HERBERT POPP

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INTRODUCTION TO THE ALGEBRAIC THEORY OF FUNDAMENTAL GROUPS

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Recently topologists became interested, in connection with monodromy and deformation problems, in calculating explicitly the topological fundamental group of the projective space $\mathbb{P}^N$ minus a hypersurface C. Compare [1].

This problem is difficult to handle, if the hypersurface C has singularities. We indicate here an attack by algebraic methods and describe the main results. A more complete treatment of the matter can be found in [6].

Let k an algebraically closed field of characteristic $p \geq 0$. V shall be an irreducible, quasi projective k-scheme which is <u>reduced</u> and <u>normal</u>.

<u>DEFINITION 1.</u> An irreducible, reduced k-scheme V' together with a surjective morphism $f:V' \longrightarrow V$, which is finite, is called a <u>covering</u> of V.

<u>DEFINITION 2.</u>

a) - For a covering $f:V' \longrightarrow V$ of V the degree of the field extension $k(V')/k(V)$ is called <u>the degree of the covering.</u> ($k(V)$ = function field of V.)

b) - A covering $f:V' \longrightarrow V$ is called <u>galois</u> if the field extension $k(V')/k(V)$ is galois and <u>abelian</u> if $k(V')/k(V)$ is abelian.

<u>DEFINITION 3.</u> A covering $f:V' \longrightarrow V$ of V is called <u>unramified</u> at $P' \in V'$ if

a) - the maximal ideal $m_{P',V'}$ of the local ring $(O_{P',V'}, m_{P',V'})$ of P' on V' is generated by the maximal ideal $m_{P,V}$ of $O_{P,V}$ where $P = f(P')$ is the image of P' and $(O_{P,V}, m_{P,V})$ the corresponding local ring of P on V.

b) - The residue field $O_{P',V'}/m_{P',V'}$ is a separable extension of the residue field $O_{P,V}/m_{P,V}$.

If every point $P' \in V'$ of a covering $f: V' \longrightarrow V$ is unramified, we say that the covering $V' \xrightarrow{f} V$ is an unramified covering of V.

Using local algebra, see [6] , lecture 1, one can prove the following propositions.

PROPOSITION 1. Let $f: V' \longrightarrow V$ be a covering of V of degree n. Then f is unramified if and only if for every closed point $P \in V$ the inverse image $f^{-1}(P)$ contains n distinct points.

PROPOSITION 2. Let $f: V' \longrightarrow V$ be an unramified covering of V. Then V' is the normalisation of V in the field k(V') and the map f is the canonical map from the normalisation onto V.

We are interested in describing the finite, unramified coverings of V by a group $\pi_1(V)$ via galois theory, i.e., we want a group $\pi_1(V)$ such that the subgroups of $\pi_1(V)$ of finite index correspond in a natural way to the (finite) unramified coverings of V. Proposition 2 can be used to obtain such a group.

Let $K = k(V)$ be the field of rational functions on V and $\bar{K}$ a separable algebraic closure of K.

If $V' \xrightarrow{f} V$ is an unramified covering of V, the function field $k(V')$ of V' can be embedded into $\bar{K}$. By proposition 2, V' is the normalisation of V in $k(V')$ and is therefore determined by $k(V')$.

Consider now the set Σ of subfields of $\bar{K}$, defined as follows

Σ ={ K' ; K' subfield of $\bar{K}$ containing K;
K'/K finite ; the normalisation of V
in K' is unramified over V. }

One shows, see [6] , lecture 1, that the composit of two fields K_1, K_2 in Σ is again in Σ. Therefore with any field $K' \in \Sigma$ one has also the galois closure of K' over K in Σ. Let $\tilde{K} = \bigcup_{K' \in \Sigma} K'$. Then $\tilde{K}$ is a galois fiels extension of K. Denote by $\pi_1(V)$ the galois group of the extension $\tilde{K}/K$. By general galois theory the subgroups of $\pi_1(V)$ of finite index are in 1-1 correspondence in a natural way with the fields $K' \in \Sigma$. Using proposition 2 we obtain that the group $\pi_1(V)$ has the desired properties.

<u>DEFINITION 4.</u> $\pi_1(V)$ is called the <u>fundamental group</u> of V.

Over the complex number field $\mathbb{C}$ what is the relation between the topological fundamental group $\pi_1^{top}(V)$ of an irreducible, reduced, normal and quasi projective $\mathbb{C}$-scheme V and the group $\pi_1(V)$?

By the general RIEMANN existence theorem due to GRAUERT and REMMERT [4] , one knows that the finite connected topological coverings of the $\mathbb{C}$-scheme V are in 1-1 correspondence with the unramified coverings of V.

in the sense of definition 1.1. This shows that $\pi_1(V)$, which is by definition a profinite group, is the profinite completion of $\pi_1^{top}(V)$.

REMARK 1. : It is well known (compare [6] , lecture 1) that for an irreducible, reduced and normal k-scheme V the connected etale coverings in the sens of [5] coincide with the unramified covering of V in the sense of definition 1. We conclude from this that π_1 (V) coincides with the π_1 (V) introduced in [5] by Grothendieck.

REMARK 2. : Serre has shown in [7] by an example that the natural map from $\pi_1^{top}(V)$ into its profinite completion is not always injective. Hence, π_1 (V) contains less information than $\pi_1^{top}(V)$.

We have already stated at the beginning that we want to determine by algebraic methods the structure of $\pi_1(\mathbb{P}^N - C)$, where C is a hypersurface of $\mathbb{P}^N$.

For simplicity we restrict from now on to the case where the ground field k has characteristic 0. In the case when characteristic $k = p > 0$, the method described can be used to obtain results on the p-prime part of the fundamental group. Details are in [6] .

THEOREM 1. Let C be an irreducible, smooth hypersurface of degree d in $\mathbb{P}^N$. Then $\pi_1(\mathbb{P}^N - C)$ is the cyclic group of order d.

PROOF : Let $X_o, \ldots, X_N$ be projective coordinates in $\mathbb{P}^N$ and H_o be the hyperplane defined by $X_o = 0$. Let $\mathbb{A}_o^N = \mathbb{P}^N - H_o$ and $x_1 = \frac{X_1}{X_o}, \ldots, x_N = \frac{X_N}{X_o}$ affine coordinates in $\mathbb{A}_o^N$. Assume that $C \not\subset H_0$. Let $f(x_1,\ldots,x_N) = f(x)$ be a defining polynomial for $C \cap \mathbb{A}_0^N$ of degree d. We describe first the abelian, unramified coverings of $\mathbb{P}^N - C$. Let $V \longrightarrow \mathbb{P}^N - C$ be such an abelian covering of $\mathbb{P}^N - C$ and $k(V) \supset k(x)$ ($k(x)$ = function field of $\mathbb{P}^N$) the corresponding field extension. Then $k(V)/k(x)$ is a galois extension with an abelian galois group and as such the composite of the cyclic subfields of $k(V)$ which contain $k(x)$. We can restrict our attention therefore to cyclic unramified coverings of $\mathbb{P}^N - C$.

Let $V \longrightarrow \mathbb{P}^N - C$ be such a cyclic unramified covering and $k(V)/k(x)$ the corresponding cyclic field extension. By general Kummer theory $k(V)$ can be generated over $k(x)$ by an element z which satisfies an irreducible equation $z^n = h(x)$ with $h(x) \in k(x)$. We can change z to $z' = \alpha . z$ with $\alpha \in k(x)$. Then z' satisfies then the equation $z'^n = \alpha^n . h$. Write $h(x) = \dfrac{h_1(x)}{h_2(x)}$ with $h_1, h_2 \in k[x]$. By taking $z' = h_2(x) z$ as a generator of $k(V)$ we find that z' satisfies an equation of the form

$$z'^n = h(x), \text{ with } h(x) \in k[x] .$$

Let $h(x) = h_1^{a_1}(x) \ldots h_r^{a_r}(x)$ be the prime factorisation of h. By changing the generator, if necessary, we may assume that the equation for the generator z is of the form

$$(1) \qquad z^n = h_1^{a_1} \ldots h_r^{a_r}$$

with $0 < a_i < n$.

We return to the cyclic field extension $k(V) \supset k(x)$ and the covering $V \longrightarrow \mathbb{P}^N - C$ where V is the normalisation of $\mathbb{P}^N - C$ in $k(V)$.

Let V^* be the normalisation of $\mathbb{P}^N$ in $k(V)$ and $f':V^* \longrightarrow \mathbb{P}^N$ the natural covering map. Then $f':V^* \longrightarrow \mathbb{P}^N$ is ramified at most along C and the restriction of f' to $V^* - f'^{-1}(C)$ is the covering $f:V \longrightarrow \mathbb{P}^N - C$. (Notice that $V^* - f'^{-1}(C)$ is isomorphic to V.)

The irreducible polynomials $h_1,\ldots,h_r$ which are divisors of h in the equation (1) define hypersurfaces $C_1,\ldots,C_r$ in $\mathbb{P}^N$ which are easily shown to be ramified in the covering $V^* \longrightarrow \mathbb{P}^N$. This implies that (1) is of the form

(2) $z^n = f^a, \quad 0 < a < n,$

where f is a defining polynomial of C of degree d.
The degree of the covering $V \longrightarrow \mathbb{P}^N - C$ is n. Therefore, the integers a and n in equation (2) are relatively prime.
Let u, v be integers such that $un + va = 1$. Then

$$(z^v)^n = f^{va} = f^{1-un}$$

or

$$(z^v . f^u)^n = f.$$

This shows that $z' = z^v f^u$ generates k(V) over k(x) and z' satisfies the equation

(3) $(z')^n = f.$

We have not considered so far the hyperplane H_0, which is by assumption unramified in the covering $V^* \longrightarrow \mathbb{P}^N$. To consider H_0 we pass to the affine space $\mathbb{A}_1^N = \mathbb{P}^N - H_1$, where H_1 is the hyperplane defined by

$X_1 = 0$. $\frac{X_0}{X_1}, \frac{X_2}{X_1}, \ldots, \frac{X_N}{X_1}$ are affine coordinates in $\mathbb{A}^N_1$. If we write (3) in terms of this coordinates, we get the equation

$$(z')^n = (X_1)^d f\left(\frac{X_0}{X_1}, \frac{X_2}{X_1}, \ldots, \frac{X_N}{X_1}\right).$$

The equation for $H_0 \cap \mathbb{A}^N_1$ is $\frac{X_0}{X_1} = 0$. By assumption H_0 is unramified in the covering $V^* \longrightarrow \mathbb{P}^N$. This implies that d is a multiple of n.

Combining we have the following proposition :

PROPOSITION 3. Let $V \longrightarrow \mathbb{P}^N - C$ be an unramified abelian covering and let $f(x_1,\ldots,x_N) = 0$ be a defining equation of $C \cap \mathbb{A}^N_0$. (The notation is as above). Then, the function field k(V) is of the form $k(x_1,\ldots,x_N)(\sqrt[n]{f})$ where n is an integer which divides the degree of C and V is the normalisation of $\mathbb{P}^N - C$ in $k(x)(\sqrt[n]{f})$.

Next we have to show that every unramified covering $V \longrightarrow \mathbb{P}^N - C$ is abelian. For this purpose it is necessary to recall some well known facts about the galois theory of local rings. Let $V' \xrightarrow{f} V$ be a normal galois covering of V, i.e., V' is normal, and let G be the galois group of the covering $f : V' \longrightarrow V$. Suppose W is an irreducible subvariety of V and W' an irreducible component of $f^{-1}(W)$. By $(O_{W',V'}, m_{W',V'})$ we denote the local ring of W' on V'.

DEFINITION 4. The group $I = I(W') = \{ \sigma \in G ;\ \sigma(x) \equiv x \text{ modulo } m_{W',V'},\ \forall x \in O_{W',V'} \}$ is called the inertia group of W'.

The significance of this group is described by the following proposition (compare [6] , lecture 1) :

PROPOSITION 4. Let k(V') be the function field of V' and $k(V')^I$ be the fixed field of I. V'^I shall be the normalisation of V in $k(V')^I$. Consider the diagram $V' \xrightarrow{f_1} V'^I \xrightarrow{f_2} V$ and let $f_1(W') = W'^I$ be the image of W' in V'^I. Then W'^I is unramified over W in the covering $V'^I \xrightarrow{f_2} V$ and V'^I is the largest normal covering of V between V' and V with this property.

We need the following 4 general theorems from algebraic geometry (compare [6], lecture 2, for comments).

THEOREM 2. (Purity of the branch locus). Let V be a smooth, irreducible k-scheme and $V' \xrightarrow{f} V$ be a normal covering. Let Δ be the branch locus of $f:V' \longrightarrow V$, i.e. $\Delta = \{P \in V;$ at least one point $P' \in V'$ lying over P is ramified in $f:V' \longrightarrow V\}$. Then Δ is a closed subvariety of V which is of pure codimension 1.

Let V be a normal irreducible k-scheme and L a linear system of effective divisors of V. Suppose dimension L $\geqslant$ 1. Let $\phi_L : V \longrightarrow \mathbb{P}^N$ be the rational map which is determined by L. We say that L is composite with a pencil if the dimension of the image variety $\phi_L(V)$ is 1. Zariski has proved the following theorem :

THEOREM 3. (Theorem of Bertini). Let L be a linear system of divisors of an irreducible normal k-scheme V. Assume that L has no fixed components and dimension ≥ 2. Then L is composite with a pencil if and only if every divisor of L is reducible.

THEOREM 4. (Connectedness Theorem). The specialisation of a connected, projective variety is connected.

THEOREM 5. (Local non splitting). Let $V' \xrightarrow{f} V$ be a (connected) normal covering of a smooth variety V, i.e. V' is connected and normal. Let $P \in \Delta$, the branch locus, and assume that Δ has at P only strong normal crossings as singularities. *) If Δ_1 is an irreducible component of Δ with $P \in \Delta_1$, and $P' \in V'$ a point lying over P, only one component of $f^{-1}(\Delta_1)$ passes trough P'.

Consider now the coverings of $\mathbb{P}^N$ which are ramified at most along C.

Let $V^* \longrightarrow \mathbb{P}^N$ be a galois covering of this type, where V^* is normal and G is the galois group.

Let $C_1^*,\dots,C_r^*$ be the irreducible components of the inverse image of C in V^* and $I_i = I_i(C_i^*)$ the inertia groups of the C_i^*.

By definition $I_i(C_i^*)$ is the inertia group of the local ring of C_i^* which is a discrete valuation ring of rank 1. This implies that $I_i(C_i^*)$ is cyclic.

CLAIM : G is equal to the subgroup $I = < I_1,\dots,I_r >$ generated by the inertia groups I_i.

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*) This means that there exist elements $x_1,\dots,x_r$ in the local ring $(O_{P,V}, m_{P,V})$ of P on V which are a part of a regular parameter system of $O_{P,V}$ such that $x_1 . x_2 \dots x_r = 0$ is a local equation for Δ at P.

PROOF : Consider $k(V^*)^I$, the fixed field of $K(V^*)$ by I. We have to show that $k(V^*)^I = k(\mathbb{P}^N)$. Let V^{*I} be the normalisation of $\mathbb{P}^N$ in $k(V^*)^I$ and consider the diagram :

$$V^* \xrightarrow{f_1} V^{*I} \xrightarrow{f_2} \mathbb{P}^N, \qquad f = f_2 \circ f_1$$

Applying proposition 4 we conclude that the general point of C is unramified in $V^{*I} \xrightarrow{f_2} \mathbb{P}^N$. Hence, $V^{*I} \xrightarrow{f_2} \mathbb{P}^N$ is unramified in codimension 1 and by theorem 2 unramified everywhere. We show in the lemma below that the $\mathbb{P}^N$ has no unramified (connected) coverings and obtain from these considerations that $V^{*I} = \mathbb{P}^N$ and $G = \langle I_1,\dots,I_r \rangle$.

Lemma : $\mathbb{P}^N$ has no non trivial unramified coverings.

PROOF : Let $V \xrightarrow{f} \mathbb{P}^N$ be an unramified covering and L the linear system of hyperplanes of the $\mathbb{P}^N$. $f^*(L)$ shall denote the pullback of L to V. By considering the rational map $\emptyset_{f^*(L)}$ determined by $f^*(L)$ we see that $\emptyset_{f^*(L)} = \emptyset_L \circ f$ and obtain that the linear system $f^*(L)$ is not composite with the pencil. This implies, because $f^*(L)$ has no fixed components, (use theorem 3), that the general member of $f^*(L)$ is irreducible. In other words, if H is a general hyperplan section of the $\mathbb{P}^N$, the inverse image $H' = f^{-1}(H)$ is an irreducible variety of V. Furthermore H' is, with the induced map $f : H' \longrightarrow H$, an irreducible, unramified covering of H of the same degree as $V \longrightarrow \mathbb{P}^N$. We can continue this argument and obtain in this way an irreducible, unramified covering $W \longrightarrow \mathbb{P}^1$ which is of the same degree as the covering $V \longrightarrow \mathbb{P}^N$.

Using Hurwitz's genus formula we conclude that the degree of $W \longrightarrow \mathbb{P}^1$ is 1.

We have shown so far that, in a normal galois covering $V \xrightarrow{f} \mathbb{P}^N$ with galois group G which is at most ramified along C, the group G is generated by the inertia groups I_i of the irreducible components of $f^{-1}(C)$. If we can show that $f^{-1}(C)$ is irreducible, we will have that G is the inertia group of $f^{-1}(C)$ and hence that G is cyclic.
Consider, for this purpose, the pullback $f^*L(C)$ of the complete linear system $L(C)$ of $\mathbb{P}^N$ which is determined by C. This system has no fixed components and is not composite with a pencil. Therefore, the general member of $f^*L(C)$ is irreducible.

By the connectedness theorem 4 we conclude that the specialisation $f^{-1}(C)$ of the general member of $f^*L(C)$ is connected.

But C was assumed to be smooth. We have, therefore, local non splitting for $f^{-1}(C)$. This together with the connectedness of $f^{-1}(C)$ implies that $f^{-1}(C)$ is irreducible.
Theorem 1 is proved.

The arguments in the proof of theorem 1 will sometimes determine the structure of the fundamental group of the complement of the curve C in $\mathbb{P}^2$, even if the curve C has singularities.

THEOREM 6. Let C be an irreducible curve in $\mathbb{P}^2$ of degree d. If C does not have "big singularities", the fundamental group $\pi_1(\mathbb{P}^2-C)$ is cyclic of order d.

PROOF. The notion of "big singularities" will be made precise during the proof of the theorem. Let $V^* \xrightarrow{f} \mathbb{P}^2$ be an irreducible, normal galois covering of $\mathbb{P}^2$, ramified at most along C. Let G be the galois group of $V^* \xrightarrow{f} \mathbb{P}^2$. We want to show that $f^{-1}(C)$ is irreducible by using the arguments that we used in the proof of theorem 1. But for C singularities are allowed. We therefore do not have theorem 5 available. We prepare the situation by resolving the singularities of C by quadratic transformations. It is known that there exists a sequence of smooth surfaces $W_1 = \mathbb{P}^2, W_2, \ldots, W_r = W$, all defined over k, such that :

1) W_i is the quadratic transform of W_{i-1} with a k-closed point $P_{i-1} \in W_{i-1}$ as center.

2) The reduced total transform of the curve C in W has only strong normal crossings as singularities. (Compare [6] , p.50).

Let C_W be the proper transform of C in W and let W^* be the normalisation of W in the function field $k(V^*)$ of V^*. $g : W^* \longrightarrow W$ shall be the covering map.

CLAIM. If $g^{-1}(C_W)$ is irreducible, $f^{-1}(C)$ is irreducible.

To prove this claim notice that by making quadratic transforms we do not change the general point of the curve C. Hence, the local rings of C and C_W on $\mathbb{P}^2$, respectively W, are the same. If $g^{-1}(C_W)$ is irreducible, only one local ring of $k(V^*) = k(W^*)$ lies over the local rings of C and C_W. But this means then that $f^{-1}(C)$ is irreducible. (Use that the irreducible components of $f^{-1}(C)$

are in 1-1 correspondence with the valuation rings of $k(V^*)$ which lie over the local ring of C on $\mathbb{P}^2$).

We can conclude the irreducibility of $g^{-1}(C_W)$, as in the proof of theorem 1, if the dimension of the complete linear system $L(C_W)$ of C_W on W is $\geqslant 2$.

We know that the dimension of the linear system L(C) of C on $\mathbb{P}^2$ is equal to $\frac{d(d+3)}{2}$, d = degree of C. However, the dimension of L(C) changes in a sequence of quadratic transformations.
In [6] , p.52 (Satz 6.3) the following is shown :

<u>PROPOSITION 5</u> . Let V be a smooth, irreducible surface over k, P a closed point of V and C be a curve on V which has multiplicity e at P. Suppose $V' \xrightarrow{\sigma} V$ is the quadratic transformation of V at P and C' the proper transform of C in V'. Then,

$$\dim L(C') \geqslant \dim L(C) - \frac{e(e+1)}{2} .$$

Proposition 5 can be used to define the weight of the singularities of C :

Let

$$P^2 \longleftarrow W_2 \longleftarrow \cdots\cdots \longleftarrow W_{r-1} \longleftarrow W_r = W$$

$$P_1 \longleftarrow P_2 \longleftarrow \cdots\cdots \longleftarrow P_{r-1}$$

$$C = C_1 \longleftarrow C_2 \longleftarrow \cdots\cdots \longleftarrow C_{r-1} \longleftarrow C_r = C_W$$

be a minimal desingularisation of the curve C. (Compare [6], p.51.)

$C_2,\dots,C_r$ are the proper transforms of C in the surfaces $W_2,\dots,W_r$ and P_i the centers of the quadratic transformations. Let e_i be the multiplicity of C_i at P_i, $i = 1,\dots,r-1$. Then we define :

DEFINITION 5. The integer $\nu(C) = \sum_{i=1}^{r-1} \frac{e_i(e_i+1)}{2}$ is called the weight of the singularities of C.

Theorem 6 can now be reformulated as follows.

THEOREM 7. Let C be an irreducible curve in $\mathbb{P}^2$ such that $\dim L(C) - \nu(C) \geq 2$. Then $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/d$, where d is the degree of C.

EXAMPLES :

1) If C has s nodes as singularities, every node contributes 3 to the weight $\nu(C)$. Hence $\nu(C) = 3s$.

2) If C has s nodes and t ordinary cusps, we see easily that

$$\nu(C) = 3s + 5t.$$

Further examples can be found in [6] , lecture 7.

REMARK 3 : It is shown in [6] , lecture 5, that one can determine explicitly the structure of the galois groups of the abelian unramified coverings of $\mathbb{P}^N - C$, C any hypersurface in $\mathbb{P}^N$. This is the result :

THEOREM 8. Let $C = C_1 \cup \dots \cup C_r$ be a hypersurface in $\mathbb{P}^N$ with $C_1,\dots,C_r$ as irreducible components. Assume that the ground field k has characteristic zero. Let $d_1,\dots,d_r$ be the degrees of $C_1,\dots,C_r$ respectively.

Then, the factor commutator group of $\pi_1(\mathbb{P}^N-C)$ is the profinite abelian group with r generators $\alpha_1,\ldots,\alpha_r$ and the relation $\alpha_1^{d_1}\cdot\ldots\cdot\alpha_r^{d_r} = 1$.

From theorem 8 and the consideration in the proof of theorem 1 follows (see [6] , lecture 5) :

THEOREM 9. Let $C = C_1 \cup \ldots \cup C_r$ be a hypersurface in $\mathbb{P}^N$, defined over a field of characteristic 0. Let C_i, $i = 1,\ldots,r$, be the irreducible components of C, and let d_i denote the degree of C_i. Assume further that C has only strong normal crossings as singularities. Then $\pi_1(\mathbb{P}^N-C)$ is the profinite abelian group with r generators $\alpha_1,\ldots,\alpha_r$ and the relation $\alpha_1^{d_1}\cdot\ldots\cdot\alpha_r^{d_r} = 1$.

REMARK 4. We have excluded so far the case where the field of definition k has characteristic $p > 0$. There the methods described can also be applied, if one restricts to coverings which are galois of a degree prime to p. Denote by $\pi_1^{(p)}(\mathbb{P}^N-C)$ the factor group of $\pi_1(\mathbb{P}^N-C)$ modulo the p-Sylow subgroups of $\pi_1(\mathbb{P}^N-C)$ ([6], lecture 1). The normal subgroups of $\pi_1^{(p)}(\mathbb{P}^N-C)$ parametrize then the finite unramified galois coverings of $\mathbb{P}^N-C$ with an order prime to p.

THEOREM 10. Let C be an irreducible, smooth hypersurface in $\mathbb{P}^N$ of degree d defined over the field k. Assume that the characteristic of k is $p > 0$. Write $d = p^a d'$, with $(d',p) = 1$. Then, $\pi_1^{(p)}(\mathbb{P}^N-C) = \mathbb{Z}/(d')$.

Further results for the group $\pi_1^{(p)}(\mathbb{P}^N-C)$ can be found in [6].

REMARK 5. The situation changes drastically if one considers, in characteristic $p > 0$, arbitrary coverings.

In characteristic 0 the affine line has no unramified covering as one checks easily using Hurwitz's genus formula. In characteristic $p > 0$ one can show the following theorem, ([6], p.119):

<u>THEOREM 11.</u> Let $\mathbb{P}^1/k$ be the projective line over the algebraic closed field k of characteristic $p > 0$ and let P be a k-valued point of $\mathbb{P}^1$. Then, every smooth, irreducible, projective curve (of any genus) defined over k is up to isomorphism an irreducible and normal covering of $\mathbb{P}^1$ ramified at most at P.

<u>REMARK 6.</u> Theorem 7 gives a method which shows that in characteristic 0 the group $\pi_1(\mathbb{P}^2-C)$ is cyclic of order d if C is irreducible with singularities of not too high weight (d = degree of C). It is of particular interest in algebraic geometry to determine $\pi_1(\mathbb{P}^2-C)$ if C is an irreducible curve with only cusps and nodes as singularities, because every smooth surface can be considered as a covering of $\mathbb{P}^2$ which is ramified at most along such a curve. Theorem 7 can handle this situation if one has not too many nodes and cusps. In this connection it is interesting that there is an other method which allows to show that $\pi_1^{(p)}(\mathbb{P}^2-C)$ is the cyclic group of order d' (d' as in theorem 10), if C is an irreducible curve of degree d with s nodes as singularities, provided s is greater than $2g^2$, where g is the genus of C. Compare [6] , lecture 14. This result follows from a close study of the behaviour of $\pi_1^{(p)}(\mathbb{P}^2-C)$, if C varies in an algebraic family of plane curves and the fact that an irreducible curve C of degree d with s nodes as singularities can be degenerated into d lines in general position if $s > 2g^2$.

It is not known that $\pi_1^{(p)}(\mathbb{P}^2 - C)$ is abelian (and therefore cyclic) if C is an arbitrary irreducible curve with only nodes as singularities. Edmunds [2] and Geyer [3] have shown the following theorem :

<u>THEOREM 12.</u> Let C be an irreducible curve in $\mathbb{P}^2$ with only nodes as singularities. Then every unramified galois covering of $\mathbb{P}^2 - C$ with a solvable galois group is abelian.

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