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SOBOLEV SPACES OF VARIABLE ORDER AND PROBLEMS OF CONVEXITY FOR PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

by André UNTERBERGER

When trying to solve an equation $P(D)u = f$, f a given distribution in an open subset of R^n, it is useful, if one can do so, to exhibit a class of functions ρ in Ω such that the equation can be solved if $f \in H^{-\rho}_{loc}(\Omega)$, for then the problem will be solvable for any distribution provided ρ is allowed to grow as fast as one pleases near the boundary of Ω. By duality, one is led to trying to get regularity theorems in the H^ρ spaces for distributions with compact support.

We do so in §3 of this paper, for some special operators, and get sufficient conditions of convexity for singular supports in a few cases.

In §1, we give two formulas which compare H^ρ norms with integrals of the type occurring in Carleman inequalities.

This is applied in §2, where we get a complete equivalence between H^ρ regularity theorems and Carleman-type inequalities, once the problem is stabilized by adding one coordinate. The remainder of §2 is devoted to some non-technical facts concerning convexity, which we hope can help to clarify some of the purely geometric aspects of the problem.

1.- Definition of the Sobolev spaces of variable order; a two-way relationship between H^{ρ}-norms and integrals of the kind occurring in Carleman inequalities.

Let ρ be a real C^{∞} function on R^n, which can be written as $\rho = \rho_{\infty} + \sigma$, where ρ_{∞} is a constant and σ belongs to the space $\mathcal{S}$ of L.Schwartz. Let s be a real constant, and k any number less than the infimum of ρ.

One can then define (as an operator, say, on $\mathcal{S}'$) the pseudo-differential operator $A^{\rho,s}$ whose symbol coincides with $|\xi|^{\rho(x)}(\mathrm{Log}\,|\xi|)^s$ for $|\xi|$ large enough, and the space $H^{\rho,s}$ of all distributions $u \in H^{k-1}$ such that $A^{\rho,s}u \in L^2$; this space depends only on (ρ,s) and can be endowed with the norm $\|u\|_{\rho,s} = \|A^{\rho,s}u\|_0 + \|u\|_{k-1}$, which, up to equivalence, depends only on (ρ,s).

When ρ is a constant and s=0, one gets the classical Sobolev space H^{ρ}; for variable ρ and σ, one has $H^{\rho} \subset H^{\sigma}$ if $\rho \geqslant \sigma$, the spaces $H^{\rho,s}$ being intermediate between H^{ρ} and $H^{\rho\pm\varepsilon}$ ($\pm$ according to the sign of s); the refinement provided by s is sometimes useful in connection with the use of partitions of unity: e.g, one gets, for operators $P(x,D)$ of order m, with simple characteristics, and suitable functions ρ, estimates of the kind
$\|u\|_{\rho+m-1,\frac{1}{2}} \leq C\left[\|P_m(x,D)u\|_{\rho} + \|u\|_{-N}\right]$: the $\frac{1}{2}$ helps to absorb what comes from the lower order terms of P, and the terms which come from a bracket $[\varphi, P_m]$.

Of course, one can define the spaces $H^{\rho,s}_{comp}(\Omega)$ and $H^{\rho,s}_{loc}(\Omega)$, in an open set Ω, assuming only that ρ is C^{∞} and real valued in Ω.

Proposition 1.1.- Let ρ and s be given according to what precedes. Let $\varphi \in \mathcal{D}(R^n)$ be real valued, and assume that:

a) there exists $M > 0$, $M > \sup \rho$, such that $\hat{\varphi}(\xi) = O(|\xi|^M)$ when $\xi \to 0$.

b) $\hat{\varphi}$ does not vanish identically on any half-straight line through the origin.

For every $t > 0$, write $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Then there exist C_1, C_2, C such that for every $u \in \mathcal{D}(R^n)$, one has

$$C_1 \int_0^{\frac{1}{2}} (\mathrm{Log}\frac{1}{t})^{2s} \frac{dt}{t} \int t^{-2\rho(x)} |(\varphi_t * u)(x)|^2 \, dx \leq \|u\|^2_{\rho,s} \leq$$
$$C_2 \int_0^{\frac{1}{2}} (\mathrm{Log}\frac{1}{t})^{2s} \frac{dt}{t} \int t^{-2\rho(x)} |(\varphi_t * u)(x)|^2 \, dx + C\|u\|^2_{\rho-1} .$$

For classical Sobolev norms (i.e ρ a constant and $s=0$), this was proved by L.Hörmander.

To prove it for general ρ, one has to show that the pseudo-differential operator $B = (A^{\rho,s})^* A^{\rho,s}$ and the (almost) pseudo-differential operator

$$S = \int_0^{\frac{1}{2}} (\mathrm{Log}\frac{1}{t})^{2s} \, \check{\varphi}_t * (t^{-2\rho(x)} \varphi_t * .) \, \frac{dt}{t}$$ "dominate" each other in the sense that $\|(Bu,u) - C_1(Su,u)\| + \|C_2(Su,u) - (Bu,u)\| \leq C\|u\|^2_{\rho-1}$ for suitable constants C, C_1, C_2.

Noting, after a commutation, that the symbol of S is like $f(x,\xi) = \int_0^{\frac{1}{2}} (\mathrm{Log}\frac{1}{t})^{2s} \, t^{-2\rho(x)} |\hat{\varphi}(t\xi)|^2 \frac{dt}{t}$, and that this function is equivalent with $|\xi|^{2\rho(x)} (\mathrm{Log}|\xi|)^{2s}$ for large $|\xi|$, one concludes thanks to a form of the Lax-Nirenberg theorem.

We shall use prop.1.1 in the following way: Assume that P is a differential operator on R^n with constant coefficients, that ρ and σ are C^∞ and real valued in an open subset Ω of R^n, that k is a real number and that for every compact subset K of Ω, there exist $C>0$ and $\tau_0>0$ such that, for every $u\in\mathcal{D}_K(\Omega)$ and every $\tau>\tau_0$, one has

(1) $$\int e^{2\tau\rho}|u|^2\,dx \leqslant \frac{C}{\tau^k}\int e^{2\tau\sigma}|P(D)u|^2\,dx.$$

Replacing u by $\varphi_t * u$ and making $\tau = \operatorname{Log}\frac{1}{t}$, and finally integrating with respect to $\frac{dt}{t}$, one gets the inequality $\|u\|_\rho \leq C\,\|P(D)u\|_{\sigma,-k/2}$.

For instance, (1) holds for every P(D) if $\rho=\sigma$ is linear: this shows that $u\in H^{<x-x_0,N>}$ if $u\in\mathcal{E}'(R^n)$ and $P(D)u\in H^{<x-x_0,N>}$. Another example is provided by the following inequality, due to F.Treves: if $\Omega\subset C^n$, and P(D) equals $P(\bar\partial)$, a polynomial in the $\frac{\partial}{\partial\bar z_j}$'s, then (1) is valid, with $\rho=\sigma$, whenever ρ is strictly pseudo-convex in Ω: one has then that $P(-\bar\partial)H^{-\rho}_{loc}(\Omega)\supset H^{-\rho}_{loc}(\Omega)$ if Ω is a domain of holomorphy; in particular, every domain of holomorphy is $P(-\bar\partial)$-strongly convex.

Proposition 1.2.- Let $\rho\in C^\infty(R^n)$ and $s\in R$; let k be an integer $\geqslant 1$, and let $\psi\in\mathcal{D}(R^k)$ be real valued and not identically 0. For every $u\in\mathcal{D}(R^n)$, and every $t>0$, define the function $u\otimes\psi^t$, on $R^n\times R^k$, as $(u\otimes\psi^t)(x,y) = t^{-k/2}u(x)\psi(y/t)$.
Assume that $\rho>0$ everywhere.
Then one has (on compact subsets of R^n, with suitable constants C_1 and C_2, and sufficiently small t):

$$C_1\|u\otimes\varphi^t\|^2_{\rho,s} \leq (\mathrm{Log}\tfrac{1}{t})^{2s}\int t^{-2\rho(x)}|u(x)|^2\,dx + \|u\|^2_{\rho,s}$$
$$\leq C_2\left[\|u\otimes\varphi^t\|^2_{\rho,s} + \|u\|^2_{\rho,s}\right].$$

To prove this, let B be the operator (on $R^n\times R^k$) of symbol $(\xi^2+\eta^2)^{\rho(x)}(\mathrm{Log}(\xi^2+\eta^2))^{2s}$; if $v\in\mathcal{D}(R^n\times R^k)$, one has $\|v\|^2_{\rho,s}\sim(Bv,v)$.

If β is a function on $R^n\times R^k$, homogeneous of order 0, with $\beta(\xi,\eta)=0$ for $|\xi|\leq|\eta|$ and $\beta(\xi,\eta)=1$ for $|\xi|\geq 2|\eta|$, B is (still thanks to the Lax-Nirenberg theorem) equivalent with B_1+B_2, where the symbols of B_1 and B_2 are respectively $\beta(\xi,\eta)|\xi|^{2\rho(x)}(\mathrm{Log}|\xi|)^{2s}$ and $(1-\beta(\xi,\eta))|\eta|^{2\rho(x)}(\mathrm{Log}|\eta|)^{2s}$.

Substituting $v_t=u\otimes\varphi^t$, one gets that (Bv_t,v_t) is, uniformly for small t, equivalent with (Ku,u), where K is the pseudo-differential operator on R^n of symbol $k=k_1+k_2$, where

$k_1(x,\xi) = |\xi|^{2\rho(x)}(\mathrm{Log}|\xi|)^{2s}\int\beta(\xi,\zeta/t)|\hat{\varphi}(\zeta)|^2\,d\zeta$ and

$k_2(x,\xi) = \int(1-\beta(\xi,\zeta/t))|\zeta/t|^{2\rho(x)}(\mathrm{Log}|\zeta/t|)^{2s}|\hat{\varphi}(\zeta)|^2\,d\zeta.$

It is not hard to see that $|(K_1u,u)|\leq C\|u\|^2_{\rho,s}$ and that (K_2u,u) can be replaced, up to an error term bounded by $C\|u\|^2_{\rho,s}$ (it is at this point that one uses the assumption $\rho>0$), by (K_3u,u), where the symbol of K_3 is

$k_3(x) = \int|\zeta/t|^{2\rho(x)}(\mathrm{Log}|\zeta/t|)^{2s}|\hat{\varphi}(\zeta)|^2\,d\zeta.$

The proposition 1.2 follows.

2.- Various types of convexity; some global properties of convex sets.

The preceding paragraph shows that, for differential operators P with constant coefficients, there should be some kind of equivalence between Carleman-type inequalities and regularity theorems in the H^ρ spaces, provided one agrees to make P operate on higher dimensional spaces.

Let us start, for brevity in further statements, with the following 4×2-matrix of definitions: being given an open set Ω in R^n, two closed subsets K and L of Ω such that $K \subset L$, and a differential operator P(D) on R^n with constant coefficients, let us say that (Ω, L, K) is P(-D)-convex if $u \in \mathcal{E}'(\Omega)$ and $\operatorname{supp}(P(D)u) \subset K$ imply $\operatorname{supp}(u) \subset L$; let us say that (Ω, L, K) is P(-D)-singular convex if the same condition holds with supports replaced by singular supports and that it is P(-D)-strongly convex if both conditions are satisfied; finally, let us say that (Ω, L, K) is P(-D)-stably convex if

(2) for every integer $k \geqslant 0$, $(\Omega \times R^k, L \times R^k, K \times R^k)$ is P(-D)-singular convex.

We shall say also that Ω is P(-D)-convex (singular-convex, strongly convex, stably convex) if for every compact subset K of Ω, there is some compact $L \subset \Omega$ such that the triple (Ω, L, K) be P(-D)-convex (singular-convex, strongly convex, stably convex).

The following theorem shows the equivalence between stable convexity and the validity of "sufficiently many" Carleman-type estimates; it also indicates (at least partially) how the stable convexity is related with the other notions of convexity.

Theorem 2.1.- Let Ω be an open subset of R^n, and K and L two compact subsets of Ω, with $K \subset L$. The following conditions are equivalent:

a) (Ω, L, K) is P(-D)-stably convex (i.e, condition (2) holds for every $k \geqslant 0$).

b) $(\Omega \times R^k, L \times R^k, K \times R^k)$ is P(-D)-strongly convex for every $k \geqslant 0$.

c) condition (2) holds for some $k \geqslant 1$.

d) for every closed set F in Ω, disjoint from L, and for every compact subset M of Ω, there exist two functions ρ and $\sigma \in C^{\infty}(\Omega)$, a real number k and a constant $C > 0$, such that $\inf_F \rho(x) > \max(0, \sup_K \sigma(x))$, and such that the inequality

$$\int e^{2\tau\rho(x)} |u(x)|^2 \, dx \leq C\left[\int e^{2\tau\sigma(x)} |P(D)u(x)|^2 \, dx + \|u\|_k^2\right] \tag{3}$$

hold for every $u \in \mathcal{D}_M(\Omega)$ and every number $\tau \geqslant 0$.

The proof goes along the cycle (abcd): $b \to c$ is trivial and $a \to b$ is almost.

To prove that $d \to a$, rewrite (3) as

$$\int e^{2\tau\rho(x)} |u(x)|^2 \, dx \leq C\left[\int e^{2\tau\sigma(x)} |P(D)u(x)|^2 \, dx + \|(1-\Delta)^m P(D)u\|_0^2\right].$$

With λ and μ real, and $\lambda \geqslant 0$, one has also, with $\rho' = \lambda\rho + \mu$, $\sigma' = \lambda\sigma + \mu$:

$$\int e^{2\tau\rho'(x)} |u(x)|^2 \, dx \leq C\left[\int e^{2\tau\sigma'(x)} |P(D)u(x)|^2 \, dx + e^{2\tau\mu} \|(1-\Delta)^m P(D)u\|_0^2\right].$$

For $v \in C^{\infty}(\Omega \times R^k)$ with support in $M \times [-1,1]^k$, one can write this inequality for the function $x \mapsto u(x,y)$ (y fixed), then integrate with respect to dy; making at the same time $\tau = \operatorname{Log}\frac{1}{t}$, one gets for $t \leq 1$:

$$\int t^{-2\rho'(x)} |v(x,y)|^2 \, dxdy \leq C\left[\int t^{-2\sigma'(x)} |P(D)v(x,y)|^2 \, dxdy + t^{-2\mu} \|P(D)v\|_{2m}^2\right].$$

Proposition 1.1 gives then $\|v\|_{\rho'}^2 \leq C\left[\|P(D)v\|_{\sigma'}^2 + \|P(D)v\|_{2m+\mu}^2\right]$ and shows that if $v \in \mathcal{E}'(\Omega \times R^k)$ has its support in $M \times [-1,1]^k$ (in fact, allow some shrinkage) and is such that $P(D)v \in H^{\sigma'} \cap H^{2m+\mu}$, then $v \in H^{\rho'}$.

Now, if $P(D)v \in H^s$ and is C^{∞} outside $K \times R^k$, it will belong to $H^{\sigma'} \cap H^{2m+\mu}$ provided that $\sigma' < s$ on K and $2m+\mu < s$; also v will be in H^t_{loc} in a neighborhood of $\{x_0\} \times R^k$ ($x_0 \in \Omega \setminus L$) provided that $\rho'(x_0) > t$. With $q=\sup_K \sigma(x)$, one has then to solve the system of inequalities (in the unknowns λ and μ, for arbitrarily given s and t):

$$\begin{cases} \lambda q+\mu < s \\ \lambda\rho(x_0)+\mu > t \\ 2m+\mu < s \\ \lambda > 0 \end{cases}$$

: the assumption $\rho(x_0) > \max(0,q)$ makes it possible.

Note that, as the proof shows, one can, in condition d), replace the closed set F by a single point.

To prove that c→d, assume that the condition (2) holds for some $k \geqslant 1$ (it will then also hold for every $k' \leqslant k$, as it is easy to see), and let F and M be given. A simple application of the closed graph theorem shows the existence of two strictly positive C^∞ bounded functions ρ and σ, such that $\inf_F \rho(x) > \sup_K \sigma(x)$, and such that the inequalities $\|v\|_\rho \leqslant C\|P(D)v\|_\sigma$ and $\|u\|_\rho \leqslant C\|P(D)u\|_\sigma$ hold, v and u having their supports in $M \times R^k$ and M respectively. Applying prop.1.2 with $v = u \otimes \psi^t$, one gets

$$\int t^{-2\rho(x)} |u(x)|^2 \, dx \leqslant C\left[\int t^{-2\sigma(x)} |P(D)u(x)|^2 \, dx + \|P(D)u\|_\sigma^2\right],$$

which proves (3) for τ large enough.

Corollary 2.1.- Assume that Ω is stably P(-D)-convex and for every compact $K \subset \Omega$, let $\hat{K}$ be the smallest compact subset of Ω such that $(\Omega, \hat{K}, K)$ be stably P(-D)-convex. Then the (non-decreasing) map $K \mapsto \hat{K}$ is upper semi-continuous, i.e being given K and an open neighborhood U of $\hat{K}$, there is a compact neighborhood K_ε of K such that $\hat{K}_\varepsilon \subset U$.

Note that the map $K \mapsto \hat{K}$ is not continuous, as the example of the laplacian on R^2, K being a circle, shows.

We now prove the corollary:

Let N be an arbitrary compact neighborhood of K and let M be a compact neighborhood of $\hat{N}$. Apply condition d) of theorem 2.1, with $F = \Omega \setminus U$. Now define K_ε in such a way that one have $\sup_{K_\varepsilon}\sigma(x) < \inf_F \rho(x)$, and $K_\varepsilon \subset N$. It follows then easily from theorem 2.1 that the triple (Ω, L, K) is $P(-D)$-stably convex for some compact $L \subset U$.

Proposition 2.1.- Let Ω be an open subset of R^{n+k}, and K and L two compact subsets of Ω with $K \subset L$. Let $P(D)$ be a differential operator with constant coefficients on R^n. Say that an affine n-subspace E of R^{n+k} is parallel to R^n if, writing $z=(x,y)$ for each $z \in R^n \times R^k$, y is constant on E. Then (Ω, L, K) is $P(-D)$-stably convex if and only if, for every E parallel to R^n, $(\Omega \cap E, L \cap E, K \cap E)$ is.

The proof is straightforward.

At this point, one should make the following remark: it is not clear whether corollary 2.1 holds with stable convexity replaced by other types of convexity. One could then bypass the difficulty by saying, for any of the four types of convexity introduced, that (Ω, L, K) is convex if and only if for each neighborhood $\tilde{L}$ of L, there is some neighborhood $\tilde{K}$ of K such that the usual implication holds: we will refer to this type of convexity as convexity "in the Alexander sense", as it is the same trick used when one replaces singular cohomology by the direct limit of the cohomology of the neighborhoods (see proposition below).

If one uses throughout convexity in the Alexander sense, one can, in the definition of convexity for supports, replace the function $u \in \mathcal{E}'(\Omega)$ by one belonging to $\mathcal{D}(\Omega)$; on the other hand, prop.2.1 will remain true if one replaces "(Ω,L,K) is stably convex" by "(Ω,L,K) is singular-convex"(provided $k \geqslant 1$).

Note also that if one is only interested in convexity for open sets (i.e not for triples), it does not make any difference whether one defines it in the Alexander sense or not.

Proposition 2.2.- Let (Ω,L,K) be a triple in R^n. Then, if (Ω,L,K) is $P(-D)$-convex for some operator $P(D)$ in R^n of order $\geqslant 1$, the inclusion homomorphism $H^n(\Omega,L) \longrightarrow H^n(\Omega,K)$ is zero, the (say, real) cohomology groups being taken in the Alexander sense.

To say that $H^n(\Omega,L) \longrightarrow H^n(\Omega,K)$ is zero means that for every neighborhood V of L, there is a neighborhood W of K such that every $(n-1)$-cycle in W which bounds in Ω bounds in V; thanks to an Alexander duality theorem, this is equivalent with saying that each connected component of $\Omega \setminus K$ which is relatively compact in Ω is contained in L.

Then, if this homomorphism is not zero, let M be the union of K and of all the connected components of $\Omega \setminus K$ which are relatively compact in Ω : it is well known that M is compact and by hypothesis, it is not contained in L; now, if $u=e^{\langle x,\zeta\rangle}$ is a solution of $P(D)u=0$ and if χ is the characteristic function of M, the support of $P(D)(\chi u)$ is included in K but the support of χu is M; hence, (Ω,L,K) is not $P(-D)$-convex.

This concludes the proof of proposition 2.2.

Now, assume that an open set Ω of $R^{n+k} (k \geq 1)$ is $P(-D)$-singular convex for some operator $P(D)$ on R^n of order ≥ 1; thus, to each compact subset K of Ω one can associate a compact L of Ω such that (Ω, L, K) is $P(-D)$-singular convex in the Alexander sense; then $(\Omega \cap E, L \cap E, K \cap E)$ is $P(-D)$-convex for each E parallel to R^n. Proposition 2.2 then shows that $H^n(\Omega \cap E, L \cap E) \longrightarrow H^n(\Omega \cap E, K \cap E)$ is zero for every E. This condition can be shown to be sufficient for certain operators $P(D)$, e.g. those which are elliptic and with simple (complex) characteristics.

In any case, the question now arises of the characterization of such open sets. For $k=1$, we give the following (though we prove only the "only if" part): for each E parallel to R^n, the boundary homomorphism $H_n(\Omega, \Omega \cap E) \longrightarrow H_{n-1}(\Omega \cap E)$ is zero, that is, every $(n-1)$-cycle in $\Omega \cap E$ which bounds in Ω must bound in $\Omega \cap E$. Indeed, let c be an n-chain in Ω such that ∂c lies in $\Omega \cap E$: we have to show that c is homologous to zero in $\Omega \cap E$. Let E be defined by the equation $y=0$. One can write $c = c_1 + c_2$, where c_1 and c_2 are carried by the half-spaces $\{y \geq 0\}$ and $\{y \leq 0\}$: this permits a reduction to the case when c is carried by $\{y \geq 0\}$. Without changing ∂c, one can replace c by a polyhedral n-chain whose elements are transversal to the hyperplanes E_y parallel to $E=E_0$: this allows to define the intersection cycle $\gamma_y = c \cap E_y$ for each $y \geq 0$, with $\gamma_0 = \partial c$. Now the carrier of γ_y is an upper semi-continuous function of y, which proves that the set of all numbers y such that γ_y bounds in $\Omega \cap E_y$ is open. On the other hand, there is some compact L such that the inclusion homomorphism $H^n(\Omega \cap E_y, L \cap E_y) \longrightarrow H^n(\Omega \cap E_y, \bar{c} \cap E_y)$ is zero for every y, $\bar{c}$ being the carrier of c: in other words, γ_y bounds in $L \cap E_y$ if it bounds in $\Omega \cap E_y$; this shows that the set of all y such that γ_y bounds in $\Omega \cap E_y$ is closed, and concludes the proof.

We now give an easy generalisation of prop.2.2. Let G(n) be the (disjoint) union of the G(n,p) $(1 \leq p \leq n)$, where G(n,p) is the set of all p-dimensional linear subspaces of R^n. Define the subset V(P) of G(n) as follows: an element ξ of G(n,p) does not belong to V(P) if and only if each distribution $u \in \mathcal{D}'(R^n)$ whose singular support is included in ξ and is such that $P(D)u \in C^\infty(R^n)$ is in fact C^∞ in R^n (according to a lecture he gave in Paris quite recently, it seems that L.Hörmander has a characterization of V(P)).

Proposition 2.3: Let P(D) be a differential operator in R^n with constant coefficients, and let $\xi \in V(P) \cap G(n,p)$. If a triple (Ω, L, K) is P(-D)-singular convex in the Alexander sense, then for each affine space E parallel to ξ the inclusion homomorphism $H^p(\Omega \cap E, L \cap E) \longrightarrow H^p(\Omega \cap E, K \cap E)$ is zero.

Assume it is not. With $\Omega' = \Omega \cap E$, $L' = L \cap E$, $K' = K \cap E$, let M' be the union of K' and of all the connected components of $\Omega' \setminus K'$ which are relatively compact in K'. Let x_0 be a point in M' not in L'. As $\xi \in V(P)$ there is some $u \in \mathcal{D}'(R^n)$ whose singular support is included in E, such that $P(D)u \in C^\infty(R^n)$, which is not C^∞ at x_0. On any given compact subset of R^n, one can assume that u is continuous since one can replace it by $k * u$, where k is a function with compact support such that k(x-y) coincides near the diagonal with the kernel of $(1-\Delta)^{-N}$ for some large N. Assume that E is defined by the equation x"=0, where x=(x',x") is a linear set of coordinates in R^n. Let $\alpha(x")$ be a C^∞ function with compact support in the set $\{|x"|<\varepsilon\}$, equal to 1 for $|x"|< \varepsilon/2$. Let $\chi(x')$ be the characteristic function of M'. Then the function $v(x',x")= \alpha(x")\chi(x')u$ is well-defined and with compact support in Ω if ε is small enough. Its singular support contains x_0, hence does not shrink towards L as $\varepsilon \to 0$. However, one can check that the singular support of P(D)v shrinks towards K' as $\varepsilon \to 0$.

3.- Sufficient conditions of convexity for some special classes of operators.

Proposition 3.1.- Let $K = Op[h]$ be a pseudo-differential operator of convolution, whose symbol h is C^∞ outside 0 and can be written as $h=h_1+h_r$, where h_1 is homogeneous of degree 1 and h_r is (non-homogeneous) of order $r<1$. Let $\rho \in C^\infty(R^n)$ be real-valued. Let Σ be the unit sphere of R^n, W an open subset of Σ, Ω an open subset of R^n, and assume that, for each $(x,\xi)\in \Omega \times W$, one has

$$(4)\qquad \sum_{jk} \frac{\partial h_1}{\partial \xi_j}\frac{\partial \bar{h}_1}{\partial \xi_k}\frac{\partial^2 \rho}{\partial x_j \partial x_k} > 0.$$

Let $\alpha(\xi)$ be a C^{∞} function on $R^n \setminus \{0\}$, homogeneous of degree 0, such that $\mathrm{supp}(\alpha) \cap \Sigma \subset W$; let $\mathrm{Op}[\alpha]$ be the associated pseudo-differential operator of convolution. Then for every distribution $u \in \mathcal{E}'(\Omega)$ such that $\mathrm{Op}[\alpha]Ku$ belongs to H^{ρ}, $\mathrm{Op}[\alpha]u$ belongs to $H^{\rho,\frac{1}{2}}$.

Using some cut-off functions, to take into account the fact that (4) holds only in $\Omega \times W$, one can reduce the proof to the case when (4) holds in $\Omega \times \Sigma$. One has then to prove the inequality

$$\|v\|^2_{\rho,\frac{1}{2}} \leq C\left[\|Kv\|^2_{\rho} + \|v\|^2_{-N}\right],\ v \in \mathcal{D}_M(\Omega)\ \ (M \text{ compact} \subset \Omega).$$

With $A^{\rho} = \mathrm{Op}\left[|\xi|^{\rho(x)}\right]$, $B = \mathrm{Op}\left[\mathrm{Log}|\xi|\right]$, one can find a pseudo-differential operator L such that $A^{\rho}K - LA^{\rho}$ is of order $-\infty$. The symbol of L is, up to a zero-order symbol,

$$h_1(x,\xi) + h_r(x,\xi) - \frac{1}{4i\pi}\sum \frac{\partial \rho}{\partial x_j}\frac{\partial h_1}{\partial \xi_j}\mathrm{Log}\,|\xi|,$$

and the inequality to be proved is equivalent with

$$\|B^{\frac{1}{2}}w\|^2_0 \leq C\left[\|Lw\|^2_0 + \|w\|^2_{-N}\right],\ w \in \mathcal{D}_{M_1}(\Omega),\ (M \subset \mathring{M}_1).$$

As $\|Lw\|^2_0 \geq ((L^*L - LL^*)w,w)$, and as the principal symbol of $L^*L - LL^*$ is

$$\frac{1}{4\pi^2}\ \left(\sum \frac{\partial h_1}{\partial \xi_j}\frac{\partial \bar{h}_1}{\partial \xi_k}\frac{\partial^2 \rho}{\partial x_j \partial x_k}\right)(\mathrm{Log}\,|\xi|),$$

the proof is finished, with the help of the Lax-Nirenberg theorem.

Using the fact that if a triple (Ω, L, K) in R^n is such that $H^n(\Omega, L) \longrightarrow H^n(\Omega, K)$ is zero, then one can for each $x_0 \in \Omega \setminus L$ find in Ω a strictly subharmonic function as large as one pleases at x_0 and as small as one pleases on K, one derives easily from prop.3.1 the following:

Proposition 3.2.- Let $A = Op[h]$ be a pseudo-differential operator of convolution, where h is C^∞ outside 0, $h=h_1+h_r$ (h_1 homogeneous of degree 1, h_r of order < 1). Let $\xi^o \in \Sigma$. Let Ω be an open subset of $R^n (n \geqslant 2)$, and K and L two compact subsets of Ω, with $K \subset L$. Assume that one at least of the three conditions a),b),c) holds:

a) $h_1(\xi^o) \neq 0$

b) $h_1(\xi^o) = 0$, and the vector $dh_1(\xi^o)$ whose components are $\frac{\partial h_1}{\partial \xi_j}(\xi^o)$ is real and non zero. Moreover, for every line F_1 parallel to $dh_1(\xi^o)$, the inclusion homomorphism $H^1(F_1 \cap \Omega, F_1 \cap L) \longrightarrow H^1(F_1 \cap \Omega, F_1 \cap K)$ is zero.

c) $h_1(\xi^o) = 0$, $dh_1(\xi^o) = a+ib$, where a and b are two independent real vectors. Moreover, for each plane F_2 parallel to the plane generated by a and b, the inclusion homomorphism $H^2(F_2 \cap \Omega, F_2 \cap L) \longrightarrow H^2(F_2 \cap \Omega, F_2 \cap K)$ is zero.

Then for each point $x_0 \in \Omega \setminus L$, and each compact subset M of Ω, there is a neighborhood W of ξ^o in Σ such that, for each function $\alpha(\xi)$ homogeneous of degree 0 and C^∞ outside zero, with $\Sigma \cap \mathrm{supp}(\alpha) \subset W$, the following holds: if a distribution $u \in \mathcal{E}'_M(\Omega)$ satisfies sing supp$(Op[\alpha] Au) \subset K$, then $Op[\alpha] u$ is C^∞ in a neighborhood of x_0.

Proposition 3.2 can be applied to give sufficient conditions of convexity for operators which, ξ-locally, can be reduced in a way to operators with simple real characteristics. The case of operators which, themselves, have only simple real characteristics is now well-known.

Instead, we shall state results for the case of operators on R^2 and some special operators on R^3.

If $P(D)$ is an operator on R^2, then the set $V(P)$ introduced before prop.2.3 can be characterized as the set of lines of equation $\xi_1 x_1 + \xi_2 x_2 = 0$, such that there exists a sequence (ζ^n) in C^2, $|\zeta^n| \to \infty$ while $|\mathrm{Im}\,\zeta^n|$ remains bounded, with $P(\zeta^n)=0$, the (complex) direction defined by (ζ_1^n, ζ_2^n) tending as $n \to \infty$ to the (real) direction (ξ_1, ξ_2) (think of Hörmander's characterization of hypoelliptic operators for a motivation). Now, any operator $P(D)$ on R^3 can be factored as a product of pseudo-differential operators of convolution. which, except for hypoelliptic factors, contains only operators of order 1 whose principal parts are operators of derivation along the directions belonging to $V(P)$. It follows then from prop.2.3 and prop.3.2 that an open set Ω in R^2 is $P(-D)$-sigular convex if and only if every line parallel to some element of $V(P)$ has a connected intersection with Ω (note that in this case, the ξ- localisation in prop.3.2 is not needed, as one can prove singular convexity for each pseudo-differential factor of $P(D)$ separately).

We shall now state a result for operators $P(D)$ on R^3 whose symbols $P(\xi)$ satisfy the following conditions: $P(\xi)$ is homogeneous and real-valued, and at every point $\xi \neq 0$ such that $P^{(j)}(\xi) = 0$ $(j=1,2,3)$, the matrix $(P^{(jk)}(\xi))$ has rank 2 (note that it cannot have rank 3, because of Euler's identity). This being assumed, let Γ be the set of equation $P(\xi) = 0$, let S be the subset of Γ defined by the equations

$P^{(j)}(\xi) = 0$ $(j=1,2,3)$ (it is easily seen to consist of a finite number of lines), and let S_1 be the subset of S generated, as a cone, by the points of $S \cap \Sigma$ which are isolated in $V \cap \Sigma$. Then the set V(P) can be characterized as the following: $V(P) \cap G(3,1)$ is the closure of the set of all lines which are orthogonal to Γ at some points of $\Gamma \setminus S$, and $V(P) \cap G(3,2)$, apart from the planes which contain some element of $V(P) \cap G(3,1)$ (this set can be neglected in the formulation of conditions of convexity), consists of the directions of planes which are orthogonal to some line belonging to S_1. Then, using Morse's lemma to be in a position to apply prop.3.2, one can show that the necessary conditions given in prop.2.3 are, in this case, also sufficient.