ROBERT KAUFMAN

M-sets and distributions

Astérisque, tome 5 (1973), p. 225-230
<http://www.numdam.org/item?id=AST_1973__5__225_0>

M-SETS AND DISTRIBUTIONS

Robert Kaufman

0. INTRODUCTION.

A closed subset S of the circle group T is called a weak Kronecker set (K_o-set) if each complex measure μ carried by S has the property $\sup|\hat{\mu}(n)| = \|\mu\|$; S is called an M-set if it carries a distribution $\tau \neq 0$ (of L. Schwartz) whose Fourier transform $\hat{\tau}(n)$ vanishes for $n = \pm\infty$; and S is called M_o if the distribution τ is a finite measure. In 1954 Pyatečkii-Šapiro [3] showed the existence of sets of type M, not of type M_o; this work is still striking because it exhibits a specific set S. Then Körner [1] showed the existence of sets of type $M \cap K_o$. In this note we modify the method of [3] to prove.

THEOREM. Each closed set S of type M contains a closed set S_1 of type $M \cap K_o$.

1. Let the N-dimensional torus T^N be represented as the product of intervals $[-\pi, \pi]$ and let V_ε^N be the set of N-tuples $(x_1, \ldots, x_N)$ such that $|x_k| \leq \varepsilon$ for at least $(1-\varepsilon)N$ indices $K = 1, 2, \ldots, N$ $(0 < \varepsilon < 1)$. We need a chain of lemmas to prove.

LEMMA 1. For $N > N_\varepsilon$ we can find a function F, continuous on T^N, vanishing off V_ε^N, such that $|\hat{F}(\chi)| < \varepsilon |\hat{F}(0)|$ for all characters $\chi \neq 0$ (in additive notation).

To prove Lemma 1 we construct a special kind of product probability measure on T^N. Let $0 \le t \le 1$ and let σ_t be the measure $(2\pi)^{-1} t dx + (1-t)\delta_o$ on t, and λ_t the N-fold product of σ_t, a probability measure on T^N. (The index N is suppressed in λ_t).

LEMMA 2. $\lambda_t(T^N \sim V_\varepsilon^N) \to 0$ as $N \to +\infty$, uniformly on the interval $0 \le t \le \frac{\varepsilon}{2}$.

This is a simple consequence of Chebyshev's inequality, because $[-\varepsilon, \varepsilon]$ has σ_t measure $\ge 1 - \frac{\varepsilon}{2}$.

LEMMA 3. Let V be an open set in a compact abelian group G, with dual Γ; suppose that $\sup\left\{|\hat{F}(\chi)| : \chi \neq 0\right\} \ge \varepsilon |\hat{F}(0)|$ for every F continuous on G and vanishing on $G \sim V$. Then there is an identity

$$1 = \sum \alpha_\chi \chi(y) \ , \ \sum |\alpha_\chi| \le \varepsilon^{-1} \ , \ \alpha_o = 0,$$

valid for all y in V.

Proof. The Fourier transform associates to each continuous F an element of the space $C_o(\Gamma)$; assuming that $\varepsilon |\hat{F}(0)| \le \sup\left\{|\hat{F}(\chi)| : \chi \neq 0\right\}$ for all F in our subspace, we can write $\hat{F}(0) = \sum' b_\chi \hat{F}(\chi)$, with $\sum |b_\chi| \le \varepsilon^{-1}$. Since V is open, this implies $1 = \sum b_\chi \overline{\chi(y)}$ identically in V.

Proof of Lemma 1. We shall prove that an equality of the type mentioned in Lemma 3 can be valid for only finitely many integers $1, 2, \ldots, N_\varepsilon$. The key to this is the formula $\lambda_t(\chi) = (1-t)^k$ $(k = 1, 2, 3, \ldots)$ for each character $\chi \neq 0$ on T^N. Suppose $\lambda_t(V_\varepsilon^N) \geq 1-\eta_N$ for $0 \leq t \leq \varepsilon/2$, and integrate the identity with respect to λ_t. Then $|1 - \sum_1^\infty C_k^N(1-t)^k| \leq \varepsilon^{-1}\eta_N$ over $0 \leq t \leq \varepsilon/2$, with $\sum |C_k^N| \leq \varepsilon^{-1}$. Since the functions $\sum_1^\infty C_k^N s^k$ form a normal family for $|s| \leq 1$ in the plane, and $\eta_N \to 0$, the identities in question are possible only for $N \leq N_\varepsilon$.

Let F_N be the function just constructed, corresponding to an $\varepsilon > 0$ and $N > N_\varepsilon$. We must replace F_N by a smooth function, since τ, being a distribution rather than a measure, does not admit multiplication by continuous functions. Let $\psi(x)$ be a smooth approximation to δ_0 vanishing outside a small interval $[-\delta, \delta]$ and let G_N be the convolution $F_N * (\psi(x_1) \ldots \psi(x_N))$. Then $\hat{G}_N(0) = \hat{F}_N(0) = 1$ (say) and G_N vanishes outside $V_{\varepsilon+\delta}^N \subseteq V_{2\delta}^N$ when $0 < \varepsilon < \delta$. Also $|\hat{G}_N(\chi)| < \varepsilon |\hat{\psi}(k_1) \ldots \hat{\psi}(k_N)|$ when $\chi \neq 0$ and χ has components $(K_1, \ldots, K_N)$.

2. Proof of Theorem. Let τ be a distribution such that $\hat{\tau}(\infty) = 0$, and $g(x)$ a real function of class $C^\infty(T)$. For integers $p \geq 1$ we are going to use distributions of the form $\tau_1 = G_N(g(x) - px, \ldots, g(x) - p^N x). \tau(dx)$, and observe first of all that the multiplier of τ is smooth on T, so the product is defined. Using the expansion of G_N as a Fourier series on T^N, we can write τ_1 as a sum

$$\sum C(k_1, \ldots, k_N) \exp i(k_1 + \ldots + k_N) g(x). \exp - i(pk_1 + \ldots + p^N k_N)x . \tau(dx).$$

The distributions with bounded Fourier transforms form a Banach space with the norm $\|\sigma\| = \sup|\hat{\sigma}|$. The sum above converges in norm, uniformly with respect to p.

For $\|\exp - ikx . \tau(x)\| = \|\tau\|$; the $C^1(T)$-norm of $\exp i(k_1 + \ldots + k_N)g(x)$ is $O(1) + O(|k_1 + \ldots + k_N|)$, and $|C|k_1, \ldots, k_N)| \leq |\hat{\psi}(k_1) \ldots \hat{\psi}(k_N)|$ with ψ in

$C^{\infty}(T)$, so $\hat{\psi}$ decreases rapidly.

We assert now that for large p $\|\tau_1 - \tau\|$ exceeds by $o(1)$ the maximum norm of the summands with $|k_1| + \ldots + |k_N| > 0$, a number bounded in turn by $\|\tau\| \cdot \sup |C(k_1, \ldots, k_N)| \cdot \|\exp i(k_1 + \ldots + k_N)g(x)\|_{C^1}$. In view of the uniform convergence mentioned above, it is sufficient to verify this for finite sums, say for $1 \le |k_1| + \ldots + |k_N| \le A$. Each distribution in the sum has a transform vanishing at infinity ; to each B, and $p > p(B)$, the values of $pk_1 + \ldots + p^N k_N$, generated by the N-tuples in question, differ by at least B. This in fact suffices for the necessary bound on $\|\tau_1 - \tau\|$.

Recall that p was chosen after N ; we now study the effect of increasing N, and assert that $|\hat{\psi}(k_1) \ldots \hat{\psi}(k_N)| \cdot |k_1 + \ldots + k_N|$ remains bounded for all N. In the argument we can assume $1 \le k_1 \le \ldots \le k_N$, and observe that $|\hat{\psi}(k)| \le 1-\eta$ for $k \ge 1$. Cancellation of k_1 effects a multiplication by at least $(1-\eta)^{-1}(1-N^{-1})$, and this exceeds 1 provided $N > \eta^{-1}$. Thus the problem is reduced to the special case $N \le \eta^{-1}$ and here the inequality $|\hat{\psi}(k)| < k^{-1}$ is at hand. Finally, for large N we have the additional factor $\varepsilon > 0$.

Before applying this to the last step, we recapitulate what has been attained. Given g in $C^{\infty}(T)$ and $\delta > 0$ we found a function H in $C^{\infty}(T)$ such that $\|H(x)\tau(x) - \tau(x)\| < \delta$. Moreover there exist integers $p \ge 1$ and $N \ge 1$ such that $H(x) = 0$ unless at least $(1-\delta)N$ of the inequalities $|g(x) - p^r x| \le 2\delta$ (modulo 2π) are fulfilled. Of course the closed support of $H(x)\tau(x)$ is contained in that of τ, and also in the set just mentioned.

Beginning with a distribution $\tau \neq 0$, we choose a sequence $(g_j)_1^{\infty}$, uniformly dense in the real Banach space $C(T)$ and perform a sequence of operations of the kind just completed. We obtain a distribution $\tau_1 \neq 0$, whose closed support S_1 is contained in the support of S. For each $j \geq 1$ there are integers p_j and N_j so that at least $(1-2^{-j})N_j$ of the N_j inequalities (with $p = p_j$, $g = g_j$, $N = N_j$) $|g(x) - p^r x| \leq 2^{-j}$ (modulo 2π), $1 \leq r \leq N$ are fulfilled at each point in S_1. Thus S_1 is a K_o-set ; S_1 has the property, somewhat stronger, that each finite measure on S_1 is nearly carried by a Kronecker set.

REFERENCES

[1] KÖRNER, T. W. A pseudofunction on a Helson set, ce volume.

[2] PYATEČKII-ŠAPIRO, I. I. On the problem of uniqueness of the expansion of a function in a trigonometric series (Russian). Moskov. Gos. Univ. Uc Zap. 155 (Math V), (1952), 54-72.

[3] Supplement to the same. Ibidem 165 (Math VII), (1954), 79-97.