ simple means that it is composed of an arc $\tau$ from $p_{z_0}$ to a point near a regular point $\tilde{p}$ of $\tilde{\Gamma}$ , a small circle around $\tilde{\Gamma}$ and then back with $\tau^{-1}$ . One can construct a cylinder in $U - \Delta \cup \tilde{\Gamma}$ whose boundaries are two circles, one being that of $w_i$ , the other being in $V - \Gamma$ . Choose an arc $\tilde{\tau}$ in $V - \Gamma$ from $p_{z_0}$ to that circle, and call $v_i$ the resulting simple loop ; if $\alpha_i$ is defined so that it follows $\tau$ and then a path along the cylinder from one circle to the other and then $\tau^{-1}$ , one realizes that $w_i = \alpha_i v_i \alpha_i^{-1}$ . So $b$ is a product of elements, each of which conjugated to an element of $\operatorname{Im} \gamma$ .

<u>COROLLARY</u>. – $\ker \alpha \cap \ker \beta$ is the minimal normal subgroup $N$ of $\ker \beta$ that contains the elements of the form $bvb^{-1}v^{-1}$ with $b \in \ker \beta$ , $v \in \operatorname{Im} \gamma$ .

<u>Proof</u>. – Let $b \in \pi_1(U - \Delta \cup \tilde{\Gamma})$ , $v \in \operatorname{Im} \gamma$ .

Then $\quad bvb^{-1} = (b.\gamma\beta(b^{-1})).(\gamma\beta(b).v.\gamma\beta(b^{-1})).(\gamma\beta(b).b^{-1}) = \bar{b}.\bar{v}.\bar{b}^{-1}$

where $\bar{v} \in \operatorname{Im} \gamma$ , $\bar{b} \in \ker \beta$ . So if $b \in \ker \alpha$ , one has from 4) and this remark that $\bar{b} = b_1 v_1 b_1^{-1} \dots b_s v_s b_s^{-1}$ with $b_i \in \ker \beta$, $v_i \in \operatorname{Im} \gamma$. Moreover $b \in \ker \beta$ implies $\beta(v_1 \dots v_s) = 1$ and hence $v_1 \dots v_s = 1$, since $\beta$ is injective on $\operatorname{Im} \gamma$. Let $n_i = b_i v_i b_i^{-1} v_i^{-1} \in N$ ; then $b = n_1 v_1 \dots n_s v_s = v_s^{-1} \dots v_1^{-1} n_1 v_1 \dots n_s v_s$. from this and the remark that $v \in \operatorname{Im} \gamma$, $n \in N$ implies that $v^{-1} n v \in N$ one deduces $b \in N$.

Let $R$ be a straight line in $V$ s. th. $\pi_1(R - \Gamma) \to \pi_1(V - \Gamma)$ is surjective and let $H$ denote $\varphi^{-1}(R)$. From the preceding result we know that $\pi_1(H - \Delta)$ and $\pi_1(U - \Delta)$ have the same generators with the same relations, hence they are isomorphic. This is the local version of a theorem of Zariski – Van Kampen on the presentation of the fondamental group of $\mathbb{P}_n$ minus a hypersurface ; compare the article of Cheniot [2] in this volume. Now $H = \mathbb{C} \times E$ where $E = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ and $\Delta_1 = \Delta \cup H$ is defined by some equation $w^m + b_1(z) w^{m-1} + \dots + b_m(z) = 0$, with the $b_i$ holomorphic on $E$. Suppose that $\pi$ is versal. Then (see [1], [6]) one may suppose that $\Delta_1$ has the following properties :

i) it is irreducible

ii) it has only cusps or ordinary double points as singularities, with distinct images on $E$.

iii) it is flat on the z-direction, i.e. $z$ is a transversal parameter at any point of $\Delta_1$.

Let $\sigma : \tilde{\Delta}_1 \to \Delta_1$ be the normalization of $\Delta_1$ and consider $\tau = \varphi \circ \sigma : \tilde{\Delta}_1 \to E$. Then $\tau$ is only ramified at points that correspond to cusps of $\Delta_1$, and the ramification index at those points is two.

<u>LEMMA</u>. – The permutation group of the Riemann surface $\Delta_1$ is the full group of permutations of $r$ elements.

<u>Proof</u>. – It is sufficient to remark that a) it is transitive (because of the irreducibility of $\Delta_1$ ) and b) it is generated by transpositions (because $\Delta_1$ is simply ramified over distinct points).

Let $\gamma$ be a simple loop in $L_{z_0} - \Delta$ from $z_0$ turning positively around some element of $L_{z_0} \cap \Delta$. Its image in $\pi_1(U - \Delta)$ will be called a geometric generator.

THEOREM 1. – i) Let $\gamma_1, \gamma_2$ be geometric generators ; there exists $\delta \in \pi_1(U - \Delta)$ s. th. $\gamma_1\delta = \delta.\gamma_2$ .

ii) Suppose that $\Delta$ is not smooth, and let $\gamma$ be a geometric generator ; there exists a geometric generator $\gamma'$ s. th. $\gamma.\gamma'.\gamma = \gamma'.\gamma.\gamma'$ .

Proof. – i) Remark first that any two loops around the same $z_i$ are conjugated in $\pi_1(L_{z_0} - \Delta)$ and hence also in $\pi_1(U - \Delta)$ . Then the preceding lemma assures that if $z_i, z_j \in L_{z_0} \cap \Delta$ and $\gamma$ is a loop around $z_i$ , then $\gamma$ is equivalent in $\pi_1(U - \Delta)$ to some loop around $z_j$ .

ii) Since $\Delta$ is not smooth, $\Delta_1$ must be ramified somewhere so that $\Delta_1$ has at least a cusp. Suppose that $z_0$ is near that ramification point and that $\gamma$ is a loop around a point near the corresponding cusp. Then the loop $\gamma'$ that goes like $\gamma$ until that cusp and then links the other point near the cusp, satisfies the required relation. In general one knows from i) that $\gamma$ is conjugated to such an element. Then it remains only to remark that each conjugated to a geometric generator is a geometric generator.

3 . PICARD-LEFSCHETZ THEORY . – This theory describes the monodromy of a semiuniversal deformation in the following way (see [5]) : the fibre over each simple point of $\Delta$ has just an isolated singularity of the type $\sum_0^r x_i^2 = 0$ ; hence to each geometric generator $\gamma$ , there is associated a vanishing cycle $e \in H_r(M, \mathbb{Z})$ uniquely determined up to the sign by $\gamma$ . The action of $\gamma$ on $H_r(M, \mathbb{Z})$ is given by the Picard-Lefschetz formula

$$(*) \qquad h \to h + (-1)^s \langle e, h\rangle e \qquad , \quad h \in H_r(M, \mathbb{Z})$$

where $s = (r+1).(r+2)/2$ and $\langle\ ,\ \rangle : H_r(M, \mathbb{Z}) \times H_r(M, \mathbb{Z}) \to \mathbb{Z}$ is the cap product. Moreover $\langle e, e\rangle$ is zero if $r$ is odd and $2.(-1)^{r.(r+1)/2}$ if $r$ is even.

Let $\{z_1, \dots, z_m\} = L_{z_o} \cap \Delta$ and choose simple loops $\gamma_1, \dots, \gamma_m$ in such a way that $\gamma_i$ turns positively around $z_i$ and $\gamma_i$, $\gamma_j$ don't intersect outside $z_o$. Call $e_i$ the vanishing cycle associated to $\gamma_i$. Then $\gamma_1, \dots, \gamma_m$ generate freely $\pi_1(L_{z_o} - \Delta)$ and $e_1, \dots, e_m$ are a base of $H_r(M, \mathbb{Z})$ over $\mathbb{Z}$.

<u>THEOREM 2</u>. – Let $I$, $J$ be a partition of $\{1, \dots, m\}$. There exist $i \in I$ and $j \in J$ s. th. $\langle e_i, e_j \rangle \neq 0$.

<u>Proof</u>. – From formula $(*)$ one gets that the images $\bar{\gamma}_i$, $\bar{\gamma}_j$ of $\gamma_i$, $\gamma_j$ in $\operatorname{Aut} H_r(M, \mathbb{Z})$ commute if $\langle e_i, e_j \rangle = 0$. Suppose that this happens for all $i \in I$ and $j \in J$. Fix $i_1 \in I$, $j_1 \in J$; because of theorem 1 one can write $\gamma_{i_1} \delta = \delta . \gamma_{j_1}$ where $\delta \in \pi_1(U - \Delta)$ and hence $\delta$ is a product of $\gamma_i$, $\gamma_j$. Since each $\bar{\gamma}_i$ commutes with each $\bar{\gamma}_j$ one can write $\bar{\delta} = \bar{\delta}_J . \bar{\delta}_I$ where $\bar{\delta}_I$ is a product of $\bar{\gamma}_i$ and $\bar{\delta}_J$ a product of $\bar{\gamma}_j$. So one has $\bar{\delta}_J^{-1} . \bar{\gamma}_{i_1} . \bar{\delta}_J = \bar{\delta}_I . \bar{\gamma}_{j_1} . \bar{\delta}_I^{-1}$ and hence $\gamma_{i_1} = \gamma_{j_1}$. This equality, with the help of formula $(*)$ and the fact that $e_{i_1} \neq e_{j_1}$, gives $\langle e_{i_1}, h \rangle = 0$ for all $h \in H_r(M, \mathbb{Z})$. This cannot happen. In fact theorem 1 says that there exists a geometric generator $\gamma'$ such that $\gamma_{i_1} . \gamma' . \gamma_{i_1} = \gamma' . \gamma_{i_1} . \gamma'$ ; if $e'$ denotes the vanishing cycle associated to $\gamma'$ one sees from formula $(*)$ that this relation is equivalent to $\langle\ ',\ {}_{i_1} \rangle = \pm 1$ and this concludes the proof.

<u>COROLLARY</u>. – The set of points where $\Delta$ is locally reducible is contained in the set of points where $\Delta$ has smaller multiplicity than at the origin.

<u>Proof</u>. – Let $t' \in \Delta$ where $\Delta$ has irreducible components $\Delta_1, \dots, \Delta_s$, $s \geq 2$. Then $\pi^{-1}(t')$ has $s$ singular points $x_1, \dots, x_s$ s.th. the multiplicity $m_i$ of $\Delta_i$ at $t'$ is the number of vanishing cycles at $x_i$, $i = 1, \dots, s$. Choose $L_{z_o}$ near $t'$ and define $\{z_1, \dots, z_m\} = \Delta \cap L_{z_o}$, $I = \{1, \dots, m\}$, $I_\alpha = \{i \in I \mid z_i \in \Delta_\alpha\}$, $\alpha = 1, \dots, s$. Suppose $\sum_1^s m_i = m$ ; it follows that $(I_\alpha)_\alpha$ is a partition of $I$. Let $e_i$ denote the vanishing cycle at $z_i$. One has $\langle e_i, e_j \rangle = 0$ for $i \in I_\alpha$, $j \in I_\beta$ and $\alpha \neq \beta$ ; in fact

$e_i$, $e_j$ have representative cycles lying in disjoint balls around $x_\alpha$, $x_\beta$ respectively. This cannot happen because of the theorem 2, so that the multiplicity $\sum_1^s m_\alpha$ of $\Delta$ at $t'$ must be less than $m$.

Remark. – This result can be expressed in the following way : if a deformed fibre of a hypersurface isolated singularity has more than one singularity, then the direct sum module of vanishing cycles at those singularities is a proper submodule of that of vanishing cycles at the original singularity.

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