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ULTRADISTRIBUTIONS, HYPERFUNCTIONS AND

LINEAR DIFFERENTIAL EQUATIONS

by Hikosaburo KOMATSU

In his lecture [18] at Lisbon in 1964, A. Martineau has shown that if a holomorphic function $F(x + iy)$ on the wedge domain $(\mathbf{R}^n + i\Gamma) \cap V$, where Γ is an open convex cone in $\mathbf{R}^n$ and V is an open set in $\mathbf{C}^n$, satisfies the estimate

$$\sup_{x \in K} |F(x + iy)| \leqslant C|y|^{-L}, \qquad y \in \Gamma' \tag{0.1}$$

for every compact set K in $\Omega = \mathbf{R}^n \cap V$ and closed subcone Γ' in Γ, then it has the boundary value

$$F(x + i\Gamma 0) = \lim_{\substack{y \to 0 \\ y \in \Gamma'}} F(x + iy) \tag{0.2}$$

in the sense of distribution and that if $\Gamma_1, \ldots, \Gamma_m$ are open convex cones in $\mathbf{R}^n$ such that the dual cones $\Gamma_1^o, \ldots, \Gamma_m^o$ cover the dual space of $\mathbf{R}^n$, then every distribution $f(x)$ on Ω can be represented as the sum of boundary values

$$F(x) = F_1(x + i\Gamma_1 0) + \ldots + F_m(x + i\Gamma_m 0) \tag{0.3}$$

of holomorphic functions $F_j(x + iy)$ on $(\mathbf{R}^n + i\Gamma_j) \cap V$ satisfying the above estimate. He has also found a necessary and sufficient condition for F_j in order that the sum of the form (0.3) is equal to zero.

We establish corresponding results for ultradistributions of Roumieu [24], [25] and Beurling [1] and apply them to a few problems of regularity and existence of linear differential equations.

1. Hyperfunctions.

A hyperfunction f on an open set Ω in $\mathbb{R}^n$ is by definition a cohomology class in the relative cohomology group $H^n_\Omega(V, \mathcal{O})$ with support in Ω, where V is an open set in $\mathbb{C}^n$ containing Ω as a relatively closed set and $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$. (For the theory of hyperfunctions in general see [26], [27], [17], [5], [8] and [30].)

There are two interpretations. According to Sato [26], [27] a hyperfunction is the sum of "boundary values" of holomorphic functions whereas Martineau [17] and Schapira [30] regard it as a locally finite sum of real analytic functionals.

Suppose that V is a (Stein) open set in $\mathbb{C}^n$ and that Γ is an open convex cone in $\mathbb{R}^n$. We have a canonical injective mapping

$$\partial_\Gamma : \mathcal{O}(V_\Gamma) \longrightarrow \mathcal{B}(\Omega), \tag{1.1}$$

where $\mathcal{O}(V_\Gamma)$ is the space of all holomorphic functions on the wedge

$$V_\Gamma = (\mathbb{R}^n + i\Gamma) \cap V \tag{1.2}$$

and

$$\mathcal{B}(\Omega) = H^n_\Omega(V, \mathcal{O}) \tag{1.3}$$

is the space of all hyperfunctions on

$$\Omega = \mathbb{R}^n \cap V. \tag{1.4}$$

If $F(x + iy) \in \mathcal{O}(V_\Gamma)$, we denote its image by $F(x + i\Gamma 0)$ and call it the <u>boundary value in the sense of hyperfunction</u>. $F(x + i\Gamma 0)$ depends only on F. Namely if Γ' is a subcone of Γ, we have

(1.5) $$F(x + i\Gamma' 0) = F(x + i\Gamma 0).$$

(Cf. Martineau [20], Komatsu [11], Sato-Kawai-Kashiwara [28], Chap. I).

If $\Gamma_1, \dots, \Gamma_m$ are open convex cones in $\mathbb{R}^n$ such that the dual cones

(1.6) $$\Gamma_j^o = \{\xi \in \mathbb{R}^n \ ; \ \langle y, \xi \rangle \geqslant 0, \quad y \in \Gamma_j\}$$

cover the dual space of $\mathbb{R}^n$, then we have

(1.7) $$\mathcal{B}(\Omega) = \partial_{\Gamma_1} \mathcal{O}(V_{\Gamma_1}) + \dots + \partial_{\Gamma_m} \mathcal{O}(V_{\Gamma_m}) \ ,$$

i. e. every hyperfunction $f \in \mathcal{B}(\Omega)$ can be written

(1.8) $$f(x) = F_1(x + i\Gamma_1 0) + \dots + F_m(x + i\Gamma_m 0)$$

for some $F_j(x + iy) \in \mathcal{O}(V_{\Gamma_j})$. We call the m-tuple $(F_1, \dots, F_m)$ of holomorphic functions a <u>defining function</u> of the hyperfunction f.

Martineau's edge of the wedge theorem [20] (cf. also Morimoto [22], [23]) asserts that

(1.9) $$F_1(x + i\Gamma_1 0) + \dots + F_\ell(x + i\Gamma_\ell 0) = 0$$

if and only if there are holomorphic functions $F_{jk}(x + iy) \in \mathcal{O}(V'_{\Gamma_{jk}})$ depending on indices j, k alternatingly such that

(1.10) $$F_j = \sum_{k=1}^{\ell} F_{jk} \ ,$$

where Γ_{jk} is the convex hull of $\Gamma_j \cup \Gamma_k$ and $V' \subset V$ is a complex neighborhood of Ω.

In particular, a hyperfunction may be identified with a class of defining functions.

If the dimension $n = 1$, the situation is particularly simple. We have only one choice of the cones Γ_j :

(1.11) $$\Gamma_1 = \{y \in \mathbf{R} \; ; \; y > 0\}, \qquad \Gamma_2 = \{y \in \mathbf{R} \; ; \; y < 0\}.$$

In this case it is convenient to call $F = (F_1 \; , -F_2) \in \mathcal{O}(V \setminus \Omega)$ a defining function.

Thus

(1.12) $$f(x) = F(x + i0) - F(x - i0).$$

The zero class is composed of the restrictions to $V \setminus \Omega$ of all holomorphic functions F on V. Hence we have

(1.13) $$\mathcal{B}(\Omega) = \mathcal{O}(V \setminus \Omega) / \mathcal{O}(V).$$

It is known that $\mathcal{B}(\Omega)$, $\Omega \subset \mathbf{R}^n$, form a flabby sheaf under the natural restriction mapping. In particular we can talk about the support of a hyperfunction.

On the other hand, let $\mathcal{A}(\Omega)$ be the space of all real analytic functions on Ω. We have

(1.14) $$\begin{aligned} \mathcal{A}(\Omega) &= \varinjlim_{V \supset \Omega} \mathcal{O}(V) \\ &= \varprojlim_{K \subset\subset \Omega} \mathcal{A}(K), \end{aligned}$$

where V runs through the complex neighborhoods of Ω. Here the space $\mathcal{O}(V)$ is a Fréchet space and $\mathcal{A}(K) = \varinjlim_{U \supset K} \mathcal{O}(U)$ is a (DFS)-space. Hence we can introduce two natural locally convex topologies. However, as Martineau [19] shows, these two topologies coincide.

The elements in the dual $\mathcal{A}'(\Omega)$ of $\mathcal{A}(\Omega)$ are called <u>real analytic functionals</u> on Ω. If $f \in \mathcal{A}'(\Omega)$, there is the smallest compact set $K \subset \Omega$ such that $f \in (\mathcal{A}(K))'$, which we call the <u>support</u> of f (Martineau [17]).

$\mathcal{A}'(\Omega)$ is naturally identified with the set of all hyperfunctions with compact support in Ω including the concept of support under Martineau's duality [17]. Since the hyperfunctions form a flabby sheaf, every $f \in \mathcal{B}(\Omega)$ can be written

(1.15) $$f = \sum f_j ,$$

where $f_j \in \mathcal{A}'(\omega)$ and $\{\operatorname{supp} f_j\}$ is locally finite.

As Martineau [17] and Schapira [30] did, we can also construct the theory of hyperfunctions starting with the definition that hyperfunctions are locally finite sums of real analytic functionals.

We note, however, that the first interpretation provides us with means to study the properties of real (généralized) functions through the behavior of holomorphic functions. In one-dimensional case this is an old idea. For example, Hardy [4] proved in 1916 the non-differentiability of Weierstrass' function

(1.16) $$\sum_{k=1}^{\infty} a^k \cos b^k \pi x , \quad 0 < a < 1, \quad ab > 1,$$

by the order of growth of dF/dz as z tends to the real axis.

Moreover, we have a theory of multiplication and restriction independent of regularity. If two hyperfunctions f and $g \in \mathcal{B}(\Omega)$ can be written

(1.17) $$f(x) = \sum_j F_j(x + i\Gamma_j 0), \quad g(x) = \sum_k G_k(x + i\Gamma'_k 0)$$

with open convex cones $\{\Gamma_1, \ldots, \Gamma_m\}$ and $\{\Gamma'_1, \ldots, \Gamma'_m\}$ such that $\Gamma_j \cap \Gamma_k \neq \emptyset$ for all j and k, then we can define the <u>product</u> fg by

(1.18) $$(fg)(x) = \sum_{j,k} (F_j G_k)(x + i(\Gamma_j \cap \Gamma'_k)0).$$

If $f \in \mathcal{B}(\Omega)$ is written as (1.17) and if H is a complex affine submanifold of $\mathbf{C}^n$ such that $\Gamma'_j = \operatorname{Im}((\mathbf{R}^n + i\Gamma_j) \cap H)$ is a nonvoid open convex cone in $\operatorname{Im} H$ for all j, then we can define the <u>restriction</u> $f|\mathbf{R}^n \cap H$ by

(1.19) $$f|\mathbf{R}^n \cap H = \sum_j F_j(x + i\Gamma'_j 0)|H.$$

This theory has been developed into the deep theory of microfunctions by

Sato-Kawai-Kashiwara [28] and Morimoto [22].

Now, Martineau's results in [18] may be summarized as follows. If we replace $\mathcal{O}(V_\Gamma)$ by the subspace $\mathcal{O}_{\mathcal{D}'}(V_\Gamma)$ of all $F \in \mathcal{O}(V_\Gamma)$ satisfying the growth condition (0.1) and the cohomology group $H^n_\Omega(V, \mathcal{O})$ by the cohomology group $H^n_{\mathcal{D}';\Omega}(V, \mathcal{O})$ with bound, then we obtain distributions instead of hyperfunctions and the boundary value (1.1) in the sense of hyperfunction coincides with the boundary value (0.2) in the sense of distribution (cf. also Köthe [15]).

As Morimoto [22], [23] points out, Martineau's theory [18], [20] of the edge of the wedge theorem has reached a point very close to the theory of microfunctions.

The motivation of our study is to develop Martineau's idea further and apply the results to regularity problems of differential equations.

2. Ultradistributions.

Let M_p, $p = 0, 1, \ldots,$ be a sequence of positive numbers. We impose the following conditions :

(M.1) $$M_p^2 \leqslant M_{p-1} M_{p+1}, \qquad p = 1, 2, \ldots ;$$

(M.2) $$M_p \leqslant A H^p \min_{0 \leqslant q \leqslant p} M_q M_{p-q}, \qquad p = 1, 2, \ldots ;$$

(M.3) $$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leqslant A p \frac{M_p}{M_{p+1}}, \qquad p = 1, 2, \ldots .$$

Here A and H are constants independent of p.

An infinitely differentiable function φ on an open set Ω in $\mathbb{R}^n$ is called an <u>ultradifferentiable function of class</u> (M_p) (resp. <u>of class</u> $\{M_p\}$) if for each compact set K in Ω and for every h 0 there is a constant C (resp. there are constants h and C) such that

$$\sup_{x\in K} |D^\alpha \varphi(x)| \leqslant Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha| = 0, 1, 2, \dots , \tag{2.1}$$

where

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{2.2}$$

and

$$|\alpha| = \alpha_1 + \dots + \alpha_n . \tag{2.3}$$

If $s > 1$, the Gevrey sequence

$$M_p = (p!)^s \quad \text{or} \quad p^{ps} \quad \text{or} \quad \Gamma(1 + ps) \tag{2.4}$$

satisfies the conditions (M.1), (M.2) and (M.3). In this case we write (s) and $\{s\}$ instead of (M_p) and $\{M_p\}$ for short.

We denote by $\mathcal{E}^{(M_p)}(\Omega)$ (resp. by $\mathcal{E}^{\{M_p\}}(\Omega)$) the space of all ultradifferentiable functions φ of class (M_p) (resp. of class $\{M_p\}$) on Ω, and by $\mathcal{D}^{(M_p)}(\Omega)$ (resp. $\mathcal{D}^{\{M_p\}}(\Omega)$) the subspace of $\mathcal{E}^{(M_p)}(\Omega)$ (resp. of $\mathcal{E}^{\{M_p\}}(\Omega)$) composed of all functions with compact support.

We have

$$\mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \subset\subset \Omega} \varprojlim_{h \to 0} \mathcal{E}^{\{M_p\},h}(K) , \tag{2.5}$$

$$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K \subset\subset \Omega} \varinjlim_{h \to \infty} \mathcal{E}^{\{M_p\},h}(K) , \tag{2.6}$$

$$\mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K \subset\subset \Omega} \varprojlim_{h \to 0} \mathcal{D}_K^{\{M_p\},h} , \tag{2.7}$$

$$\mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K \subset\subset \Omega} \varinjlim_{h \to \infty} \mathcal{D}_K^{\{M_p\},h} , \tag{2.8}$$

where $\mathcal{E}^{\{M_p\},h}(K)$ is the Banach space of all infinitely differentiable functions φ in the sense of Whitney on the regular compact set K satisfying condition (2.1) and $\mathcal{D}_K^{\{M_p\},h}$ is its closed subspace composed of all functions φ on $\mathbb{R}^n$ with support in K.

Thus we can introduce natural locally convex topologies in these spaces. We denote by $\mathcal{D}^{(M_p)'}(\Omega)$ (by $\mathcal{D}^{\{M_p\}'}(\Omega)$) the strong dual of $\mathcal{D}^{(M_p)}(\Omega)$ (of $\mathcal{D}^{\{M_p\}}(\Omega)$) and call its elements <u>ultradistributions of class</u> (M_p) (<u>of class</u> $\{M_p\}$).

$\mathcal{D}^{\{M_p\}'}(\Omega)$ is exactly the space of ultradistributions discussed by Roumieu [24], [25] and if M_p is the Gevrey sequence, $\mathcal{D}^{(M_p)'}(\Omega)$ coïncides with the space of generalized distributions of Beurling and Björck [1].

By the Denjoy-Carleman-Mandelbrojt theorem there are sufficiently many functions in the space $\mathcal{D}^*(\Omega)$, where $* = (M_p)$ or $\{M_p\}$. Hence we can construct the theory of ultradistributions in the same way as Schwartz' theory of distributions. In particular, $\mathcal{D}^{*'}(\Omega)$, $\Omega \subset R^n$, form a soft sheaf and the dual $\mathcal{E}^{*'}(\Omega)$ of $\mathcal{E}^*(\Omega)$ is identified with the subspace of $\mathcal{D}^{*'}(\Omega)$ composed of all ultradistributions with compact support in Ω. The spaces $\mathcal{D}^*(\Omega)$, $\mathcal{E}^*(\Omega)$, $\mathcal{D}^{*'}(\Omega)$ and $\mathcal{E}^{*'}(\Omega)$ are all complete barrelled bornologic nuclear spaces ([13] cf. also Schapira [29]).

We have

$$\mathcal{A}(\Omega) \subset \mathcal{E}^*(\Omega) \subset \mathcal{E}(\Omega) \tag{2.9}$$

and the inclusion mappings are continuous and of dense range. Hence we may consider

$$\mathcal{E}'(\Omega) \subset \mathcal{E}^{*'}(\Omega) \subset \mathcal{A}'(\Omega). \tag{2.10}$$

Since these inclusion mappings keep support (Harvey [5]), they can be extended to the inclusions

$$\mathcal{D}'(\Omega) \subset \mathcal{D}^{*'}(\Omega) \subset \mathcal{B}(\Omega). \tag{2.11}$$

If $t > s > 1$, we have

$$\mathcal{D}'(\Omega) \underset{\neq}{\subset} \mathcal{D}^{\{t\}'}(\Omega) \underset{\neq}{\subset} \mathcal{D}^{(t)'}(\Omega) \underset{\neq}{\subset} \mathcal{D}^{\{s\}'}(\Omega) \underset{\neq}{\subset} \mathcal{D}^{(s)'}(\Omega) \underset{\neq}{\subset} \mathcal{B}(\Omega). \tag{2.12}$$

The theory of multiplication and convolution is the same as for distributions.

A differential operator

$$P(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \tag{2.13}$$

of infinite order is said to be an ultradifferential operator of class (M_p) (of class $\{M_p\}$) if there are constants L and C (for every $L > 0$ there is C) such that

$$|a_\alpha| \leq CL^{|\alpha|}/M_{|\alpha|}, \quad |\alpha| = 0, 1, \dots . \tag{2.14}$$

Ultradifferential operators of class $*$ are continuous operators in the spaces of class $*$ and sheaf homomorphisms in the sheaves $\mathcal{E}^*$ and $\mathcal{D}^{*\prime}$.

THEOREM 2.1 (First structure theorem [13]).

$f \in \mathcal{D}^{(M_p)\prime}(\Omega)$ $(\in \mathcal{D}^{\{M_p\}\prime}(\Omega))$ if and only if on every relatively compact open set G of Ω (on Ω)

$$f = \sum_{|\alpha|=0}^{\infty} D^\alpha f_\alpha \tag{2.15}$$

with measures f_α on G (on Ω) such that

$$\|f_\alpha\|_{C'(\bar{G})} \leq C\, L^{|\alpha|}/M_{|\alpha|}, \quad |\alpha| = 0, 1, \dots \tag{2.16}$$

for some L and C (for every relatively compact open set G, every $L > 0$ and some C).

For the class $\{M_p\}$ this is due to Roumieu [24], [25]. However, it is not certain whether or not the topology of $\mathcal{D}^{\{M_p\}}(\Omega)$ he employed in his proof coincides with the above natural topology.

THEOREM 2.2 (Second structure theorem [13]).

$f \in \mathcal{D}^{*\prime}(\Omega)$ if and only if for every relatively compact convex open set G in Ω there are an ultradifferential operator $P(D)$ of class $*$ and a measure g on G such that

$$f = P(D)g . \tag{2.17}$$

The proofs of Theorems 2.1 and 2.2 for the class $\{M_p\}$ are complicated. We employ a result by De Wilde [3] and Komatsu [9] on the duals of the inductive limits of weakly compact sequences of Banach spaces.

We define

(2.18) $$M^*(\rho) = \sup_p \log \frac{\rho^p p! M_0}{M_p} .$$

If $M_p = (p!)^s$, $M^*(\rho)$ is equivalent to $\rho^{1/(s-1)}$.

THEOREM 2.3 ([13]).

If $F(x + iy)$ is a holomorphic function on V_Γ and if for every compact set K in Ω and closed subcone $\Gamma' \subset \Gamma$ there are constants L and C (for every $L > 0$ there is a constant C) such that

(2.19) $$\sup_{x \in K} |F(x + iy)| \leqslant C \exp M^*(L/|y|) \quad \text{for } y \in \Gamma' ,$$

then the boundary value $F(x + i\Gamma 0)$ in the sense of hyperfunction is in $\mathcal{D}^{(M_p)'}(\Omega)$ (in $\mathcal{D}^{\{M_p\}'}(\Omega)$) and (0.2) holds in the topology of $\mathcal{D}^{*'}(\Omega)$.

The assumption may considerably be relaxed. Namely employing the coordinate transformation which was used in the proof of a local version of Bochner's tube theorem [12], we can show that if (2.19) holds for a ray Γ' in Γ, then it holds for any closed subcone Γ'.

From the second structure theorem we obtain

THEOREM 2.4 ([13]).

Let $f \in \mathcal{D}^{*'}(\Omega)$ and G be a relatively compact open subset of Ω. Then f can be represented as (1.8) on G with holomorphic functions F_j satisfying the condition of Theorem 2.3.

If the dimension $n = 1$, the difference of two defining functions of the same hyperfunction is a holomorphic function on a neighborhood of Ω. Thus if a defining function satisfies estimate (2.19), any other defining function satisfies it also.

Moreover, Painlevé's theorem holds also in the topology of ultradistributions. Therefore we have

THEOREM 2.4.

Let $n = 1$. If $F(x + iy)$ is a holomorphic function on V_Γ and if its boundary value $F(x + i\Gamma 0)$ either in the sense of hyperfunction or in the sense of ultradistribution is in $\mathcal{D}^{(M_p)'}(\Omega)$ (in $\mathcal{D}^{\{M_p\}'}(\Omega)$), then $F(x+iy)$ satisfies the condition of Theorem 2.3.

Our conjecture is that this is true also in the case where $n > 1$.

3. Ordinary differential equations.

First we consider the single linear ordinary differential equation

$$(3.1) \qquad (a_m(x)\frac{d^m}{dx^m} + a_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \dots + a_0(x))u(x) = f(x) ,$$

where $a_j(x)$ are real analytic functions on the open interval $\Omega = (a , b)$ and $a_m(x) \not\equiv 0$.

Choose a complex neighborhood V of Ω to which $a_j(x)$ are continued analytically and let $F(x + iy)$ and $U(x + iy) \in \mathcal{O}(V \setminus \Omega)$ be the defining functions of f and u respectively. We denote by $P(x , d/dx)$ the differential operator on the left hand side of (3.1) and by $P(z , d/dz)$ its analytic continuation to V. Then equation (3.1) is equivalent to

$$(3.2) \qquad P(z , \frac{d}{dz})U(z) \equiv F(z) \mod \mathcal{O}(V).$$

We can choose V so that V and $V \setminus \Omega$ are simply connected and that $V \setminus \Omega$ is free from zeros of $a_m(z)$. Hence we obtain :

THEOREM 3.1 (Sato [26]).

For every hyperfunction f on Ω there is a hyperfunction solution u of

(3.1) on Ω.

THEOREM 3.2 (Sato [26], Komatsu [10]).

If f is a hyperfunction on Ω, any hyperfunction solution u_1 of (3.1) on a subinterval Ω_1 can be prolonged to a hyperfunction solution u on Ω.

THEOREM 3.3 (Komatsu [10]).

There are

(3.3) $$m + \sum_{x \in \Omega} \mathrm{ord}_x\, a_m(x)$$

linearly independent hyperfunction solutions u on Ω of the homogeneous equation

(3.4) $$P(x\ ,\ \frac{d}{dx})u(x) = 0\ ,$$

where $\mathrm{ord}_x\, a_m(x)$ is the order of zero of $a_m(x)$ at x.

The last theorem is derived from the index formula

(3.5) $$\chi(P_V) = m\,\chi(V) - \sum_{z \in V} \mathrm{ord}_z\, a_m(z)$$

for the operator

(3.6) $$P_V = P(z\ ,\ d/dz)\ :\ \mathcal{O}(V) \longrightarrow \mathcal{O}(V).$$

Here $\chi(V)$ denotes the Euler characteristic of the open set V in $\mathbf{C}$ ([10]) (1).

Now it is easy to prove the following.

THEOREM 3.4 ([14])(2). The following are equivalent :

(i) Every hyperfunction solution on Ω of the homogeneous equation is real analytic ;

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(1) We were informed at the conference that B. Malgrange had obtained index formula (3.5) independently about a year later than us. See B. Malgrange, Remarques sur les points singuliers des équations différentielles, C. R. Acad. Sc. Paris, Sér. A, 273 (1971), 1136-1137.

(2) Combining the results of Y. Kannai at this conference and Theorem 3.5, we can prove Theorem 3.4 with "real analytic" replaced by "infinitely differentiable".

(ii) $a_m(x) \neq 0$ for all $x \in \Omega$;

(iii) If $Pu \in \mathcal{A}(\Omega)$, then $u \in \mathcal{A}(\Omega)$.

If x_0 is a singular point of $P(x , d/dx)$ or a zero of $a_m(z)$, we define the irregularity(1) σ of x_0 to be the maximal gradient of the highest convex polygon below the points $(j , \mathrm{ord}_{x_0} a_j(x))$, $j = 0, 1, \dots, m$. x_0 is a regular singular point if $\sigma \leq 1$ and an irregular singular point if $\sigma > 1$.

THEOREM 3.5 ([14]). The following are equivalent :

(i) Every hyperfunction solution on Ω of the homogeneous equation is a distribution ;

(ii) All singular points in Ω are regular ;

(iii) If $Pu \in \mathcal{D}'(\Omega)$, then $u \in \mathcal{D}'(\Omega)$.

THEOREM 3.6 ([14]).

Let $s > 1$. Then the following are equivalent :

(i) Every hyperfunction solution on Ω of the homogeneous equation is an ultradistribution of class (s) ;

(ii) The irregularity σ of any singular point in Ω does not exceed $s/(s-1)$;

(iii) If $Pu \in \mathcal{D}^{(s)'}(\Omega)$, then $u \in \mathcal{D}^{(s)'}(\Omega)$.

Sketch of proof.

(i) $\Rightarrow$ (ii). Let 0 be an irregular singular point. Then, Hukuhara [6] and Malmquist [16] show that there is a holomorphic solution $U(z)$ of $P(z , d/dz)U(z) = 0$ either in the upper half plane or in the lower such that

$$\sup_{x \in K} |U(x + iy)| \geq C \exp\{(L/|y|)^{\sigma-1}\}, \tag{3.6}$$

with $C > 0$. Hence its boundary value $U(x \pm i0)$ can not belong to $\mathcal{D}^{(s)'}(\Omega)$ for any

(1) Our definition of irregularity is different from that of Malgrange, loc. cit.

$s > \sigma/(\sigma-1)$.

(ii) $\Longrightarrow$ (iii). Set $\tau = 1$ for Theorem 3.5 and $\tau = s/(s-1)$ for Theorem 3.6. Statement (ii) means that the irregularity $\sigma \leqslant \tau$ at any singular point.

Let x_0 be an arbitrary point in Ω. If we set

$$W^j(z) = ((z - x_0)^{\tau} \frac{d}{dz})^{j-1} U(z), \qquad j = 1, \ldots, m, \tag{3.7}$$

the vector $W(z) = {}^t(W^1(z), \ldots, W^m(z))$ satisfies the equation

$$((z - x_0)^{\tau} \frac{d}{dz} - B(z))W(z) = F'(z) , \tag{3.8}$$

where $B(z)$ is an $m \times m$ matrix of holomorphic functions bounded in a neighborhood of x_0 and the components F^j of F' satisfy the estimate

$$\sup_{x \in K} |F^j(x + iy)| \leqslant C|y|^{-L} \qquad \text{or} \qquad \leqslant C \exp\{(L/|y|)^{1/(s-1)}\}. \tag{3.9}$$

Changing the independent variable into

$$t = \begin{cases} \log(\frac{\pm i}{z - x_0}), & \tau = 1 \\ (\frac{\pm i}{z - x_0})^{\tau - 1}, & \tau > 1, \end{cases}$$

and integrate (3.8) along the curve $\Gamma^1 \cup \Gamma^2$, where Γ^1 is a segment joining $x_0 + id$ and $x_0 + ir$ and Γ^2 is an arc joining $x_0 + ir$ and $z = x_0 + ire^{i\theta}$ with center at x_0. Then we can easily show that $W^j(z)$ and hence $U(z)$ satisfy estimate (3.9).

(iii) $\Longrightarrow$ (i). Trivial.

Combining Theorems 3.5 and 3.6 with Theorems 3.1, 3.2 and 3.3, we obtain

THEOREM 3.7.

If $P(x, d/dx)$ satisfies the equivalent conditions of Theorem 3.5 (resp. Theorem 3.6), then Theorems 3.1, 3.2 and 3.3 hold with hyperfunction replaced by distribution (resp. ultradistribution of class (s)).

The above results are extended to the first order system :

(3.10) $$(A_1(x)\frac{d}{dx} + A_0(x))u(x) = f(x) ,$$

where $A_0(x)$ and $A_1(x)$ are $m \times m$ matrices of real analytic functions on Ω such that $\det A_1(x)$ is not identically zero.

In fact, Theorems 3.1, 3.2 and 3.3 hold good if we replace (3.3) by

(3.11) $$m + \sum_{x \in \Omega} \mathrm{ord}_x \det A_1(x)$$

([10]). Hence Theorem 3.4 holds with $a_m(x)$ in (ii) replaced by $\det A_1(x)$.

To define the irregularity σ of a singular point, which we assume to be the origin, we employ Hukuhara's canonical form. Let

(3.12) $$(A_1(z)\frac{d}{dz} + A_0(z))U(z) = F(z)$$

be the equation on $V \setminus \Omega$ for the defining function. Hukuhara [6] shows that there is a transformation matrix $T(z)$ whose elements are polynomials in $z^{\mp 1/q}$ for some integer q such that $W(z) = T^{-1}(z)U(z)$ satisfies the equation

(3.13) $$(\frac{d}{dz} - B(z))W(z) = F'(z) ,$$

where $F'(z)$ satisfies essentially the same growth condition as $F(z)$ and $B(z)$ has the form

(3.14) $$B(z) = \Lambda'(z) + z^{-1}C(z)$$

with a diagonal matrix $\Lambda'(z)$ whose (j,j)-element

(3.15) $$\lambda_j(z) = \lambda_j z^{-\sigma} + \ldots + \omega_j z^{-1-1/q}$$

is either 0 or z^{-1} times a polynomial in $z^{-1/q}$ and a matrix $C(z)$ of holomorphic functions in $z^{1/q}$.

$x_0 = 0$ is called a <u>regular singular point</u> if $\det A_1(x_0) = 0$ but all $\lambda_j(z) = 0$

and an irregular singular point of irregularity $\sigma\,(>1)$ if some $\lambda_j \neq 0$ in (3.15).

Then Theorems 3.5 and 3.6 and hence Theorem 3.7 hold for the system (3.10). Methée's result [21] is a special case of ours.

4. Existence of singular solutions.

Let

$$P(D) = \sum a_\alpha D^\alpha \tag{4.1}$$

be a differential operator with constant coefficients possibly of infinite order. $P(D)$ is said to be elliptic if there exists a constant A such that we have

$$|\eta| \geqslant A^{-1}|\xi| \quad \text{for} \quad |\xi| \geqslant A \tag{4.2}$$

for those $\xi + i\eta \in \mathbb{R}^n + i\mathbb{R}^n$ which satisfy $P(\xi + i\eta) = 0$.

Chou [2], BJörck [1], Harvey [5] and Kawai [7] have proved under various assumptions that if $P(D)$ is not elliptic, then the equation

$$P(D)u(x) = 0 \tag{4.3}$$

has always a singular hyperfunction (or ultradistribution) solution u.

Modifying the method of [7] we prove the following.

THEOREM 4.1. *Suppose that there exists a sequence* $\zeta^{(j)} = \xi^{(j)} + i\eta^{(j)} \in \mathbb{R}^n + i\mathbb{R}^n$ *of zeros of* $P(\zeta)$ *such that*

$$\xi_1^{(j)} \longrightarrow \infty, \tag{4.4}$$

$$|(\xi_2^{(j)}, \ldots, \xi_n^{(j)})| + |\eta^{(j)}| \leqslant C(\xi_1^{(j)})^\sigma \tag{4.5}$$

for some constants $0<\sigma<1$ *and* $C>0$.

Then (4.3) *has a solution* $u \in \mathcal{D}^{(1/\sigma)'}(\mathbb{R}^n)$ *whose singular support in the sense of Sato* [28] *contains* $0 \times (1, 0, \ldots, 0)\infty$ *and is contained in* $\mathbb{R}^n \times (1, 0, \ldots, 0)\infty$.

Proof. We may assume that $\xi_1^{(j)} \geqslant 2j$. Let

$$F(x + iy) = \sum_{j=1}^{\infty} \exp\left\{\sigma^{-1}(\xi_1^{(j)}) + i \langle x + iy \, , \, \xi^{(j)} \rangle\right\}. \tag{4.6}$$

Suppose that for some $K < \infty$

$$|x| \leqslant K, \quad |y| \leqslant K \quad \text{and} \quad y_1 > 0.$$

Then we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} \left|\exp\left\{\sigma^{-1}(\xi_1^{(j)})^{\sigma} + i\langle x+iy \, , \, \xi^{(j)}\rangle\right\}\right| \\
&\leqslant \sum_{j=1}^{\infty} \exp\left\{\sigma^{-1}(\xi_1^{(j)})^{\sigma} - y_1\, \xi_1^{(j)} + KC(\xi_1^{(j)})^{\sigma}\right\} \\
&\leqslant \exp \sup_{\xi>0}(\sigma^{-1} L' \xi^{\sigma} - y_1\, \xi/2) \cdot \sum_{j=1}^{\infty} \exp(-y_1\, \xi_1^{(j)}/2) \\
&\leqslant \exp\left\{(L''/y_1)^{\sigma/(1-\sigma)}\right\}/(1-e^{-y_1}) \\
&\leqslant C \exp\left\{(L/y_1)^{1/(\sigma^{-1}-1)}\right\} \qquad \text{for } y_1 \leqslant K'.
\end{aligned}$$

Hence (4.6) converges absolutely and locally uniformly in $\mathbb{R}^n + i\Gamma$, where $\Gamma = \left\{y \in \mathbb{R}^n \, ; \, y_1 > 0\right\}$. By Theorem 2.3 $F(x + i\Gamma 0) \in \mathcal{D}^{(1/\sigma)'}(\mathbb{R}^n)$. It is clear that $P(D)F(x+iy) = 0$ so that $P(D)F(x+i\Gamma 0) = 0$.

If $x = 0$ and $y = (y_1, 0, \dots, 0)$, we have

$$\begin{aligned}
F(x + iy) &= \sum_{j=1}^{\infty} \exp\left\{\sigma^{-1}(\xi_1^{(j)})^{\sigma} - y_1\, \xi_1^{(j)}\right\} \\
&\geqslant \exp\left\{\sigma^{-1}(\xi_1^{(j)})^{\sigma} - y_1\, \xi_1^{(j)}\right\}.
\end{aligned}$$

Thus there is a sequence $y_1^{(j)} \longrightarrow 0$ such that

$$|F(x + iy)| \geqslant \exp\left\{(\sigma^{-1} - 1)(y_1^{(j)})^{\sigma/(\sigma-1)}\right\} .$$

Therefore $F(x + i\Gamma 0)\big|_{x_2=\dots=x_n=0}$ does not belong to $\mathcal{D}^{(s)'}$ for any $s > \sigma^{-1}$ on any neighborhood of the origin.

If the conjecture after Theorem 2.4 is true, the inequality shows that $F(x+i\Gamma 0)$ itself does not belong to $\mathcal{D}^{(s)'}$ for any $s > \sigma^{-1}$ on any neighborhood of the origin.

We note that if $P(D)$ is a non-elliptic operator of finite order, then after an affine coordinate transformation the assumptions of Theorem 4.1 are satisfied.

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