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VANISHING THEOREMS ON THE COHOMOLOGIES OF SOLUTION SHEAVES OF SYSTEMS OF PSEUDO-DIFFERENTIAL EQUATIONS

by

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Recently, the theory of linear (pseudo)-differential equations has much developed, in particular, their algebraic structures and their classifications at generic points.

It is a remarkable fact that we can easily manipulate the constant multiple operators by using (pseudo)-differential operators of infinite order.

In this report, we present one of their applications, that is, the criterion of the vanishing of the cohomologies of solution sheaves of systems of pseudo-differential equations. Professor GUILLEMIN talked about similar results at the NICE Congress in 1970 ([2]). For the precise theory given here, we refer the reader to [4] .

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We fix a real analytic manifold M and its complex neighborhood X. We denote by $P^*X = (T^*X - X)/\mathbb{C}^\times$ the cotangential projective bundle. The complex structure of X gives the canonical decomposition :

$$T^*X \underset{X}{\times} M = T^*M \oplus \sqrt{-1}\, T^*M.$$

It implies that the co-normal bundle $T^*_M X = \mathrm{Ker}(T^*X \underset{X}{\times} M \to T^*M)$ is canonically isomorphic with $\sqrt{-1}\, T^*M$.

We denote by $\sqrt{-1}\, S^*M$ the sphere bundle $(T^*_M X - M)/\mathbb{R}^+$, where $\mathbb{R}^+$ is the multiplicative group of positive numbers. The injection $\sqrt{-1}\, T^*M \to T^*X$ induces a canonical map $p : \sqrt{-1}\, S^*M \to P^*X$. It is clear that $\sqrt{-1}\, S^*M \to P^*X \underset{X}{\times} M$ is the two-sheeted covering. Therefore, we identify frequently P^*X with the complex neighborhood of $\sqrt{-1}\, S^*M$. We denote by the same letter π the projections $\sqrt{-1}\, S^*M \to M$ and $P^*X \to X$. $\mathcal{A} = \mathcal{A}_M$ and $\mathcal{B} = \mathcal{B}_M$ are the sheaves of real analytic functions and hyperfunctions respectively. As shown in [4], we can construct a flabby sheaf $\mathcal{C} = \mathcal{C}_M$ on $\sqrt{-1}\, S^*M$ so that there exists a canonical exact sequence of sheaves on M

$$(D1) \qquad 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \pi_*\mathcal{C} \longrightarrow 0 \ .$$

We denote by $\mathcal{D} = \mathcal{D}_X$ and $\mathcal{P} = \mathcal{P}_X$ the sheaves of rings of differential operators of infinite order and pseudo-differential operators of infinite order respectively. $\mathcal{D}$ is a sheaf on X and $\mathcal{P}$ is a sheaf on P^*X. $\pi^{-1}\mathcal{D}$ is a sub-ring of $\mathcal{P}$. Moreover, we have $\mathcal{D} \xrightarrow{\sim} \pi_* \mathcal{P}$ if the dimension of X is strictly greater than 1. Note that the sheaf $\mathcal{O} = \mathcal{O}_X$ of holomorphic functions on X is a left $\mathcal{D}$-Module and that $\mathcal{A}$ and $\mathcal{B}$ are left $(\mathcal{D}|_M)$-Modules. Moreover, $\mathcal{C}$ is a left $p^{-1}\mathcal{P}$-Module and (D1) is a $(\mathcal{D}/_M)$-linear sequence. The system of differential equations signifies a left admissible $\mathcal{D}$-Module, and that of pseudo-differential equations signifies a left admissible $\mathcal{P}$-Module. (See the remark at the end of this report). For instance, let $\mathcal{M}$ be an admissible $\mathcal{D}$-Module and

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{r_0} \xleftarrow[P_0]{} \mathcal{D}^{r_1} \xleftarrow[P_1]{} \cdots \xleftarrow[P_{N-1}]{} \mathcal{D}^{r_N} \leftarrow 0$$

be a free resolution of $\mathcal{M}$, where P_j is a matrix of differential operators of size $r_{j+1} \times r_j$. Let $\mathcal{F}$ be another $\mathcal{D}$-Module (frequently, $\mathcal{F} = \mathcal{O}$ or $\mathcal{A}$ or $\mathcal{B}$ or $\mathcal{B}/\mathcal{A}$).

Then :

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{M};\mathcal{F}) = \mathrm{Ker}(\mathcal{F}^{r_0} \xrightarrow{P_0} \mathcal{F}^{r_1})$$
$$\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M};\mathcal{F}) = \mathrm{Ker}(\mathcal{F}^{r_j} \xrightarrow{P_j} \mathcal{F}^{r_{j+1}}) / \mathrm{Im}(\mathcal{F}^{r_{j-1}} \xrightarrow{P_{j-1}} \mathcal{F}^{r_j}).$$

It means that $\mathcal{H}om_{\mathcal{D}}(\mathcal{M};\mathcal{F})$ is the solution sheaf of P_0 and each cohomology group $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M};\mathcal{F})$ gives the obstruction of the solvability of P_{j-1} with compatibility condition given by P_j. The study of the extension groups $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{M};\mathcal{F})$ is accordingly important in the theory of differential equations.

In this report, we will give some criterion of the vanishing of the cohomology groups $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M};\mathcal{C})$ or more precisely $\mathcal{E}xt^j_{p^{-1}\mathcal{P}}(p^{-1}\mathcal{M};\mathcal{C})$ of the system of pseudo-differential equations with respect to the sheaf $\mathcal{C}$ of microfunctions. As you know later, this is an extension of the concept of q-convexity and concavity in the theory of analytic functions with several complex variables (cf. [1], [3]).

Before stating the main theorems, we will give some notations and terminologies.

A homogeneous analytic function on an open set U of $\sqrt{-1}\ S^*M$ (resp. homogeneous holomorphic function on an open set U of P^*X) of degree r is an analytic function f (resp. holomorphic function) defined on $q^{-1}(U)$, where q is the projection $\sqrt{-1}\ T^*M-M \longrightarrow \sqrt{-1}\ S^*M$ (resp. $T^*X-X \longrightarrow P^*X$), satisfying $f(cx^*) = c^r f(x^*)$ for every x^* in $q^{-1}(U)$ and for every strictly positive number c (resp. for every non zero complex number c). If $f(x^*)$ is a homogeneous analytic function defined on an open set U in $\sqrt{-1}S^*M$, we denote by $f^c(x^*)$ the homogeneous analytic function defined by $f^c(x^*) = \overline{f(x^*)}$.

Now, let $\mathcal{M}$ be a system of pseudo-differential equations (that is, an admissible $\mathcal{P}$-Module). Then, the support of $\mathcal{M}$ is an analytic set of the complex manifold P^*X, that we will denote by V. We denote by $V_{\mathbb{R}}$ the intersection of V and $\sqrt{-1}S^*M$. Let $f_1..f_r$ be a system of (homogeneous) defining functions of V in a neighborhood of a point x^* in $\sqrt{-1}S^*M$. Then :

$$L(\xi) = \sum_{1\leqslant\nu,\mu\leqslant r} \{f_\nu, f^c_\mu\}\ (x^*)\ \xi_\nu\ \bar{\xi}_\mu$$

is a hermitian form, which we call the Levi form of $\mathcal{M}$ at x^*. Note that the Poisson bracket $\{\ \}$ is the one corresponding to the contact structure of P^*X. We have :

$$\{f\ ,\ g^c\ \}^c = \{g\ ,\ f^c\}\ .$$

The number of strictly positive eigenvalues and that of strictly negative eigenvalues of the Levi form $L(\xi)$ is evidently independent of the choice of the system of defining functions $f_1, \ldots, f_r$. In fact, if $g_1, \ldots, g_N$ are homogeneous functions vanishing on V, then the hermitian form :

$$\sum \{g_j, g^c_k\}(x^*)\ \xi_j\ \bar{\xi}_k$$

has strictly positive eigenvalues and strictly negative eigenvalues less than the Levi form $L(\xi)$.

DEFINITION 1.

If the number of negative eigenvalues of Levi form of the system $\mathcal{M}$ at $x^* \in \sqrt{-1}S^*M$ exceeds q, then we say that $\mathcal{M}$ is q-convex at x^*. Note that if $\mathcal{M}$ is q-convex at x^*, then $\mathcal{M}$ is also q-convex at every point sufficiently near x^*.

Our first theorem is the following :

THEOREM 1.

If $\mathcal{M}$ is q-convex, then we have :

(F.1) $$\mathcal{E}xt_{\mathcal{P}}^{j}(\mathcal{M}, \mathcal{C}) = 0 \quad \text{for } j < q.$$

Using homological algebra, we can deduce from (F.1) that

(F.2) $$\mathrm{Ext}^{j}_{\mathcal{P},Z}(\Omega ; \mathcal{M}, \mathcal{C}) = 0 \quad \text{for } j < q,$$

every open set Ω in $\sqrt{-1}S^*M$ where $\mathcal{M}$ is q-convex and the closed subset Z in Ω. $\mathrm{Ext}^{j}_{\mathcal{P},Z}(\Omega ; \ )$ is a relative extension group with support Z.

We call reader's attention on the fact that q-convexity depends only on the support of $\mathcal{M}$. Therefore, we can say about the vanishing of the extension groups merely knowing the support of the system.

Theorem 1 gives us the information about vanishing of cohomology groups in the lower terms. We will explain some terminologies in order to state the next theorem that tells us when the cohomology groups in the upper terms vanish.

Let $\mathcal{M}$ be a system of pseudo-differential equations and x^* be a point in P^*X. The least number d satisfying the following equivalent conditions is called the local projective dimension of $\mathcal{M}$ at x^*, and denoted by $\mathrm{proj.dim}_{x^*} \mathcal{M}$:

a) $\mathcal{E}xt^j_{\mathcal{P}} (\mathcal{M} ; \mathcal{P})_{x^*} = 0$ for every $j > d$,

b) $\mathrm{Ext}^j_{\mathcal{P}_{x^*}} (\mathcal{M}_{x^*}; N) = 0$ for every $j > d$ and every $\mathcal{P}_{x^*}$-module N.

c) There is an exact sequence in a neighborhood of x^*

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{P}^{N_0} \leftarrow \cdots \leftarrow \mathcal{P}^{N_{d-1}} \leftarrow \mathcal{L} \leftarrow 0$$

such that $\mathcal{L}$ is a direct summand of a free Module.

DEFINITION 2.

$\mathcal{M}$ is said to be q-concave at $x^* \in \sqrt{-1}S^*M$ if the number of strictly positive eigenvalues of Levi form $L(\xi)$ of $\mathcal{M}$ at x^* is at least $\mathrm{proj.dim.}_{x^*} \mathcal{M} - q$.

The second Theorem is formulated as follows :

THEOREM 2.

If $\mathcal{M}$ is q-concave at any point in an open set Ω of $\sqrt{-1}S^*M$, then we have :

(i) $\mathcal{E}xt^j_{\mathcal{P},Z} (\mathcal{M}, \mathcal{C}) = 0$ for any closed set Z of Ω and every $j > q$,

(ii) $\mathrm{Ext}_{\mathcal{P},Z}(\Omega ; \mathcal{M}, \mathcal{C}) = 0$ for any locally closed set Z of Ω and every $j > q$.

These two Theorems are proved by induction on the dimension of M, using the fact that the initial value problem or the boundary value problem is well-posed.

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