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SOLVABILITY OF PARTIAL DIFFERENTIAL EQUATIONS
IN THE TRACES OF ANALYTIC SOLUTIONS OF THE HEAT EQUATION

by

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N. ARONSZAJN introduced in his lecture at this colloqium [1] an abstract FRECHET space $\mathcal{H}$: "the traces" of the analytic solutions of the heat equation. In this talk, we give additional properties and discuss the solvability of partial differential equations in $\mathcal{H}$. As examples, we prove the solvability in this space of some first order operators which are solvable neither in the space of distributions, nor in the space of SATO-MARTINEAU hyperfunctions .

The complete proofs will be published elsewhere ([2]).

I- Définitions, notations and basic properties

We denote by $\mathbb{R}^n$(resp. $\mathbb{C}^n$) the n dimensional real (resp.complex) space. We introduce the following notations:

$$\mathbb{C}_+ = \{x \in \mathbb{C}^1 \ , \ \mathrm{Re}\ x > 0\} \quad , \quad \overline{\mathbb{C}}_+ = \{ x \in \mathbb{C}^1 \ , \ \mathrm{Re}\ x \geqslant 0\}$$

$$\mathbb{C}^n_+ = \mathbb{C}^{n-1} \times \mathbb{C}_+ \quad , \quad \overline{\mathbb{C}}^n_+ = \mathbb{C}^{n-1} \times \overline{\mathbb{C}}_+ \ .$$

If $x \in \mathbb{C}^n$, $x = (x_1, \ldots, x_n) = (x', x_n)$

with
$$x' = (x_1, \ldots, x_{n-1}) \; ,$$
$$x^2 = \sum_{i=1}^{n} x_i^2 \qquad |x|^2 = \sum_{i=1}^{n} |x_i|^2$$

where $|x_i|$ denotes the modulus of the complex number x_i . If α is a multi-index, $\alpha = (\alpha_1 \ldots \alpha_n)$, α_i integer > 0 we denote
$$D_x^\alpha = D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \; , \quad x \in \mathbb{C}^n \text{ or } x \in \mathbb{R}^n \; .$$

If Ω is an open set in $\mathbb{C}^n$, we denote by $H(\Omega)$ the space of analytic functions defined in Ω , with the usual topology ; and $H'(\Omega)$ its dual, the space of analytic functionals in Ω .

We denote
$$E(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{x^2}{4t}\right) , \; (x,t) \in \mathbb{C}_n^{n+1}$$

Let $\mathcal{H} = \mathcal{H}_n$ be the space of analytic solutions u of the heat equation

(I.1)
$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

defined in $\mathbb{C}_+^{n+1}$. $\mathcal{H}$ is a closedsubspace of $H(\mathbb{C}_+^{n+1})$.

The mapping

(I.2)
$$\mathcal{H}_n \to (H(\mathbb{C}_+^n))^2$$
$$u \mapsto (u_0 , u_1)$$

where u_0 and u_1 are the CAUCHY data defined by
$$u_0(x',t) = u(x',0,t)$$
$$u_1(x't) = \frac{\partial u}{\partial x_n}(x',0,t) \quad ,$$

is a topological isomorphism. (The inverse mapping is given by the solution of the corresponding global CAUCHY problem).

If f is, say a tempered distribution defined on $\mathbb{R}^n$, we denote

$$\tilde{f}(x,t) = \langle f(y) \, , \, E(x-y,t)\rangle \tag{I.3}$$

In fact, (I.3) defines a topological embedding of $\mathcal{S}'(\mathbb{R}^n)$, $H'(\mathbb{C}^n)$ into $\mathcal{H}_n$ with dense range. Therefore, we have a natural topological embedding of the dual $\mathcal{H}'_n$ into $\mathcal{S}(\mathbb{R}^n) \cap H(\mathbb{C}^n)$. We will give in the next section a complete characterization of $\mathcal{H}'_n$, as subspace of $\mathcal{S}(\mathbb{R}^n) \cap H(\mathbb{C}^n)$.

From now on, we condider $\mathcal{H}$ as an abstract space which contains $\mathcal{S}'(\mathbb{R}^n), H'(\mathbb{C}^n)$, etc... . If $u \in \mathcal{H}$, we denote

$$\tilde{u}(x,t)$$

the value of the corresponding solution of the heat equation at $(x,t) \in \mathbb{C}^{n+1}_+$. An element in $\mathcal{H}$ is called a "trace".

We refer to [1] where the space $\mathcal{H}$ is introduced and where other properties are discussed.

II. Characterization of the dual space, the multipliers and the convolutors

We consider $\mathcal{H}'_n$ as a subspace of $\mathcal{H}_n$. We have the following

PROPOSITION II.1.-

1)- A trace u is in $\mathcal{H}'_n$ if and only if there exists $F \in H'(\mathbb{C}^{n+1}_+)$ (non-unique) such that, for $(x,t) \in \mathbb{C}^{n+1}_+$

$$\tilde{u}(x,t) = \langle F(y,\tau) \, , \, E(x-y \, , \, t+\tau)\rangle \, . \tag{II.1}$$

2)- For any $u \in \mathcal{H}'_n$, there exists a unique pair $(F_0,F_1) \in [H'(\mathbb{C}^n_+)]^2$ such that, for $(x,t) \in \mathbb{C}^{n+1}_+$.

$$\tilde{u}(x,t) = \langle F_0(y',\tau) \otimes \delta(y_n) + F_1(y',\tau) \otimes \delta'_0(y_n) \, , \, E(x-y,t+\tau)\rangle \tag{II.2}$$

where δ is the DIRAC measure in one variable, and δ' its derivative.

Part 2) follows, in particular from the isomorphism (1.2).

THEOREM II.1.-

Let f be an entire function defined in $\mathbb{C}^n$. $f \in \mathcal{H}'$ if an only if there exist $C \geq 0$, $M \geq 0$, $A > B \geq 0$ such that for any $x \in \mathbb{C}^n$

$$|f(x)| < \exp\left(M|x| + B|x^2| - A \operatorname{Re} x^2\right). \tag{II.3}$$

PROOF 1°) Necessity of (II.3)

From (II.1) we get

$$|\tilde{u}(x)| = |\langle F(y,\tau) , E(x-y,\tau)\rangle| \leq C \sup_{(y,\tau) \in K} E(x-y,\tau)$$

where $C \geq 0$ and K is a compact set in $\mathbb{C}_+^{n+1}$.

The inequality (II.3) follows easily for $\tilde{u}(x)$.

2°) Sufficiency of (II.3)

If F is an anlytic functional in $\mathbb{C}^n$, let us denote

$$\overset{*}{F}(\zeta) = \langle F(x) , \exp(-ix.\zeta)\rangle$$

its Fourier-BOREL transform. If $f \in \mathcal{H}'$, it follows easily from (II.1) and (II.2) that one can obtain the representations :

$$f(x) = \overset{*}{G}(x, -ix^2) \quad , \quad G \in H'(\mathbb{C}_+^{n+1}) \tag{II.4}$$

or

$$f(x) = \overset{*}{G}_0(x', -ix^2) + x_n \overset{*}{G}_1(x', -ix^2) \ , \ G_0, G_1 \in H'(\mathbb{C}_+^n) \tag{II.5}$$

In order to prove the sufficiency of (II.3), we assume that a given entire function f satisfies (II.3). We shall show that f may be written in the form (II.5). The latter result is a consequence of the following lemmas.

LEMMA II.1.-

Let f be an entire function in n complex variables even with respect to x_n (i.e $f(x',x_n) = f(x',-x_n)$). There exists a unique entire function g defined in C^n such that

$$f(x',x_n) = g(x', x^2).$$

In addition, f satisfies (II.3) if an only if there exist $C' \geq 0$, $M' \geq 0$, $A' > B' \geq 0$ such that for $y \in \mathbb{C}^n$ and $\tau \in \mathbb{C}$.

$$|g(y,\tau)| < C' \exp (M'|y| + B'|\tau| - A' \operatorname{Re} \tau). \qquad \text{(II.6)}$$

We observe here, that the condition (II.6) is equivalent to say that there exists $G \in H'(\mathbb{C}^n_+)$ such that

$$g(y,\tau) = \hat{G}(y, -i\tau)$$

(see[5], [6]).

LEMMA(II.2).-

Let f be an entire function satisfying (II.3), and P a polynomial in n variables with complex coefficients. If $\frac{f}{P}$ is entire it also satisfies (II.3).

The proof of lemma (II.2) uses the inequality

$$\left|\frac{f(x)}{P(x)}\right| \leq K \sup_{\substack{\zeta \in C^n \\ |\zeta| \leq 1}} |f(x+\zeta)|$$

(see[5]).

Remark II.1.- Using the characterization (II.1) and (II.4), it is readily seen that $\mathcal{H}\mathcal{C}'$ is closed under the FOURIER transform.

From the characterization (II.3) one can obtain

THEOREM II.2.-

The multipliers of $\mathcal{H}$ are the entire functions $u(x)$ in n variables which satisfy the condition :

For any $\varepsilon>0$, there exist $C\geqslant 0$, $M\geqslant 0$, $A\geqslant B-\varepsilon$ such that, for any $x \in \mathbb{C}^n$

(II.7) $$|u(x)| \leqslant C \exp (M|x| + B|x^2| - A \operatorname{Re} x^2).$$

Let us observe in particular, that the space of the multipliers contains the space of entire functions of exponential type.

Let us denote by $\mathcal{M}$ the space of multipliers, namely the space of entire functions which satisfy (II.7).

The space $\hat{\mathcal{M}} = \mathcal{C}$ is the space of convolutors. We have in particular the inclusion

$$H'(\mathbb{C}^n) \subset \mathcal{C}.$$

Let $t_0>0$, we denote by Σ_{t_0} the subspace of $\mathcal{H}$ defined by :

$$u \in \Sigma_{t_0} \iff \tilde{u}(x,t) = \tilde{v}(x,t + t_0)$$

where $v \in \mathcal{H}$.

The space Σ_{t_0} is the space of sections at t_0 . Σ_{t_0} is provided with the topology of uniform convergence on compact sets of $\mathbb{C}^n \times \{t \in \mathbb{C} \ , \operatorname{Re} t>-t_0\}$.

We define the space of sections :

$$\Sigma = \operatorname*{ind.lim.}_{\substack{t_0 \to 0 \\ t_0 > 0}} \Sigma_{t_0}$$

We have the following result :

THEOREM II.3.-

The dual Σ' of the space of sections Σ is equal as a subspace of ${}_c\mathcal{H}$, to the space of convolutors $\mathcal{C} = {}_c\hat{\mathcal{U}}$.

In addition, a trace u is in Σ' if and only if there exists $F \in H'(\overline{\mathbb{C}}^{n+1}_+)$ (non-unique) such that, for $(x,t) \in \mathbb{C}^{n+1}_+$

$$\tilde{u}(x,t) = \langle F(y,\tau) , E(x-y,t+\tau)\rangle.$$

III.- Solvability of P.D.E. with polynomial coefficients

We consider first, in this section, the constant coefficient case. We prove the possibility of the division by a polynomial in ${}_c\mathcal{H}$. Using the Fourier transform, we get the solvability of P.D.E. with constant coefficients, as well as the approximation of the solution of homogeneous equations by exponential-polynomials.

The proofs are based on LEMMA(II.2). In the distribution case, similiar ideas are used in [5].

We have the following results.

THEOREM III.I.-

Let $P \neq 0$ be a polynominal in n variables, with complex coefficients. For any $f \in {}_c\mathcal{H}_n$, there exists $u \in {}_c\mathcal{H}_n$ such that

$$Pu = f .$$

COROLLARY III.1.-

Let $P(D) \neq 0$ be a partial differential operator in n variables with complex coefficients. For any $f \in {}_c\mathcal{H}_n$, there exists $u \in {}_c\mathcal{H}_n$ such that

$$P(D)u = f$$

Let us call exponential-polynomial any entire function f of the form

$$f(x) = Q(x)\ \exp.(a.x) \quad ,$$

where Q is a polynomial and $a \in \mathbb{C}^n$. We denote by $V(P(D))$ the space of traces spanned by the exponential-polynomials f satisfying

$$P(D)f = 0 \quad .$$

THEOREM III.2.-

The space $V(P(D))$ is dense in the space

$$\{ u \in \mathcal{H},\ P(D)\ u = 0 \}$$

We shall consider now the polynomial coefficient case. Let P be a differential operator in n variables $x_1, \ldots, x_n$ with polynomial coefficients

(III.1) $$P = P(x, D_x) = \Sigma\ a_\alpha(x) D^\alpha$$

$$a_\alpha \in C\,[X_1, \ldots . X_n].$$

We shall discuss the solvability in $\mathcal{H}$ of the equation

(III.2) $$P(x, D_x)\ u = f \ .$$

Let us first consider the following operator

(III.3) $$\tilde{P} = \tilde{P}(x,t,D_x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_k(x, D_x)$$

with $$L_o(x, D_x) = P(x, D_x)$$

$$L_{k+1}(x, D_x) = \Delta_x L_x(x, D_x) - L_k(x, D_x)\Delta_x$$

$$(\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2})$$

L_k vanishes for large k and the series (III.3) is in fact a finite sum.

The operator $\tilde{P}$ satisfies the following properties :

(III.4) $$\tilde{P}(x, 0, D_x) = P(x, D_x)$$

$\tilde{P}$ commutes with the heat operator $\mathbb{H} = \frac{\partial}{\partial t} - \Delta_x$

(III.5) $$\mathbb{H}\,\tilde{P} = \tilde{P}\,\mathbb{H}$$

For any trace u ; $x \in \mathbb{C}^n$, $t \in \mathbb{C}_+$:

$$(\widetilde{Pu})(x,t) = \tilde{P}(x,t,D_x)\tilde{u}(x,t) \quad .$$

Let us write $\tilde{P}$ in the following form

(III.6) $$\tilde{P}(x,t,D_x) = \sum_k M_k(x,t,D_{x'})D_{x_n}^{2k} + N_k(x,t,D_{x'})D_{x_n}^{2k+1} \quad .$$

where M_k and N_k are differential operators in x' , their coefficient being polynomial in x and t .

THEOREM III.3.-

The solvability of (III.2) in $\mathcal{H}$ is equivalent to the solvability of the following problem :

For any $(f_0,f_1) \in (H(\mathbb{C}^n_+))^2$ find

$(u_0,u_1) \in (H(\mathbb{C}^n_+))^2$ such that

(III.7) $$\sum_k M_k(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^k u_0 + N_k(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^k u_1 = f_0$$

$$\sum_k (\frac{\partial M_k}{\partial x_n})(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^k u_0 + N_k(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^{k+1} u_0 +$$
$$(\frac{\partial N_k}{\partial x_n})(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^k u_1 + M_k(x',0,t,D_{x'})(\frac{\partial}{\partial t} - \Delta_{x'})^k u_1 = f_1 \; (*)$$

The isomorphism (I.2) is used in the proof of theorem (III.3). This theorem reduces the solvability of (III.2) in $\mathcal{H}$ to the solvability of a system of two P.D.E. with polynomial coefficients in the space of homolorphic functions in $\mathbb{C}^n_+$, for which global CAUCHY-KOVALEVSKY type theorems may be used.

(*) if $Q(x,D) = \Sigma q_\alpha(x)D^\alpha$ is a partial differential operator, we denote

$$(\frac{\partial Q}{\partial x_i})(x,D) = \Sigma \frac{\partial q_\alpha}{\partial x_i}(x)\, D^\alpha$$

Example 1 : Let us consider the operator

$$\frac{\partial}{\partial x_1} + ix_1 \frac{\partial}{\partial x_2} .$$

It is well known that it is not solvable (even locally) in the space of distributions.([4],[7],...). For the non-solvability in the space of hyperfunctions see [8]. (see also [9]).

THEOREM III.4

The operator

$$P = \frac{\partial}{\partial x_1} + ix_1 \frac{\partial}{\partial x_2}$$

is solvable in $\mathcal{H}_2$.

After a permissible change of variables, the operator $\tilde{P}$ defined in (III.3) becomes in this case

$$\tilde{P} = it\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) + P$$

The system (III.7) is of the form

$$\text{(III.8)} \qquad \frac{\partial^2 u_0}{\partial x_1^2} = g_0 + Q_0(u_0,u_1)$$

$$\frac{\partial^2 u_1}{\partial x_1^2} = g_1 + Q_1(u_0,u_1)$$

where g_0 and g_1 are given homolorphic functions in $\mathbb{C}^2_+$ and Q_0 , Q_1 are differential operators of order 1, acting on u_0 and u_1 , with respect to the variables x_1 and t , with homolorphic coefficients in $\mathbb{C}^2_+$. The solvability of (III.8) in $(H(\mathbb{C}^2_+))^2$ may be proved using a global CAUCHY-KOVALEVSKY type theorem (see[3] for similar techniques).

Example 2.-

THEOREM III.5.-

Let α be a complex number.

The operator

$$P = x_2 \frac{\partial}{\partial x_1} + \alpha\, x_1 \frac{\partial}{\partial x_2}$$

is solvable in $\mathcal{H}_2$ if and only if $\alpha \neq -1$.

The idea of the proof is similar to that used in theorem (III.4). The non - solvability of (III.9) for $\alpha = -1$ was pointed out by R. MOYER.

Remark III.1. - The operator

(III.10) $$x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

is not solvable in $\mathcal{H}_2$. However it is possible to prove that, if we consider the "traces of analytic solutions" of the operator

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \lambda \frac{\partial^2}{\partial x_2^2} \quad ; \quad \lambda > 0 \quad , \lambda \neq 1$$

instead of the heat operator, (III.10) is solvable in this new space.

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