Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. X (2011), 683-710

Markov uniqueness of degenerate elliptic operators

DEREK W. ROBINSON AND ADAM SIKORA

Abstract. Let Ω be an open subset of $\mathbb{R}^d$ and $H_\Omega = -\sum_{i,j=1}^d \partial_i\, c_{ij}\, \partial_j$ be a second-order partial differential operator on $L_2(\Omega)$ with domain $C_c^\infty(\Omega)$, where the coefficients $c_{ij} \in W^{1,\infty}(\Omega)$ are real symmetric and $C = (c_{ij})$ is a strictly positive-definite matrix over Ω. In particular, H_Ω is locally strongly elliptic. We analyze the submarkovian extensions of H_Ω, *i.e.*, the self-adjoint extensions that generate submarkovian semigroups. Our main result states that H_Ω is Markov unique, *i.e.*, it has a unique submarkovian extension, if and only if $\mathrm{cap}_\Omega(\partial\Omega) = 0$ where $\mathrm{cap}_\Omega(\partial\Omega)$ is the capacity of the boundary of Ω measured with respect to H_Ω. The second main result shows that Markov uniqueness of H_Ω is equivalent to the semigroup generated by the Friedrichs extension of H_Ω being conservative.

Mathematics Subject Classification (2010): 47B25 (primary); 47D07, 35J70 (secondary).

1. Introduction

The Markov uniqueness problem [9] consists of finding conditions which ensure that a diffusion operator has a unique submarkovian extension, *i.e.* an extension that generates a submarkovian semigroup. An operator with this property is said to be Markov unique. Our aim is to analyze this problem for the class of second-order, divergence-form, elliptic operators with real Lipschitz continuous coefficients acting on an open subset of Ω of $\mathbb{R}^d$. Each of these operators has at least one submarkovian extension, the Friedrichs extension [16]. This extension corresponds to Dirichlet boundary conditions on $\partial\Omega$ and alternative boundary conditions can lead to different submarkovian extensions. Our principal results establish that Markov uniqueness is equivalent to the boundary $\partial\Omega$ having zero capacity, Theorem 1.2, or to conservation of probability, Theorem 1.3.

Define H_Ω as the positive symmetric operator on $L_2(\Omega)$ with domain $D(H_\Omega) = C_c^\infty(\Omega)$ and action

$$H_\Omega\varphi = -\sum_{i,j=1}^d \partial_i\, c_{ij}\, \partial_j\varphi = -\sum_{i,j=1}^d c_{ij}\, \partial_i\, \partial_j\varphi - \sum_{i,j=1}^d (\partial_i c_{ij})\, \partial_j\varphi \tag{1.1}$$

Received February 4, 2010; accepted May 26, 2010.

where the $c_{ij} = c_{ji} \in W^{1,\infty}(\Omega)$ are real, $C = (c_{ij})$ is a non-zero, positive-definite matrix over Ω and $\partial_i = \partial/\partial x_i$. We assume throughout that $C(x) = (c_{ij}(x)) > 0$ for all $x \in \Omega$. This ensures that H_Ω is locally strongly elliptic, *i.e.* for each compact subset K of Ω there is a $\mu_K > 0$ such that $C(x) \geq \mu_K I$ for all $x \in K$. This ellipticity property is fundamental as it ensures that the various possible self-adjoint extensions of H_Ω differ only in their boundary behaviour (see Section 2).

The Markov uniqueness problem has been considered in a variety of contexts (see [9] for background material and an extensive survey). It is related to a number of other uniqueness problems. For example, the operator H_Ω, which can be viewed as an operator on $L_p(\Omega)$ for each $p \in [1, \infty]$, is defined to be L_p-unique if it has a unique extension which generates an L_p-continuous semigroup. In particular H_Ω is L_2-unique if and only if it is essentially self-adjoint (see [9, Corollary 1.2]). Then the self-adjoint closure is automatically submarkovian and H_Ω is Markov unique. Moreover, if H_Ω is L_1-unique then it is Markov unique [9, Lemma 1.6]. In Theorem 1.3 we will establish a converse to this statement for the class of operators under consideration. As a byproduct of our analysis of Markov uniqueness we also derive criteria for various other forms of uniqueness.

In the sequel we extensively use the theory of positive closed quadratic forms and positive self-adjoint operators (see [21, Chapter 6]) and the corresponding theory of Dirichlet forms and submarkovian operators (see [4, 15, 23]). First we introduce the quadratic form h_Ω associated with H_Ω by

$$h_\Omega(\varphi) = \sum_{i,j=1}^{d} \int_\Omega dx\, c_{ij}(x)\, (\partial_i\varphi)(x)(\partial_j\varphi)(x) \tag{1.2}$$

with domain $D(h_\Omega) = D(H_\Omega) = C_c^\infty(\Omega)$. The form h_Ω is closable with respect to the graph norm $\varphi \mapsto \|\varphi\|_{D(h_\Omega)} = (h_\Omega(\varphi) + \|\varphi\|_2^2)^{1/2}$ and its closure $\overline{h}_\Omega$ is a Dirichlet form. The positive self-adjoint operator corresponding to $\overline{h}_\Omega$ is the Friedrichs extension H_Ω^F of H_Ω. It is automatically submarkovian. Moreover, it is the largest positive self-adjoint extension of H_Ω with respect to the usual ordering of self-adjoint operators. Krein [22] established that H_Ω also has a smallest positive self-adjoint extension. But the Krein extension is not always submarkovian. For example, if Ω is bounded the Krein extension of the Laplacian restricted to $C_c^\infty(\Omega)$ is not submarkovian (see [15, Theorem 3.3.3]). Our first aim is to establish that H_Ω also has a smallest submarkovian extension. Then the Markov uniqueness problem is reduced to finding conditions which ensure that this latter extension coincides with the Friedrichs extension (see [9, Chapter 3]).

Define l_Ω by setting

$$l_\Omega(\varphi) = \int_\Omega dx \sum_{i,j=1}^{d} c_{ij}(x)\, (\partial_i\varphi)(x)(\partial_j\varphi)(x) \tag{1.3}$$

where the $\partial_i\varphi$ denote the distributional derivatives and the domain $D(l_\Omega)$ of the form is defined to be the space of all $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for which the integral is finite.

It is clear that l_Ω is an extension of $\overline{h}_\Omega$ but it is not immediately obvious that l_Ω is closed and that the corresponding operator L_Ω is an extension of H_Ω. These properties were established in [15] for operators of the form (1.1) but with smooth coefficients. Our first result is a generalization for operators with Lipschitz coefficients which also incorporates some regularity and domination properties.

Recall that the positive semigroup S_t is defined to dominate the positive semigroup T_t if $S_t\varphi \geq T_t\varphi$ for all positive $\varphi \in L_2(\Omega)$ and all $t > 0$. Moreover, $D(k_\Omega)$ is defined to be an order ideal of $D(l_\Omega)$ if the conditions $0 \leq \varphi \leq \psi$, $\psi \in D(k_\Omega)$ and $\varphi \in D(l_\Omega)$ imply $\varphi \in D(k_\Omega)$. (See [24], Chapter 2, for these and related concepts.)

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^d$ and $H_\Omega = -\sum_{i,j=1}^d \partial_i\, c_{ij}\, \partial_j$ be a second-order partial differential operator on $L_2(\Omega)$ with domain $C_c^\infty(\Omega)$, where the $c_{ij} \in W^{1,\infty}(\Omega)$ are real symmetric and $C(x) = (c_{ij}(x)) > 0$ for all $x \in \Omega$.*

Then the following are true.

I. *l_Ω is a Dirichlet form and $D(l_\Omega) \cap C^\infty(\Omega)$ is a core of l_Ω.*

II. *The submarkovian operator L_Ω associated with l_Ω is an extension of H_Ω.*

III. *If K_Ω is any submarkovian extension of H_Ω and k_Ω the corresponding Dirichlet form then $l_\Omega \supseteq k_\Omega \supseteq h_\Omega$. Therefore $0 \leq L_\Omega \leq K_\Omega \leq H_\Omega^F$ in the sense of operator order.*

IV. *If K_Ω is any self-adjoint extension of H_Ω then $C_c^\infty(\Omega)D(K_\Omega) \subseteq D(\overline{H}_\Omega)$ and if K_Ω is a submarkovian extension then $C_c^\infty(\Omega)D(k_\Omega) \subseteq D(\overline{h}_\Omega)$.*

V. *If K_Ω is a submarkovian extension of H_Ω then $D(\overline{h}_\Omega)$ is an order ideal of $D(k_\Omega)$ and e^{-tK_Ω} dominates $e^{-tH_\Omega^F}$. Moreover, e^{-tL_Ω} dominates e^{-tK_Ω} if and only if $D(k_\Omega)$ is an order ideal of $D(l_\Omega)$.*

The first three statements are a generalization of [15, Lemma 3.3.3 and Theorem 3.3.1]. They establish that the operator L_Ω is the smallest submarkovian extension of H_Ω, but not necessarily the smallest self-adjoint extension. The fourth statement is an interior regularity property. It establishes, in particular, that every submarkovian extension of H_Ω is a Silverstein extension in the terminology of [30] (see [9, Definition 1.4]).

The third statement of the theorem implies that H_Ω is Markov unique if and only if $l_\Omega = \overline{h}_\Omega$, *i.e.* if and only if $D(l_\Omega) = D(\overline{h}_\Omega)$. It is this criterion that has been used extensively in the analysis of the Markov uniqueness problem (see [2] [15] and [9, Chapter 3]). But the fourth statement implies that all the Dirichlet form extensions coincide in the interior of Ω and consequently differ only on the boundary. Our first criterion for Markov uniqueness is in terms of the capacity of the boundary.

The (relative) capacity of a measurable subset $A \subset \overline{\Omega}$ is defined by

$$\begin{aligned}\mathrm{cap}_\Omega(A) = \inf\Big\{ \|\psi\|^2_{D(l_\Omega)} : \psi \in D(l_\Omega) \text{ and there exists an open set}\\ U \subset \mathbb{R}^d \text{ such that } U \supseteq A \text{ and } \psi = 1 \text{ a. e. on } U \cap \Omega \Big\}\,.\end{aligned} \tag{1.4}$$

Thus cap_Ω is directly related to the capacity occurring in the theory of Dirichlet forms [4, 15].

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, the following conditions are equivalent:*

I. *H_Ω is Markov unique,*
II. $\mathrm{cap}_\Omega(\partial\Omega) = 0$.

It should be emphasized that there is no comparable geometric or potential-theoretic characterization of essential self-adjointness, *i.e.* L_2-uniqueness. Folklore would suggest that H_Ω is L_2-unique if and only if the Riemannian distance to the boundary $\partial\Omega$, measured with respect to the metric C^{-1}, is infinite. But this is not true in dimension one (see Example 6.5).

Our second result on Markov uniqueness is based on a conservation property. The submarkovian semigroup S_t^F generated by the Friedrichs extension H_Ω^F is defined to be conservative on $L_\infty(\Omega)$ if $S_t^F \mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t \geq 0$.

Theorem 1.3. *Adopt the assumptions of Theorem* 1.1*. Let S_t^F denote the semigroup generated by the Friedrichs extension H_Ω^F of H_Ω.*

The following conditions are equivalent:

I. *H_Ω is Markov unique,*
II. *S_t^F is conservative,*
III. *H_Ω is L_1-unique.*

The implications II $\Leftrightarrow$ III $\Rightarrow$ I are already known under slightly different hypotheses. The equivalence of Conditions II and III was established by Davies [7, Theorem 2.2], for a different class of second-order operators with smooth coefficients. His proof is based on an earlier result of Azencott [3]. The implication III $\Rightarrow$ I is quite general and is given by [9, Lemma 1.6]. Moreover, the implication I $\Rightarrow$ II follows from [9, Corollary 3.4], if $|\Omega| < \infty$. The proof of this implication for general Ω is considerably more complicated (see Section 5). In the broader setting of second-order operators acting on weighted spaces considered in [9] this implication is not always valid. The weights can introduce singular boundary behaviour (see [9, Remark following Corollary 3.4]).

Combination of the foregoing theorems gives the conclusion that Markov uniqueness, L_1-uniqueness and the conservation property are all characterized by the capacity condition $\mathrm{cap}_\Omega(\partial\Omega) = 0$. This is of interest since the latter condition can be estimated in terms of the boundary behaviour of the coefficients c_{ij} and the geometric properties of $\partial\Omega$. In Section 4 we derive estimates in terms of the order of degeneracy of the coefficients and the Minkowski dimension of the boundary (see Proposition 4.2).

The proofs of Theorems 1.1, 1.2 and 1.3 will be given in Sections 3, 4 and 5, respectively. In Section 6 we demonstrate that versions of the capacity estimates also give sufficient conditions for L_p-uniqueness for all $p \in [1, 2]$ and we establish that the semigroup S_t^F is irreducible if and only if Ω is connected.

ACKNOWLEDGEMENTS. Our interest in the characterization of operator extensions by their boundary behaviour originated in a series of interesting discussions with Ricardo Weder and Gian Michele Graf whilst the first author was visiting the Institut für Theoretische Physik at the Eidgenössische Technische Hochschule, Zürich in 2007. The author is indebted to Jürg Fröhlich and Gian Michele Graf for facilitating this visit and to the ETH for providing financial support. Both authors are grateful to Michael Röckner and Masayoshi Takeda for comments on an earlier draft of the paper.

2. Elliptic regularity

In this section we derive some basic regularity properties of the operators H_Ω defined by (1.1). Since H_Ω is symmetric its adjoint H_Ω^* is an extension of its closure $\overline{H}_\Omega$ and the domain $D(K_\Omega)$ of each self-adjoint extension K_Ω of H_Ω satisfies $D(\overline{H}_\Omega) \subseteq D(K_\Omega) \subseteq D(H_\Omega^*)$. The principal observation is that $D(\overline{H}_\Omega)$ and $D(H_\Omega^*)$ only differ on the boundary $\partial\Omega$. Hence the various possible extensions are distinguished by their boundary behaviour.

The comparison of $D(\overline{H}_\Omega)$ and $D(H_\Omega^*)$ can be articulated in various ways but it is convenient for the sequel to express it as a multiplier property.

Theorem 2.1. *Adopt the assumptions of Theorem* 1.1. *Then* $C_c^\infty(\Omega)D(H_\Omega^*) \subseteq D(\overline{H}_\Omega)$.

Proof. The principal step in the proof consists of establishing that $D(H_\Omega^*) \subseteq W^{1,2}_{\mathrm{loc}}(\Omega)$. Once this is achieved the rest of the proof is given by the following argument.

Let Ω' be a bounded open subset of Ω which is strictly contained in Ω, *i.e.* $\overline{\Omega'} \subset \Omega$. (Strict containment will be denoted by $\Omega' \Subset \Omega$.) If $\psi \in D(H_\Omega^*)$ and $D(H_\Omega^*) \subseteq W^{1,2}_{\mathrm{loc}}(\Omega)$ then $\psi \in W^{1,2}(\Omega')$. Set $\xi = H_\Omega^*\psi$ then $\xi \in L_2(\Omega')$ and

$$\sum_{i,j=1}^{d} (c_{ij}\partial_j\eta, \partial_i\psi) = (\eta, \xi)$$

for all $\eta \in C_c^\infty(\Omega')$, *i.e.* ψ is a weak solution of the elliptic equation $H_{\Omega'}\psi = \xi$ on Ω'. Since $H_{\Omega'}$ is strongly elliptic on $L_2(\Omega')$ it follows by elliptic regularity (see, for example, [20] Theorem 8.8) that $\psi \in W^{2,2}(\Omega'')$ for all $\Omega'' \Subset \Omega'$. Thus $\psi \in W^{2,2}_{\mathrm{loc}}(\Omega') \subseteq W^{2,2}_{\mathrm{loc}}(\Omega)$. Therefore $D(H_\Omega^*) \subseteq W^{2,2}_{\mathrm{loc}}(\Omega)$. Hence if $\eta \in C_c^\infty(\Omega)$ then $\eta\,\psi \in W^{2,2}_0(\Omega)$. But $W^{2,2}_0(\Omega) \subseteq D(\overline{H}_\Omega)$, because the coefficients are bounded, and the statement of the theorem is established.

It remains to prove that $D(H_\Omega^*) \subseteq W^{1,2}_{\mathrm{loc}}(\Omega)$.

First, fix $\eta \in C_c^\infty(\Omega)$ and set $K = \operatorname{supp}\eta$. Next let $\Omega' \Subset \Omega$ be a bounded open subset which contains K. Since $c_{ij} \in W^{1,\infty}(\Omega)$ and $C(x) > 0$ for all $x \in \Omega$ the restriction $H_{\Omega'}$ of H_Ω to $C_c^\infty(\Omega')$ is strongly elliptic.

Secondly, $\varphi \in D(H_\Omega^*)$ if and only if there is an $a > 0$ such that

$$|(\varphi, H_\Omega \psi)| \leq a \, \|\psi\|_2$$

for all $\psi \in C_c^\infty(\Omega)$. In particular if $\varphi \in D(H_\Omega^*)$ then these bounds are valid for all $\psi \in C_c^\infty(\Omega')$. Thus the restriction $\mathbf{1}_{\Omega'}\varphi$ of φ to Ω' is in $D(H_{\Omega'}^*)$ and $H_{\Omega'}^*(\mathbf{1}_{\Omega'}\varphi) = \mathbf{1}_{\Omega'}(H_\Omega^*\varphi)$. But $\eta\,\varphi = \mathbf{1}_{\Omega'}(\eta\,\varphi) = \eta\,(\mathbf{1}_{\Omega'}\varphi)$. In particular if $\eta\,(\mathbf{1}_{\Omega'}\varphi) \in D(\overline{H}_{\Omega'})$ then $\eta\,\varphi \in D(\overline{H}_\Omega)$. Thus $\eta D(H_\Omega^*) \subseteq D(\overline{H}_\Omega)$ if and only if $\eta D(H_{\Omega'}^*) \subseteq D(\overline{H}_{\Omega'})$ for all possible choices of η and Ω'. Therefore it suffices to prove $D(H_{\Omega'}^*) \subseteq W_{\mathrm{loc}}^{1,2}(\Omega')$ for the strongly elliptic operator $H_{\Omega'}$ on $L_2(\Omega')$ for all bounded open subsets $\Omega' \Subset \Omega$.

Thirdly, we extend $H_{\Omega'}$ to a strongly elliptic operator L on $L_2(\mathbb{R}^d)$ with coefficients $\hat{c}_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ such that $H_{\Omega'} = L|_{C_c^\infty(\Omega')}$. This is achieved in two steps. Since the c_{ij} are continuous on Ω, $C(x) \geq \mu I$ for all $x \in \Omega'$ and $C(x) > 0$ for all $x \in \Omega$ one may choose an Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$ and $C(x) \geq (\mu/2)I$ for all $x \in \Omega''$. Then one may choose a $\chi \in C^\infty(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $x \in \Omega'$ and $\chi(x) = 0$ if x is in the complement of Ω''. Then set $\widehat{C} = \chi\, C + (1 - \chi)\,(\mu/2)I$. It follows that $\widehat{C} \geq (\mu/2)I$. Now let L be the divergence form operator on $L_2(\mathbb{R}^d)$ with the matrix of coefficients $\widehat{C} = (\hat{c}_{ij})$. It is strongly elliptic, $\hat{c}_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ and $L|_{C_c^\infty(\Omega')} = H_{\Omega'}$ by construction. Therefore the proof is completed by the following lemma. □

Lemma 2.2. *Let $H_{\Omega'} = L|_{C_c^\infty(\Omega')}$ where L is a strongly elliptic operator, with coefficients $\hat{c}_{ij} \in W^{1,\infty}(\mathbb{R}^d)$, acting on $L_2(\mathbb{R}^d)$. Then $D(H_{\Omega'}^*) \subseteq W_{\mathrm{loc}}^{1,2}(\Omega')$.*

Proof. The proof exploits some basic properties of strongly elliptic operators with Lipschitz continuous coefficients summarized in Proposition A.1 of the appendix. In particular L is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ and its self-adjoint closure $\overline{L}$ has domain $D(\overline{L}) = W^{2,2}(\mathbb{R}^d)$.

Let $\mathcal{D}(\Omega')$ denote $C_c^\infty(\Omega')$ equipped with the Frechet topology and $\mathcal{D}'(\Omega')$ the dual space, *i.e.* the space of distributions on Ω'. If $\psi \in L_2(\Omega')$ then $\varphi \in C_c^\infty(\Omega') \mapsto (\psi, H_{\Omega'}\varphi)$ is a continuous linear function over $\mathcal{D}(\Omega')$. Thus for each $\psi \in L_2(\Omega')$ there is a distribution $H_{\Omega'}(\psi) \in \mathcal{D}'(\Omega')$ such that

$$(H_{\Omega'}(\psi), \varphi) = (\psi, H_{\Omega'}\varphi)$$

for all $\varphi \in C_c^\infty(\Omega')$. Similarly for each $\psi \in L_2(\mathbb{R}^d)$ there is a distribution $L(\psi) \in W^{-2,2}(\mathbb{R}^d)$, the dual of $W^{2,2}(\mathbb{R}^d)$, such that

$$(L(\psi), \varphi) = (\psi, L\varphi)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. But by assumption

$$(\psi, H_{\Omega'}\varphi) = (\psi, L\varphi) = (L(\psi), \varphi)$$

for all $\psi \in L_2(\Omega')$ and all $\varphi \in C_c^\infty(\Omega')$. Therefore

$$(H_{\Omega'}(\psi), \varphi) = (L(\psi), \varphi)$$

for all $\varphi \in C_c^\infty(\Omega')$. In particular $\psi \in D(H_{\Omega'}^*)$ if and only if $L(\psi) \in L_2(\Omega')$.

Next fix $\psi \in D(H_{\Omega'}^*)$. Then $L(\psi) \in L_2(\Omega')$. Moreover, if $\eta \in C_c^\infty(\Omega')$ then $\eta\,\psi \in D(H_{\Omega'}^*)$ if and only if $L(\eta\,\psi) \in L_2(\Omega')$. But one has the distributional relation

$$L(\eta\,\psi) = \eta L(\psi) + L(\eta)\psi + \Psi_\eta \tag{2.1}$$

where

$$\Psi_\eta = -2 \sum_{i,j=1}^{d} \hat{c}_{ij}\,(\partial_j \eta)\,(\partial_i \psi)\ .$$

Since $\psi, L(\psi) \in L_2(\Omega')$ and $\eta, L(\eta) \in L_\infty(\Omega')$ it follows that the first two terms on the right of (2.1) are in $L_2(\Omega')$. But there is an $a > 0$ such that

$$|(\Psi_\eta, \varphi)| \le a\,\|\psi\|_2\,\|\varphi\|_{1,2}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ where $\|\cdot\|_{1,2}$ denotes the $W^{1,2}$-norm. Therefore $\Psi_\eta \in W^{-1,2}(\mathbb{R}^d)$, the dual of $W^{1,2}(\mathbb{R}^d)$. Hence it follows from (2.1) that $L(\eta\,\psi) \in W^{-1,2}(\mathbb{R}^d)$. But

$$\eta\,\psi = (I+L)^{-1}\eta\,\psi + L(I+L)^{-1}\eta\,\psi = (I+L)^{-1}\eta\,\psi + (I+L)^{-1}L(\eta\,\psi)$$

and $\eta\,\psi \in W^{1,2}(\mathbb{R}^d)$ by Proposition A.1.III of the appendix applied to the strongly elliptic operator L. Since $\eta \in C_c^\infty(\Omega')$ it follows that $\eta\,\psi \in W_0^{1,2}(\Omega')$.

Finally let K be an arbitrary compact subset of Ω'. If $\eta_1 \in C_c^\infty(\Omega')$ with $\eta_1 = 1$ on K it follows that $\partial_i(\eta_1\psi)|_K = \partial_i\psi|_K$. Thus $\partial_i\psi \in L_2(K)$ for all $i \in \{1, \ldots, d\}$. Therefore $\psi \in W_{\mathrm{loc}}^{1,2}(\Omega)$. □

One can also draw a conclusion about the domain of a general self-adjoint extension of H_Ω and partially establish Statement IV of Theorem 1.1.

Corollary 2.3. *If K_Ω is any self-adjoint extension of H_Ω then $C_c^\infty(\Omega)D(K_\Omega) \subseteq D(\overline{H}_\Omega)$.*

Proof. Since $D(\overline{H}_\Omega) \subseteq D(K_\Omega) \subseteq D(H_\Omega^*)$ one has $C_c^\infty(\Omega)D(K_\Omega) \subseteq C_c^\infty(\Omega)D(H_\Omega^*) \subseteq D(\overline{H}_\Omega)$ by Theorem 2.1. □

Note that $D(\overline{H}_\Omega) \subseteq D(H_\Omega^*) \subseteq W_{\mathrm{loc}}^{2,2}(\Omega)$. Therefore $C_c^\infty(\Omega)D(K_\Omega) \subseteq D(\overline{H}_\Omega) \subseteq D(K_\Omega) \subseteq W_{\mathrm{loc}}^{2,2}(\Omega)$ for each self-adjoint extension K_Ω.

3. The minimal Markov extension

In this section we prove Theorem 1.1. The proof of the first parts of the theorem broadly follows the reasoning used in [15] to prove the analogous result, Theorem 3.3.1, for operators with C^∞-coefficients. The essential new ingredient is the elliptic regularity properties of Theorem 2.1 and its corollaries.

Proof of Theorem 1.1. I. The Markov property of l_Ω follows by the calculations of [15, Example 1.2.1]. Moreover, the form is closed with respect to the graph norm by the arguments in [23, Section II.2.b]. The latter arguments depend crucially on the local strong ellipticity property. Therefore the form l_Ω is a Dirichlet form. Finally $D(l_\Omega) \cap C^\infty(\Omega)$ is a core of l_Ω by the proof of [15, Lemma 3.3.3].

II. The proof that L_Ω is an extension of H_Ω is identical to the proof of [15, Lemma 3.3.4] modulo a regularity argument.

Let $\varphi \in L_2(\Omega)$. Then $\psi = (I + L_\Omega)^{-1}\varphi \in D(L_\Omega) \subseteq D(l_\Omega)$. Moreover,

$$(\varphi, \eta) = (\psi, \eta) + l_\Omega(\psi, \eta) = (\psi, \eta) + \int \sum_{i,j=1}^{d} c_{ij}\,(\partial_i\psi)(\partial_j\eta)$$

for all $\eta \in C_c^\infty(\Omega)$. Now fix an $\eta_1 \in C_c^\infty(\Omega)$ such that $\eta_1 = 1$ on the support of η. Then $\psi_1 = \eta_1\psi \in D(l_\Omega)$ by a straightforward estimate and $\psi_1 \in W_0^{1,2}(\Omega)$ by local strong ellipticity. Therefore

$$(\varphi, \eta) = (\psi_1, \eta) + \int \sum_{i,j=1}^{d} c_{ij}\,(\partial_i\psi_1)(\partial_j\eta) = (\psi_1, (I + H_\Omega)\eta) = (\psi, (I + H_\Omega)\eta)$$

by partial integration. Hence $\psi \in D(I + H_\Omega^*)$ and $(I + H_\Omega^*)\psi = \varphi$. Thus $D(L_\Omega) \subseteq D(H_\Omega^*)$ and H_Ω^* is an extension of L_Ω. So $D(H_\Omega) \subseteq D(L_\Omega)$ and L_Ω is an extension of H_Ω.

III. This is the lengthiest part of the proof. We divide it into two steps.

Step 1. First, we prove that $D(k_\Omega) \cap D(H_\Omega^*) \subseteq D(l_\Omega)$ (see proof of [15, Lemma 3.3.5]). Clearly it suffices to prove that

$$k_\Omega(\varphi) \geq \int_\Omega dx \sum_{i,j=1}^{d} c_{ij}(x)\,(\partial_i\varphi)(x)(\partial_j\varphi)(x) \tag{3.1}$$

for all $\varphi \in D(k_\Omega) \cap D(H_\Omega^*)$.

Set $R_\lambda = (\lambda I + K_\Omega)^{-1}$ for all $\lambda > 0$ and introduce the bounded forms

$$k_\Omega^{(\lambda)}(\varphi) = \lambda\,(\varphi, (I - \lambda R_\lambda)\varphi) \tag{3.2}$$

for all $\varphi \in L_2(\Omega)$. The $k_\Omega^{(\lambda)}$ are Dirichlet forms and

$$k_\Omega(\varphi) = \sup_{\lambda>0} k_\Omega^{(\lambda)}(\varphi) = \limsup_{\lambda\to\infty} k_\Omega^{(\lambda)}(\varphi) \tag{3.3}$$

with $D(k_\Omega)$ the subspace of $L_2(\Omega)$ for which the supremum is finite (see, for example [15, Lemma 1.3.4(ii)]).

Next for $\eta \in C_c^\infty(\Omega)$ with $0 \le \eta \le 1$ define the truncated form

$$k_{\Omega,\eta}^{(\lambda)}(\varphi) = k_\Omega^{(\lambda)}(\varphi, \eta\varphi) - 2^{-1} k_\Omega^{(\lambda)}(\eta, \varphi^2)$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$. It then follows from the Dirichlet form structure that

$$k_\Omega^{(\lambda)}(\varphi) \ge k_{\Omega,\eta}^{(\lambda)}(\varphi) \tag{3.4}$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ (see [4, Proposition I.4.1.1]). Moreover,

$$k_{\Omega,\eta}^{(\lambda)}(\varphi) = \lambda\,(\varphi, (I - \lambda R_\lambda)\eta\varphi) - 2^{-1}\lambda\,(\varphi(I - \lambda R_\lambda)\eta, \varphi) \tag{3.5}$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ since $(I - \lambda R_\lambda)\eta \in L_\infty(\Omega)$ by the submarkovian property of K_Ω. Then, however, (3.5) extends to all $\varphi \in L_2(\Omega)$ by continuity. Combination of (3.2), (3.3), (3.4) and (3.5) immediately gives

$$k_\Omega(\varphi) \ge \lambda\,(\varphi, (I - \lambda R_\lambda)\eta\varphi) - 2^{-1}\lambda\,(\varphi(I - \lambda R_\lambda)\eta, \varphi) \tag{3.6}$$

for all $\varphi \in D(k_\Omega)$. Now we consider the limit $\lambda \to \infty$.

If $\varphi \in D(H_\Omega^*)$ then $\eta\varphi \in D(\overline{H}_\Omega) \subseteq D(K_\Omega)$ by Theorem 2.1. Therefore

$$\lim_{\lambda\to\infty} \lambda\,(\varphi, (I - \lambda R_\lambda)\eta\varphi) = (\varphi, K_\Omega\eta\varphi) = (\varphi, \overline{H}_\Omega\eta\varphi)\;.$$

Now let S denote the submarkovian semigroup generated by K_Ω on $L_2(\Omega)$ and $S^{(\infty)}$ the corresponding weak* semigroup on $L_\infty(\Omega)$. Further let $K_\Omega^{(\infty)}$ denote the generator of $S^{(\infty)}$ and $R_\lambda^{(\infty)} = (\lambda I + K_\Omega^{(\infty)})^{-1}$ the resolvent. Then $\eta \in D(H_\Omega) \cap L_\infty(\Omega) \subseteq D(K_\Omega) \cap L_\infty(\Omega)$ and $K_\Omega\eta = H_\Omega\eta \in L_\infty(\Omega)$. Therefore $\eta \in D(K_\Omega^{(\infty)})$ and $K_\Omega^{(\infty)}\eta = H_\Omega\eta$. Consequently

$$H_\Omega\eta = K_\Omega^{(\infty)}\eta = \text{weak}^* \lim\nolimits_{\lambda\to\infty}\lambda(I - \lambda R_\lambda^{(\infty)})\eta$$

and one concludes that

$$\lim_{\lambda\to\infty} \lambda\,(\varphi(I - \lambda R_\lambda)\eta, \varphi) = \lim_{\lambda\to\infty} \lambda((I - \lambda R_\lambda^{(\infty)})\eta, \varphi^2) = (K_\Omega^{(\infty)}\eta, \varphi^2) = (\varphi H_\Omega\eta, \varphi)\;.$$

Then it follows from taking the limit $\lambda \to \infty$ in (3.6) that

$$k_\Omega(\varphi) \ge (\varphi, \overline{H}_\Omega\eta\varphi) - 2^{-1}(\varphi H_\Omega\eta, \varphi)$$

for all $\varphi \in D(k_\Omega) \cap D(H_\Omega^*)$. Since $\eta \in C_c^\infty(\Omega)$ and $D(H_\Omega^*) \subseteq W_{\rm loc}^{2,2}(\Omega)$, by the proof of Theorem 2.1, it follows by direct calculation that

$$k_\Omega(\varphi) \geq \int_\Omega dx \sum_{i,j=1}^d \eta(x)\, c_{ij}(x)\, (\partial_i\varphi)(x)(\partial_j\varphi)(x)$$

for all $\varphi \in D(k_\Omega) \cap D(H_\Omega^*)$ and $\eta \in C_c^\infty(\Omega)$ with $0 \leq \eta \leq 1$. But $k_\Omega(\varphi)$ is independent of η. Therefore taking the limit over a sequence of η which converges monotonically upward to $\mathbf{1}_\Omega$ one deduces that (3.1) is valid by the Lebesgue dominated convergence theorem.

Step 2. Next we argue that the inclusion $D(k_\Omega) \cap D(H_\Omega^*) \subseteq D(l_\Omega)$ established by Step 1 implies $D(k_\Omega) \subseteq D(l_\Omega)$.

By definition $D(\overline{h}_\Omega)$ is a subspace of $D(k_\Omega)$. But the orthogonal complement of $D(\overline{h}_\Omega)$ with respect to the graph norm $\|\cdot\|_{D(k_\Omega)}$ is $D(k_\Omega) \cap \mathcal{N}$ where

$$\mathcal{N} = \{\varphi \in D(H_\Omega^*) : (I + H_\Omega^*)\varphi = 0\}$$

(see [15, Lemma 3.3.2(ii)]). Therefore each $\varphi \in D(k_\Omega)$ has a unique decomposition $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in D(\overline{h}_\Omega)$ and $\varphi_2 \in D(k_\Omega) \cap \mathcal{N}$ such that

$$\|\varphi\|^2_{D(k_\Omega)} = \|\varphi_1\|^2_{D(\overline{h}_\Omega)} + \|\varphi_2\|^2_{D(k_\Omega)}\ .$$

But $\varphi_2 \in D(k_\Omega) \cap D(H_\Omega^*)$. So $\varphi_2 \in D(l_\Omega)$ and $k_\Omega(\varphi_2) \geq l_\Omega(\varphi_2)$ by Step 1. Therefore

$$\|\varphi\|^2_{D(k_\Omega)} \geq \|\varphi_1\|^2_{D(\overline{h}_\Omega)} + \|\varphi_2\|^2_{D(l_\Omega)} = \|\varphi_1\|^2_{D(l_\Omega)} + \|\varphi_2\|^2_{D(l_\Omega)} = \|\varphi\|^2_{D(l_\Omega)}\ .$$

The last equality follows because $l_\Omega(\varphi_1, \varphi_2) + (\varphi_1, \varphi_2) = 0$. It follows immediately that $D(k_\Omega) \subseteq D(l_\Omega)$. This completes the proof of Statement III of Theorem 1.1.

IV. The inclusion $C_c^\infty(\Omega)D(K_\Omega) \subseteq D(\overline{H}_\Omega)$, was established in Corollary 2.3. But if $\eta \in C_c^\infty(\Omega)$ and $\varphi \in D(K_\Omega)$ then $\eta\varphi \in D(\overline{H}_\Omega) \subseteq D(K_\Omega) \subseteq D(k_\Omega) \subseteq D(l_\Omega)$ and

$$\begin{aligned}\overline{h}_\Omega(\eta\varphi) = l_\Omega(\eta\varphi) &\leq 2\,\|\eta\|_\infty^2\, l_\Omega(\varphi) + 2\,\|\Gamma(\eta)\|_\infty\,\|\varphi\|_2^2 \\ &\leq 2\,(\|\Gamma(\eta)\|_\infty + \|\eta\|_\infty^2)\,\|\varphi\|^2_{D(k_\Omega)}\end{aligned}$$

where $\Gamma(\eta) = \sum_{i,j=1}^d c_{ij}\,(\partial_i\eta)\,(\partial_j\eta)$, *i.e.* Γ is the *carré du champ* as defined in [4], Section I.8. Since $D(K_\Omega)$ is a core of k_Ω with respect to the $D(k_\Omega)$-graph norm it follows that $C_c^\infty(\Omega)D(k_\Omega) \subseteq D(\overline{h}_\Omega)$ by continuity.

V. First let $D(k_\Omega)_c$ denote the subspace of functions with compact support in $D(k_\Omega)$. If $\varphi \in D(k_\Omega)_c$ then by regularization one can construct a sequence $\varphi_n \in C_c^\infty(\Omega)$ such that $\|\varphi_n - \varphi\|_{D(k_\Omega)} \to 0$ as $n \to \infty$. Since k_Ω is an extension of

h_Ω it follows that $\varphi \in D(\overline{h}_\Omega)$. Therefore $D(k_\Omega)_c \subseteq D(\overline{h}_\Omega)$. Now the first part of Statement V follows from [10, Proposition 2.1]. But then $D(\overline{h}_\Omega)$ is an ideal (see [24, Definition 2.19]) of $D(k_\Omega)$ by [24, Corollary 2.22]. In particular it is an order ideal.

For the proof of the second part of Statement V we again appeal to [24, Corollary 2.22]. First if e^{-tL_Ω} dominates e^{-tK_Ω} then it follows from this corollary that $D(k_\Omega)$ is an ideal of $D(l_\Omega)$. Secondly, for the converse statement, it suffices to prove that $D(k_\Omega)$ is an ideal of $D(l_\Omega)$. Then the domination property follows from another application of [24, Corollary 2.22]. Thus if $\psi \in D(k_\Omega)$, $\varphi \in D(l_\Omega)$ and $|\varphi| \leq |\psi|$ then one must deduce that $\varphi \operatorname{sgn} \psi \in D(k_\Omega)$. But $\varphi, \psi \in D(l_\Omega)$. Therefore $\varphi \operatorname{sgn} \psi \in D(l_\Omega)$ by [24, Proposition 2.20]. (See the remark following this proposition.) Moreover, $|\psi| \in D(k_\Omega)$ and $(\varphi \operatorname{sgn} \psi)_+ \in D(l_\Omega)$ because k_Ω and l_Ω are Dirichlet forms. Since

$$0 \leq (\varphi \operatorname{sgn} \psi)_+ \leq |\varphi| \leq |\psi|$$

and since $D(k_\Omega)$ is an order ideal of $D(l_\Omega)$ it follows that $(\varphi \operatorname{sgn} \psi)_+ \in D(k_\Omega)$. Applying the same argument to $-\varphi$ one deduces that $(\varphi \operatorname{sgn} \psi)_- \in D(k_\Omega)$. Therefore $\varphi \operatorname{sgn} \psi \in D(k_\Omega)$ and $D(k_\Omega)$ is an ideal of $D(l_\Omega)$. □

We note in passing that the existence of l_Ω gives a criterion for uniqueness of the submarkovian extension of H_Ω similar to the standard criterion for essential self-adjointness.

Proposition 3.1. *The following conditions are equivalent:*

I. *H_Ω has a unique submarkovian extension.*
II. $\ker(I + H_\Omega^*) \cap D(l_\Omega) = \{0\}$.

Proof. Condition I is equivalent to $l_\Omega = \overline{h}_\Omega$ by Theorem 1.1, *i.e.* equivalent to $D(l_\Omega) = D(\overline{h}_\Omega)$. But $D(l_\Omega) = D(\overline{h}_\Omega) \oplus \mathcal{H}_\Omega$ with $\mathcal{H}_\Omega = \ker(I + H_\Omega^*) \cap D(l_\Omega)$ by Lemma 3.3.2(ii) in [15]. Therefore the equivalence of the conditions of the proposition is immediate. □

Statement III of Theorem 1.1 establishes that H_Ω is Markov unique if and only if $l_\Omega = \overline{h}_\Omega$ or, equivalently, $D(l_\Omega) = D(\overline{h}_\Omega)$. This is the criterion used extensively in the analysis of Markov uniqueness (see [9], Chapter 3). It will also be used to prove Theorem 1.2.

Statement IV of the theorem establishes that each submarkovian extension of H_Ω is a Silverstein extension (see [30] or [9, Definition 1.4]). Therefore Markov uniqueness of H_Ω and Silverstein uniqueness are equivalent.

Statement V gives an alternative approach to establishing Markov uniqueness of H_Ω if the submarkovian semigroup generated by the Friedrichs extension is conservative. This will be discussed in Section 5.

4. Markov Uniqueness

In this section we prove Theorem 1.2. Throughout the section we assume that the coefficients c_{ij} are real, symmetric, Lipschitz continuous and $C(x) = (c_{ij}(x)) > 0$ for all $x \in \Omega$.

Proof of Theorem 1.2. It follows from Theorem 1.1 that H_Ω has a unique submarkovian extension if and only if $l_\Omega = \overline{h}_\Omega$, *i.e.* if and only if $C_c^\infty(\Omega)$ is a core of l_Ω. Therefore Theorem 1.2 is a direct corollary of the following proposition which is a variation of [26, Proposition 3.2].

Proposition 4.1. *Under the assumptions of Theorem* 1.2*, the following conditions are equivalent:*

I. $\mathrm{cap}_\Omega(\partial\Omega) = 0$.
II. $C_c^\infty(\Omega)$ *is a core of* l_Ω.

Proof. I $\Rightarrow$ II First, since l_Ω is a Dirichlet form $D(l_\Omega) \cap L_\infty(\Omega)$ is a core of l_Ω. Therefore it suffices that each $\varphi \in D(l_\Omega) \cap L_\infty(\Omega)$ can be approximated by a sequence $\varphi_n \in C_c^\infty(\Omega)$ with respect to the graph norm $\|\cdot\|_{D(l_\Omega)}$. Now fix $\varphi \in D(l_\Omega) \cap L_\infty(\Omega)$.

Secondly, let $\rho_n \in C_c^\infty(\mathbb{R}^d)$ be a sequence of functions with $0 \le \rho_n \le 1$, $\|\nabla\rho_n\|_\infty \le n^{-1}$ and such that $\rho_n \to \mathbf{1}$ pointwise as $n \to \infty$. Then $\mathbf{1} - \rho_n \in W^{1,\infty}(\mathbb{R}^d)$. But $W^{1,\infty}(\mathbb{R}^d)D(l_\Omega) \subseteq D(l_\Omega)$. Therefore $(\mathbf{1} - \rho_n)\varphi \in D(l_\Omega) \cap L_\infty(\Omega)$. It then follows from Leibniz' rule and the Cauchy–Schwarz inequality that

$$\begin{aligned}\|\varphi - \rho_n\varphi\|_{D(l_\Omega)}^2 &\le 2\int_\Omega \varphi^2\,\Gamma(\rho_n) + 2\int_\Omega (\mathbf{1}-\rho_n)^2\,\Gamma(\varphi) + \|(\mathbf{1}-\rho_n)\varphi\|_2^2 \\ &\le 2\,n^{-2}\|C\|\,\|\varphi\|_2^2 + \int_\Omega (\mathbf{1}-\rho_n)^2\,(2\,\Gamma(\varphi) + \varphi^2)\end{aligned}$$

where $\|C\|$ is the supremum over the matrix norms $\|C(x)\|$. Clearly the first term on the right hand side tends to zero as $n \to \infty$. Moreover, $0 \le (\mathbf{1} - \rho_n)^2 \le 1$, $(\mathbf{1} - \rho_n)^2 \to 0$ pointwise as $n \to \infty$ and $2\,\Gamma(\varphi) + \varphi^2 \in L_1(\Omega)$. Therefore the second term on the right hand side also tends to zero as $n \to \infty$ by the Lebesgue dominated convergence theorem. Thus φ is approximated by the sequence $\rho_n\varphi$ in the graph norm.

Thirdly, since $\mathrm{cap}_\Omega(\partial\Omega) = 0$ one may choose $\chi_n \in D(l_\Omega) \cap L_\infty(\Omega)$ and open subsets $U_n \supset \partial\Omega$ such that $0 \le \chi_n \le 1$, $l_\Omega(\chi_n) + \|\chi_n\|_2^2 \le n^{-1}$ and $\chi_n = 1$ on $U_n \cap \Omega$. Now set $\varphi_n = (1 - \chi_n)\rho_n\varphi$. Then

$$\|\varphi - \varphi_n\|_{D(l_\Omega)}^2 \le 2\,\|\varphi - \rho_n\varphi\|_{D(l_\Omega)}^2 + 2\,\|\chi_n\rho_n\varphi\|_{D(l_\Omega)}^2$$

and the first term on the right hand side converges to zero as $n \to \infty$ by the previous discussion. Moreover,

$$\|\chi_n\rho_n\varphi\|_{D(l_\Omega)}^2 = l_\Omega(\chi_n\rho_n\varphi) + \|\chi_n\rho_n\varphi\|_2^2 \tag{4.1}$$

and the second term on the right tends to zero because $\|\chi_n\rho_n\varphi\|_2 \le \|\chi_n\|_2\|\varphi\|_\infty$. But the first term can be estimated by

$$\begin{aligned}
l_\Omega(\chi_n\rho_n\varphi) &\le 2\int_\Omega \rho_n^2\varphi^2\,\Gamma(\chi_n) + 2\int_\Omega \chi_n^2\,\Gamma(\rho_n\varphi)\\
&\le 2\int_\Omega \varphi^2\,\Gamma(\chi_n) + 4\int_\Omega \chi_n^2\,\varphi^2\,\Gamma(\rho_n) + 4\int_\Omega \chi_n^2\,\rho_n^2\,\Gamma(\varphi)\\
&\le 2\,\|\varphi\|_\infty^2\,l_\Omega(\chi_n) + 4\,\|C\|\,\|\nabla\rho_n\|_\infty^2\|\varphi\|_2^2 + 4\int_\Omega \chi_n^2\,\Gamma(\varphi)\ .
\end{aligned}$$

Since $l_\Omega(\chi_n) \to 0$ and $\|\nabla\rho_n\|_\infty \to 0$ as $n \to \infty$ the first two terms on the right hand side tend to zero. But if $A_m = \{x \in \Omega : \Gamma(\varphi) > m\}$ one has the equicontinuity estimate

$$\int_\Omega \chi_n^2\,\Gamma(\varphi) \le m\,\|\chi_n\|_2^2 + \int_{A_m} \Gamma(\varphi)$$

because $0 \le \chi_n \le 1$. Since $\|\chi_n\|_2 \to 0$ and $\Gamma(\varphi) \in L_1(\Omega)$ the integral also tends to zero as $n \to \infty$. Thus both terms on the right hand side of (4.1) tend to zero as $n \to \infty$ and one now concludes that φ is approximated by the sequence φ_n in the graph norm.

Finally $\operatorname{supp}\varphi_n$ is contained in the set $\Omega_n = ((\operatorname{supp}\rho_n) \cap \Omega) \cap (\Omega\backslash(U_n \cap \Omega))$. Hence Ω_n is a bounded subset which is strictly contained in Ω, *i.e.* $\Omega_n \Subset \Omega$. Then since $C(x) > 0$ for all $x \in \Omega$ one has an estimate $l_\Omega(\varphi_n) \ge \mu_n\,\|\nabla\varphi_n\|_2^2$ with $\mu_n > 0$. Therefore $\varphi_n \in W_0^{1,2}(\Omega_n)$ and it follows that it can be approximated in the $W^{1,2}(\Omega_n)$-norm by a sequence of C_c^∞-functions. Then because $l_\Omega(\psi) \le \|C\|\,\|\nabla\psi\|_2^2$ for all $\psi \in D(l_\Omega)$ it follows that φ_n, and hence φ, can be approximated by a sequence of C_c^∞-functions in the graph norm $\|\cdot\|_{D(l_\Omega)}$.

II $\Rightarrow$ I Let $\psi \in D(l_\Omega) \cap C^\infty(\Omega)$ with $\psi = 1$ on $U \cap \Omega$ where U is an open subset containing $\partial\Omega$. One may assume $0 \le \psi \le 1$. Then by Condition II there is a sequence $\psi_n \in C_c^\infty(\Omega)$ such that $\|\psi_n - \psi\|_{D(l_\Omega)} \to 0$. In particular $\psi \in W^{1,2}(\Omega)$. Since ψ_n has compact support in Ω it also follows that there is an open subset U_n containing $\partial\Omega$ such that $\psi_n = 0$ on $U_n \cap \Omega$. Therefore $\psi - \psi_n = 1$ on $(U \cap U_n) \cap \Omega$ and one must have $\operatorname{cap}_\Omega(\partial\Omega) = 0$. □

The condition $\operatorname{cap}_\Omega(\partial\Omega) = 0$ is of interest as a criterion for Markov uniqueness since the capacity can be estimated by elementary means. The estimates depend on two gross features, the order of degeneracy of the coefficients at the boundary and the dimension of the boundary. There are various ways of assigning a dimension to a measurable subset A (see, for example [14, Chapters 2 and 3]). In the next proposition we characterize the dimension in terms of the volume growth of the δ-neighbourhoods

$$B_\delta = \{x \in \mathbb{R}^d : \inf_{y\in B} |x - y| < \delta\}$$

of the bounded measurable subsets B of A.

Proposition 4.2. *Let A be a measurable subset of $\overline{\Omega}$ with $|A| = 0$ and d_A the Euclidean distance to A. Assume* 1. *there are $a > 0$ and $\gamma \geq 0$ such that $0 < C(x) \leq a\,(d_A(x) \wedge 1)^\gamma$ for all $x \in \Omega$,* 2. *there is a $d(A) \in [0, d]$ such that $\sup_{\delta\in\langle 0,1]} \delta^{-(d-d(A))}|B_\delta| < \infty$ for each bounded measurable $B \subset A$.*

It follows that $\operatorname{cap}_\Omega(A) = 0$ *if* $\gamma \geq 2-(d-d(A))$. *In particular* $\operatorname{cap}_\Omega(A) = 0$ *if* $\gamma \geq 2$.

Proof. Let B be a bounded measurable subset of A. Then $d_A \leq d_B$. Now introduce the functions $x > 0 \mapsto \chi_n(x) \in [0, 1]$ by $\chi_n(x) = 1$ if $x \in \langle 0, n^{-1}]$, $\chi_n(x) = -\log x/\log n$ if $x \in \langle n^{-1}, 1]$ and $\chi_n(x) = 0$ if $x > 1$. Set $\eta_n = \chi_n \circ d_B$. It follows that $\eta_n(x) = 0$ if $x \notin B_1$. Now $\|\eta_n\|_2 \to 0$ as $n \to \infty$ since $|B| = 0$. Moreover,

$$\begin{aligned} l_\Omega(\eta_n) &\leq a \int_{B_1} dx\, d_B(x)^\gamma\, |\nabla\eta_n(x)|^2 \\ &\leq a\,(\log n)^{-2} \int dx\, \mathbf{1}_{\{x: n^{-1}\leq d_B(x)\leq 1\}}\, d_B(x)^{-(2-\gamma)}. \end{aligned}$$

Thus if $\gamma \geq 2$ then $l_\Omega(\eta_n) \leq a\,(\log n)^{-2}|B_1| \to 0$ as $n \to \infty$. Then $\operatorname{cap}_\Omega(B) = 0$. If, however, $\gamma < 2$ then $d_B(x)^{-(2-\gamma)} = 1 + (2-\gamma)^{-1}\int_{d_B(x)}^1 d\delta\, \delta^{-(3-\gamma)}$ and one deduces that

$$l_\Omega(\eta_n) \leq a'(\log n)^{-2}\Big(1 + (2-\gamma)^{-1}\int_{n^{-1}}^1 d\delta\, \delta^{-3+\gamma+d-d(A)}(\delta^{-(d-d(A))}|B_\delta|)\Big)\,.$$

Thus if $\gamma \geq 2-(d-d(A))$ then $l_\Omega(\eta_n) \leq a''(\log n)^{-1} \to 0$ as $n \to \infty$. It follows that $\operatorname{cap}_\Omega(B) = 0$. Then $\operatorname{cap}_\Omega(A) = 0$ by the general additivity properties of the capacity. □

Remark 4.3. 1. The 'dimension' $d(A)$ introduced in Proposition 4.2 is not uniquely defined. But it is closely related to the upper Minkowski, or upper box, dimension (see [14, Chapter 3]). The upper Minkowski dimension $\overline{d_M}(B)$ of each bounded $B \subset A$ is given by

$$\begin{aligned} \overline{d_M}(B) &= -\limsup_{\delta\to 0} \log(\delta^{-d}|B_\delta|)/\log\delta \\ &= \inf\Big\{\hat{d} \in [0, d] : \sup_{\delta\in\langle 0,1]} \delta^{-(d-\hat{d})}|B_\delta| < \infty\Big\}\,. \end{aligned} \tag{4.2}$$

and then $\overline{d_M}(A) = \sup\{\overline{d_M}(B) : B \subseteq A\,, B \text{ bounded}\}$. It is clear from the second relation that $\overline{d_M}(B) \leq d(B)$ for all choices of $d(B)$. Moreover, one may choose $d(B) = \overline{d_M}(B)$ if and only if the infimum in (4.2) is attained. For example, if A is locally the graph of a Lipschitz function then one may choose $d(A) = \overline{d_M}(A) = d-1 = d_H(A)$ where $d_H(A)$ is the Hausdorff dimension.

2. If the growth condition $\sup_{\delta\in\langle 0,1]} \delta^{-(d-d(A))}|B_\delta| < \infty$ in Proposition 4.2 is replaced by the weaker condition $\sup_{n\geq 1}(\log n)^{-1}\int_{n^{-1}}^1 d\delta\,\delta^{-1}\,\delta^{-(d-d(A))}|B_\delta| < \infty$

then the conclusion of the proposition is unchanged. This follows by a minor modification of the foregoing proof. The integral bound is of interest as it is automatically satisfied for a large family of self-similar fractal sets (see [19, Theorem 2.3]) with $d(A)$ replaced by $d_H(A)$.

The estimates of Proposition 4.2 have two simple implications.

Corollary 4.4. *Assume $|\partial\Omega| = 0$ and $d(\partial\Omega) = d - 1$. If the coefficients $c_{ij} \in W_0^{1,\infty}(\Omega)$ are real symmetric and $C(x) > 0$ for all $x \in \Omega$ then H_Ω is Markov unique.*

Proof. Since the coefficients c_{ij} are in $W_0^{1,\infty}(\Omega)$ they extend by continuity to $\overline{\Omega}$ and the extended coefficients are zero on the boundary. But then by Lipschitz continuity $|c_{ij}(x)| \leq a\,(d_{\partial\Omega}(x)\wedge 1)$ for all $x \in \Omega$. Therefore $\mathrm{cap}_\Omega(\partial\Omega) = 0$ by Proposition 4.2 applied with $A = \partial\Omega$, $\gamma = 1$ and $d(A) = d - 1$. Hence H_Ω is Markov unique by Theorem 1.2. □

Corollary 4.5. *Assume $|\partial\Omega| = 0$. If the coefficients $c_{ij} \in W_0^{2,\infty}(\Omega)$ are real symmetric and $C(x) > 0$ for all $x \in \Omega$ then H_Ω is Markov unique.*

Proof. Since the coefficients c_{ij} are in $W_0^{2,\infty}(\Omega)$ they again extend to $\overline{\Omega}$, the extensions are zero on the boundary and one now has bounds $|c_{ij}(x)| \leq a\,(d_{\partial\Omega}(x) \wedge 1)^2$ for all $x \in \Omega$. Then $\mathrm{cap}_\Omega(\partial\Omega) = 0$, by Proposition 4.2 applied with $A = \partial\Omega$ and $\gamma = 2$. Hence H_Ω is Markov unique by Theorem 1.2. □

Note that the second result is universal in the sense that it does not depend on the geometry of Ω. In particular it does not depend on the dimension of $\partial\Omega$. Moreover, it suffices that the coefficients $c_{ij} \in W^{1,\infty}(\Omega) \cap W_0^{2,\infty}(U \cap \Omega)$ for some open set $U \supset \partial\Omega$. In fact if $c_{ij} \in W_0^{2,\infty}(\Omega)$ then the weaker ellipticity condition $C(x) \geq 0$ for $x \in \Omega$ suffices to deduce that H_Ω is L_2-unique (see [25, Section 6], or [11, Proposition 2.3]). In this latter case the coefficients can be extended to $\mathbb{R}^d$ by setting $c_{ij}(x) = 0$ if $x \in \Omega^{\mathrm{c}}$ and then the operator is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ and the self-adjoint extension leaves $L_2(\Omega)$ invariant.

Finally we emphasize that the condition $\mathrm{cap}_\Omega(\partial\Omega) = 0$ does not necessarily imply that the coefficients $c_{ij}(x) \to 0$ as $x \to \partial\Omega$. In fact Proposition 4.2 establishes that if $A \subset \partial\Omega$ and $d(A) \leq d - 2$ then $\mathrm{cap}_\Omega(A) = 0$ independently of the boundary behaviour of the coefficients. Nevertheless if Ω has a locally Lipschitz boundary, and consequently $d(\partial\Omega) = d - 1$, one can argue that $\mathrm{cap}_\Omega(\partial\Omega) = 0$ if and only if $c_{ij}(x) \to 0$ as $x \to \partial\Omega$.

5. Conservation criteria

In this section we prove Theorem 1.3. This theorem is to a large extent known and we concentrate on the new feature, Markov uniqueness implies semigroup conservation. An integral part in this proof is played by an approximation criterion for conservation which is also useful for the discussion of L_p-uniqueness (see Section 6).

Lemma 5.1. *Assume there exists a sequence $\eta_n \in C_c^\infty(\Omega)$ with $0 \leq \eta_n \leq \mathbf{1}_\Omega$ such that $\|(\eta_n - \mathbf{1}_\Omega)\psi\|_2 \to 0$ for all $\psi \in L_2(\Omega)$ and $h_\Omega(\eta_n) \to 0$ as $n \to \infty$. Then S_t^F is conservative.*

Proof. First it follows that $((\eta_n - \mathbf{1}_\Omega), \psi) \to 0$ as $n \to \infty$ for all $\psi \in L_1(\Omega) \cap L_2(\Omega)$. Fix φ in the L_1-dense set $D(H_\Omega^F) \cap L_1(\Omega)$. Then $S_t^F \varphi \in L_1(\Omega) \cap L_2(\Omega)$ and

$$
\begin{aligned}
|(\mathbf{1}_\Omega, S_t^F \varphi) - (\mathbf{1}_\Omega, \varphi)| &= \lim_{n\to\infty} |(\eta_n, S_t^F \varphi) - (\eta_n, \varphi)| \\
&= \lim_{n\to\infty} \Big| \int_0^t ds\, (\eta_n, S_s^F H_\Omega^F \varphi) \Big| \\
&\leq \lim_{n\to\infty} t\, h_\Omega(\eta_n)^{1/2}\, \overline{h}_\Omega(\varphi)^{1/2} = 0\ .
\end{aligned}
$$

Therefore S_t^F is conservative on $L_\infty(\Omega)$. □

Now we turn to the proof of the theorem

Proof of Theorem 1.3. I $\Rightarrow$ II The proof is in five steps.

Step 1. The first step consists of proving the implication for bounded Ω by constructing a sequence of η_n of the type occurring in Lemma 5.1.

Assume Ω is bounded. It follows from the Markov uniqueness and Theorem 1.2 that $\mathrm{cap}_\Omega(\partial\Omega) = 0$ and $l_\Omega = \overline{h}_\Omega$. Therefore there exist a decreasing sequence of open subsets U_n of $\mathbb{R}^d$ with $\partial\Omega \subset U_n$ and a sequence $\chi_n \in D(l_\Omega)$ with $0 \leq \chi_n \leq 1$ and $\chi_n = 1$ on $U_n \cap \Omega$ such that $\|\chi_n\|_2 \to 0$ and $l_\Omega(\chi_n) \to 0$ as $n \to \infty$. Since Ω is bounded it follows that $\mathbf{1}_\Omega \in D(l_\Omega)$. Therefore $\eta_n = (\mathbf{1}_\Omega - \chi_n) \in D(l_\Omega)$. But then

$$
\|(\eta_n - \mathbf{1}_\Omega)\psi\|_2 = \|\chi_n \psi\|_2 \leq \|\chi_n\|_2\, \|\psi\|_\infty \to 0\ .
$$

for all ψ in the L_2-dense subset $L_2(\Omega) \cap L_\infty(\Omega)$. Thus the first convergence property of the η_n is satisfied. Then, however, $l_\Omega(\eta_n) = l_\Omega(\chi_n)$ and the second condition is also satisfied. Finally $\operatorname{supp} \eta_n \Subset \Omega$ for each n. Hence by regularization one may construct a second sequence of $C_c^\infty(\Omega)$-functions $\eta_n \in D(l_\Omega)$ with similar boundedness and convergence properties.

Therefore it follows from Lemma 5.1 that the semigroup S_t^F is conservative.

Step 2. The second step consists of proving the theorem for unbounded Ω but for a family of cutoff operators.

Fix $\rho \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \rho \leq 1$, $\rho(x) = 1$ if $|x| \leq 1$ and $\rho(x) = 0$ if $|x| \geq 2$. Then introduce the sequence ρ_n by $\rho_n(x) = \rho(n^{-1}x)$. Thus $\rho_n(x) = 1$ if $|x| \leq n$ and $\rho(x) = 0$ if $|x| \geq 2n$. Set $B_n = \{x \in \mathbb{R}^d : |x| < 2n\}$ and $\Omega_n = \Omega \cap B_n$. Note that Ω_n is bounded.

Now define a family of truncations $h_{\Omega,n}$ of h_Ω by $D(h_{\Omega,n}) = C_c^\infty(\Omega_n)$ and

$$h_{\Omega,n}(\varphi) = h_\Omega(\varphi, \rho_n\varphi) - 2^{-1}h_\Omega(\rho_n, \varphi^2)$$

for all $\varphi \in C_c^\infty(\Omega_n)$. The truncation $h_{\Omega,n}$ is the Markovian form corresponding to the symmetric operator with $H_{\Omega,n}$ coefficients $\rho_n c_{ij}$ acting on $L_2(\Omega_n)$. Let $l_{\Omega,n}$ denote the extended form corresponding to $H_{\Omega,n}$. The form $h_{\Omega,n}$ is automatically closable, the closure $\overline{h}_{\Omega,n}$ is a Dirichlet form and the corresponding self-adjoint operator $H^F_{\Omega,n}$ is the Friedrichs extension of $H_{\Omega,n}$. The form $l_{\Omega,n}$ is a Dirichlet form which in principle differs from $\overline{h}_{\Omega,n}$. But we next argue that $H_{\Omega,n}$ is Markov unique. Hence $l_{\Omega,n} = \overline{h}_{\Omega,n}$.

Let $\mathrm{cap}_{\Omega,n}(A)$ denote the capacity of the measurable subset A of $\overline{\Omega}_n$ measured with respect to $l_{\Omega,n}$. Since $\Omega_n = \Omega \cap B_n$ it follows that $\partial\Omega_n = (\partial\Omega \cap \overline{B}_n) \cup (\partial B_n \cap \overline{\Omega})$. Hence

$$\mathrm{cap}_{\Omega,n}(\partial\Omega_n) = \mathrm{cap}_{\Omega,n}(\partial\Omega \cap \overline{B}_n) + \mathrm{cap}_{\Omega,n}(\partial B_n \cap \overline{\Omega}) .$$

But $l_{\Omega,n} \le l_\Omega$ and $\mathrm{cap}_\Omega(\partial\Omega) = 0$ by Markov uniqueness of H_Ω . Therefore

$$\mathrm{cap}_{\Omega,n}(\partial\Omega \cap \overline{B}_n) \le \mathrm{cap}_\Omega(\partial\Omega \cap \overline{B}_n) \le \mathrm{cap}_\Omega(\partial\Omega) = 0 .$$

Moreover, $\mathrm{cap}_{\Omega,n}(\partial B_n \cap \overline{\Omega}) = 0$ because the C^∞-cutoff function ρ_n and all its derivatives are zero on the boundary ∂B_n. Thus $\mathrm{cap}_{\Omega,n}(\partial\Omega_n) = 0$ and $H_{\Omega,n}$ is Markov unique by Theorem 1.2. Hence the semigroup generated by the Friedrichs extension $H^F_{\Omega,n}$ of the cutoff operator $H_{\Omega,n}$ is conservative on $L_\infty(\Omega_n)$ by Step 1.

Step 3. The third and fourth steps consist of removing the cutoff by a suitable limit $n \to \infty$, first by L_2-arguments and then by L_1-arguments.

It is convenient to view H_Ω and $H_{\Omega,n}$ as symmetric operators on $L_2(\mathbb{R}^d)$. Since the coefficients c_{ij} of H_Ω are in $W^{1,\infty}(\Omega)$ the operator can be extended to a symmetric operator on the domain $C_c^\infty(\mathbb{R}^d)$. The extension corresponds to the operator $H_\Omega \oplus 0$ with domain $C_c^\infty(\Omega) \oplus L_2(\Omega^{\mathrm{c}})$. The Markov uniqueness of H_Ω on $L_2(\Omega)$ implies that the extended operator has a unique submarkovian extension $H^F_\Omega \oplus 0$ on $L_2(\mathbb{R}^d)$ and for simplicity of notation we set $H = H^F_\Omega \oplus 0$. Similarly, since $H_{\Omega,n}$ is Markov unique by Step 2 there is a unique submarkovian operator $H_n = H^F_{\Omega,n} \oplus 0$ which extends $H_{\Omega,n}$. We let h and h_n denote the corresponding Dirichlet forms on $L_2(\mathbb{R}^d)$.

The ρ_n form an increasing sequence of functions on $\mathbb{R}^d$, by definition. Therefore the h_n are a monotonically increasing family of forms on $L_2(\mathbb{R}^d)$. This implicitly uses the Markov uniqueness through the identification $l_\Omega = \overline{h}_\Omega$ and hence $l_{\Omega,n} = \overline{h}_{\Omega,n}$. Therefore one can define h_∞ by $D(h_\infty) = \bigcap_{n\ge1} D(h_n)$ and $h_\infty(\varphi) = \sup_{n\ge1} h_n(\varphi)$ for all $\varphi \in D(h_\infty)$. The form h_∞ is closed (see, for example [21, Section VIII.3.4]) and since the h_n are Dirichlet forms the supremum h_∞ is also a Dirichlet form. Moreover, by direct calculation $h_\infty(\varphi) = h(\varphi)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Hence $h_\infty \supseteq h$. Then it follows from the monotone convergence of the forms h_n

that one has strong L_2-convergence of the resolvents $(\lambda I + H_n)^{-1}$ to the resolvent $(\lambda I + H_\infty)^{-1}$ for all $\lambda > 0$ where H_∞ is the submarkovian operator corresponding to the form h_∞. Hence

$$\begin{aligned} H_\infty(I + \varepsilon H_\infty)^{-1}\varphi &= \lim_{n\to\infty} H_n(I + \varepsilon H_n)^{-1}\varphi \\ &= \lim_{n\to\infty} (I + \varepsilon H_n)^{-1} H\varphi = (I + \varepsilon H_\infty)^{-1} H\varphi \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Since $(I+\varepsilon H_\infty)^{-1}$ converges strongly to the identity operator as $\varepsilon \to 0$ it follows that $C_c^\infty(\mathbb{R}^d) \subseteq D(H_\infty)$ and $H_\infty\varphi = H\varphi$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Thus H_∞ is a submarkovian extension of H and by Markov uniqueness one has $H_\infty = H$.

The foregoing arguments establish that the H_n converge to H in the strong resolvent sense on $L_2(\mathbb{R}^d)$. Therefore the submarkovian semigroups $S_t^{(n)}$ generated by the H_n converge strongly on $L_2(\mathbb{R}^d)$ to the submarkovian semigroup S_t generated by H.

Note that by construction the semigroup $S_t^{(n)}$ leaves both $L_2(\Omega_n)$ and the orthogonal complement $L_2(\Omega^c)$ invariant. The semigroup is conservative on $L_\infty(\Omega)$ by Step 2 and is equal to the identity semigroup on the orthogonal complement. Therefore the $S_t^{(n)}$ are conservative semigroups on $L_\infty(\mathbb{R}^d)$ which are strongly L_2-convergent to S_t. But this is not sufficient to ensure that S_t is conservative. For this one needs L_1-convergence.

Step 4. The fourth step in the proof consists in proving that the semigroups $S_t^{(n)}$ are strongly convergent on $L_1(\mathbb{R}^d)$ to S_t (see [27], Proposition 6.2, for a similar result).

Since the semigroups $S_t^{(n)}$ and S_t are all submarkovian it suffices to prove convergence on a subset of L_1 whose span is dense. In particular it suffices to prove convergence on $L_1(A) \cap L_2(A)$ for each bounded open subset A of Ω. Moreover one can restrict to positive functions.

Fix $A \subset \Omega$ and $\varphi_A \in L_1(A) \cap L_2(A)$. Assume φ_A is positive. Next let $B \supset A$ be a bounded closed set of $\mathbb{R}^d$. Then

$$\begin{aligned} \|(S_t^{(n)} - S_t)\varphi_A\|_1 &\leq \|\mathbf{1}_B(S_t^{(n)} - S_t)\varphi_A\|_1 + \|\mathbf{1}_{B^c}S_t^{(n)}\varphi_A\|_1 + \|\mathbf{1}_{B^c}S_t\varphi_A\|_1 \\ &\leq |B|^{1/2}\|(S_t^{(n)} - S_t)\varphi_A\|_2 + |(\mathbf{1}_{B^c}, S_t^{(n)}\varphi_A)| \\ &\quad + |(\mathbf{1}_{B^c}, S_t\varphi_A)| \end{aligned} \tag{5.1}$$

where we have used the positivity of the semigroups and the functions to express the L_1-norms as pairings between L_1 and L_∞. Therefore it suffices to prove that the last two terms can be made arbitrarily small, uniformly in n, by suitable choice of B. Then the L_1-convergence follows from the L_2-convergence of Step 3. But the uniform estimate follows by Davies–Gaffney bounds using the arguments of [13, Proposition 3.6]. We briefly sketch the proof.

First one can associate a set theoretic (quasi-)distance with a quite general Dirichlet form (see, for example, [1, 5, 29] or [12]). Specifically we introduce the family of $d_n(X\,;Y)$ of distances corresponding to the Dirichlet forms h_n and a similar distance $d(X\,;Y)$ corresponding to h following the definitions of [12, Section 1]. Here X and Y are measurable subsets of $\mathbb{R}^d$ and $d_n(X\,;Y)\in[0,\infty]$. The definition of the distances is quite technical and $d_n(X\,;Y)$ takes the value $+\infty$ if X or Y is not a subset of Ω_n. But since $h_n\leq h$ one has $d_n(X\,;Y)\geq d(X\,;Y)$ and since $h(\varphi)\leq\|C\|\,\|\nabla\varphi\|_2^2$ one also has

$$d(X\,;Y)\geq\|C\|^{-1/2}|X-Y|=\|C\|^{-1/2}\inf_{x\in X,\,y\in Y}|x-y|$$

(see [12, Section 5], for a discussion of the monotonicity properties of the distances). Then the Davies–Gaffney bounds [1, 8, 18, 29] as presented in [12, Theorem 2] give

$$\begin{aligned}|(\varphi_X,S_t^{(n)}\varphi_Y)|&\leq e^{-d_n(X;Y)^2(4t)^{-1}}\|\varphi_X\|_2\|\varphi_Y\|_2\\&\leq e^{-d(X;Y)^2(4t)^{-1}}\|\varphi_X\|_2\|\varphi_Y\|_2\\&\leq e^{-|X-Y|^2(4\|C\|t)^{-1}}\|\varphi_X\|_2\|\varphi_Y\|_2\end{aligned}\tag{5.2}$$

for all $\varphi_X\in L_2(X)$ and $\varphi_Y\in L_2(Y)$. These bounds are uniform in n and are conveniently expressed in terms of the Euclidean distance.

Now choose R sufficiently large that $A\subseteq B_R=\{x:|x|<R\}$ and let $B=\overline{B_{2R}}$. Then one can separate $B^{\rm c}$ into annuli $B_{(n+1)R}\backslash B_{nR}$ and make a quadrature estimate, as in the proof of [13, Proposition 3.6], to find

$$e^{-|A-B^{\rm c}|^2(4\|C\|t)^{-1}}\leq\sum\nolimits_{n\geq2}e^{-|B_R-B_{nR}^{\rm c}|^2(4\|C\|t)^{-1}}|B_{(n+1)R}|^{1/2}\leq a\,R^{d/2}e^{-bR^2t^{-1}}$$

with $a,b>0$. Therefore combining these bounds with (5.1) and (5.2) one obtains the equicontinuous bounds

$$\|(S_t^{(n)}-S_t)\varphi_A\|_1\leq a'\,R^{d/2}\,\|(S_t^{(n)}-S_t)\varphi_A\|_2+2\,a\,R^{d/2}e^{-bR^2t^{-1}}$$

where $a',a,b>0$ are all independent of n. It follows immediately that $\|(S_t^{(n)}-S_t)\varphi_A\|_1\to0$ as $n\to\infty$. Thus the $S_t^{(n)}$ converge strongly to S_t on $L_1(\mathbb{R}^d)$ and in particular on the invariant subspace $L_1(\Omega)$.

Step 5. Finally we combine the conclusions of Steps 1 and 4 to deduce that S_t^F is conservative on $L_\infty(\Omega)$.

It follows from Step 1 that the semigroup generated by the Friedrichs extension $H_{\Omega,n}^F$ of the cutoff operator $H_{\Omega,n}$ is a conservative semigroup on $L_\infty(\Omega_n)$. Therefore the extension $S_t^{(n)}$ of the semigroup to $L_\infty(\mathbb{R}^d)$ is also conservative since

$$S_t^{(n)}\mathbf{1}=(S_t^{(n)}\oplus I)(\mathbf{1}_{\Omega_n}\oplus\mathbf{1}_{\Omega_n^{\rm c}})=S_t^{(n)}\mathbf{1}_{\Omega_n}\oplus\mathbf{1}_{\Omega_n^{\rm c}}=\mathbf{1}_{\Omega_n}\oplus\mathbf{1}_{\Omega_n^{\rm c}}=\mathbf{1}\;.$$

Then, however,

$$(\mathbf{1},\varphi)=\lim_{n\to\infty}(\mathbf{1},S_t^{(n)}\varphi)=(\mathbf{1},S_t\varphi)$$

for all $\varphi \in L_1(\mathbb{R}^d)$ by Step 4. Hence S_t is conservative on $L_\infty(\mathbb{R}^d)$ and its restriction S_t^F to the invariant subspace $L_\infty(\Omega)$ is conservative.

II $\Leftrightarrow$ III This follows by an argument of Davies, [7] Theorem 2.2, which was given for operators with smooth coefficients but which is also valid for operators with Lipschitz coefficients. In fact Davies argues that S_t^F is conservative if and only if $C_c^\infty(\Omega)$ is a core for the generator of the semigroup acting on $L_1(\Omega)$. But this is equivalent to L_1-uniqueness (see [9], Section 1b). Davies arguments need a slight modification to cover the operator H_Ω but this is not difficult by the discussion of elliptic regularity properties in Section 2. We omit further details.

III $\Rightarrow$ I This is a general feature which is proved in [9], Lemma 1.6. □

Note that the implication II $\Rightarrow$ I, which is an indirect consequence of the foregoing proof, can be easily deduced from Theorem 1.1. Let S_t denote the semigroup generated by L_Ω. Then $S_t^F\varphi \leq S_t\varphi$ for all positive $\varphi \in L_2(\Omega)$ and all $t \geq 0$ by Theorem 1.1.V. But if $\varphi \in L_2(\Omega)$ and $0 \leq \varphi \leq \mathbf{1}_\Omega$ then

$$\mathbf{1}_\Omega = S_t^F\mathbf{1}_\Omega = S_t^F\varphi + S_t^F(\mathbf{1}_\Omega - \varphi) \leq S_t\varphi + S_t(\mathbf{1}_\Omega - \varphi) = S_t\mathbf{1}_\Omega \leq \mathbf{1}_\Omega\ .$$

Therefore the inequalities are equalities and $S_t^F\varphi = S_t\varphi$ for all positive $\varphi \in L_2(\Omega)$ such that $0 \leq \varphi \leq \mathbf{1}_\Omega$ and for all $t \geq 0$. It follows immediately that $S_t^F = S_t$ for all $t \geq 0$. Therefore $H_\Omega^F = L_\Omega$ and H_Ω is Markov unique by Theorem 1.1.III. (This argument follows the latter part of the proof of [9, Corollary 3.4].)

6. Concluding remarks

In this concluding section we discuss various results and examples concerning L_p-uniqueness, sets of capacity zero and irreducibility properties.

6.1. L_p-uniqueness

First note that Lemma 5.1 gives a condition, in terms of an approximation to the identity, which ensures that S_t^F is conservative, and consequently H_Ω is L_1-unique. But if $p \in [1, 2]$ there is a similar sufficient condition for L_p-uniqueness.

Proposition 6.1. *Assume $p \in [1, 2\,]$. If there exists a sequence $\eta_n \in C_c^\infty(\Omega)$ such that $0 \leq \eta_n \leq \mathbf{1}_\Omega$, $\|(\eta_n - \mathbf{1}_\Omega)\psi\|_2 \to 0$ for all $\psi \in L_2(\Omega)$ and $\|\Gamma(\eta_n)\|_{p/(2-p)} \to 0$ as $n \to \infty$ then H_Ω is L_p-unique.*

In the case $p = 2$ Davies has established similar criteria (see [7], Theorems 3.1 and 3.2). (If $p = 2$ then $p/(2-p)$ is understood to be ∞.) Moreover, if $p = 1$ then $\|\Gamma(\eta_n)\|_1 = h_\Omega(\eta_n)$ and the condition for L_1-uniqueness agrees with the condition in Lemma 5.1.

Proposition 6.1 is essentially a corollary of the following.

Lemma 6.2. *If* $\varphi \in D(H_\Omega^*)$ *and* $\eta \in C_c^\infty(\Omega)$ *then*

$$(\eta^2\varphi, H_\Omega^*\varphi) \geq -(\varphi, \Gamma(\eta)\varphi) \tag{6.1}$$

where Γ *is the carré du champ associated with* H_Ω. *Therefore if* $(I + H_\Omega^*)\varphi = 0$ *then*

$$\|\eta\,\varphi\|_2^2 \leq (\varphi, \Gamma(\eta)\varphi) \tag{6.2}$$

for all $\eta \in C_c^\infty(\Omega)$.

Proof. First, if $\eta \in C_c^\infty(\Omega)$ and $\varphi \in D(H_\Omega^*)$ then $\eta\varphi, \eta^2\varphi \in D(\overline{H}_\Omega)$ by Theorem 2.1. Therefore

$$\begin{aligned}
2\,(\eta^2\varphi, H_\Omega^*\varphi) &= 2\ \mathrm{Re}(\eta^2\varphi, H_\Omega^*\varphi)\\
&= (\overline{H}_\Omega\eta^2\varphi, \varphi) + (\varphi, \overline{H}_\Omega\eta^2\varphi)\\
&\geq (\overline{H}_\Omega\eta^2\varphi, \varphi) + (\varphi, \overline{H}_\Omega\eta^2\varphi) - 2\,(\overline{H}_\Omega\eta\varphi, \eta\varphi)
\end{aligned}$$

since $\overline{H}_\Omega \geq 0$. But if $\Omega' \Subset \Omega$ is bounded and $\operatorname{supp}\eta \subset \Omega'$ then one may construct the strongly elliptic extension L of $H_{\Omega'}$ to $L_2(\mathbb{R}^d)$ as in the proof of Theorem 2.1. Then since $\overline{H}_\Omega\eta\varphi = \overline{H}_{\Omega'}\eta\varphi$ etc. one has

$$\begin{aligned}
(\overline{H}_\Omega\eta^2\varphi, \varphi) &+ (\varphi, \overline{H}_\Omega\eta^2\varphi) - 2\,(\overline{H}_\Omega\eta\varphi, \eta\varphi)\\
&= (\varphi, \eta^2 L(\varphi)) + (\varphi, L(\eta^2\varphi)) - 2\,(\varphi, \eta L(\eta\varphi))\\
&= (\varphi, L(\eta^2)\varphi) - 2\,(\varphi, \eta L(\eta)\varphi) = -2\,(\varphi, \Gamma(\eta)\varphi)
\end{aligned}$$

where we have used the distributional relation (2.1) several times. Combination of the last two estimates immediately yields (6.1). □

Remark 6.3. The essence of the foregoing calculation is the formal double commutator identity

$$(\mathrm{ad}\,\eta)^2(H_\Omega) = [\eta, [\eta, H_\Omega]] = -2\,\Gamma(\eta)\ .$$

Double commutator estimates of a different nature were used to prove general self-adjointness results in [25], *e.g*. Theorem 2.10, (see also [11, Proposition 2.3]).

Proof of Proposition 6.1. It suffices to prove that the range of $I + H_\Omega$ is dense in $L_p(\Omega)$. Therefore assume that $\varphi \in L_q(\Omega)$, the dual space of $L_p(\Omega)$, and $(I + H_\Omega^*)\varphi = 0$. Since $q \in [2, \infty]$ it follows that $\eta_n\varphi = -\eta_n H_\Omega^*\varphi \in L_2(\Omega)$ and then (6.1) gives

$$\|\eta_n\varphi\|_2^2 \leq (\varphi, \Gamma(\eta_n)\varphi) = \int \Gamma(\eta_n)\,\varphi^2 \leq \|\varphi\|_q^2\,\|\Gamma(\eta_n)\|_{p/(2-p)}\ .$$

Taking the limit $n \to \infty$ one deduces that $\|\varphi\|_2 = 0$ so $\varphi = 0$ and the range is dense. □

If $p = 2$ then the statement of Proposition 6.1 be strengthened.

Corollary 6.4. *Assume there exists a sequence* $\eta_n \in C_c^\infty(\Omega)$ *such that* $0 \leq \eta_n \leq \mathbf{1}_\Omega$, $\|(\eta_n - \mathbf{1}_\Omega)\psi\|_2 \to 0$ *for all* $\psi \in L_2(\Omega)$ *and* $\sup_{n\geq 1} \|\Gamma(\eta_n)\|_\infty < \infty$. *Then* H_Ω *is* L_2*-unique,* i.e. H_Ω *is essentially self-adjoint.*

Proof. It suffices to prove that the range of $I + \varepsilon H_\Omega$ is dense in $L_2(\Omega)$ for all small $\varepsilon > 0$. But if $\varphi \in D(H_\Omega^*)$ and $(I + \varepsilon H_\Omega^*)\varphi = 0$ then the foregoing argument gives

$$\|\eta_n\varphi\|_2^2 \leq \varepsilon\,(\varphi, \Gamma(\eta_n)\varphi) \leq \varepsilon \sup_{n\geq 1} \|\Gamma(\eta_n)\|_\infty \|\varphi\|_2^2 .$$

Therefore $\|\varphi\|_2 = 0$ for all small $\varepsilon > 0$. Thus $\ker(I + \varepsilon\, H_\Omega) = \{0\}$ and the range of $I + \varepsilon H_\Omega$ is dense. □

Example 6.5. Let $\Omega = \mathbb{R}^d$. Then the operator $H = -\sum_{i,j=1}^d \partial_i c_{ij} \partial_j$ acting on $C_c^\infty(\mathbb{R}^d)$ with $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ and $(c_{ij}) > 0$ is L_2-unique as a consequence of Proposition 6.1. It suffices to choose $\eta_n \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \eta_n \leq 1$, $\eta_n(x) = 1$ if $|x| \leq n$ and $\|\nabla\eta_n\|_\infty \leq a\,n^{-1}$. Then the η_n converge pointwise to the identity as $n \to \infty$ and $\|\Gamma(\eta_n)\|_\infty \leq a\,n^{-2}\|C\| \to 0$. The L_2-uniqueness implies that H is Markov unique. Therefore H is also L_1-unique and S_t^F is conservative by Theorem 1.3.

Example 6.6. Assume that $c_{ij} \in W^{1,\infty}(\Omega)$ and $0 < C(x) \leq a\, d_{\partial\Omega}(x)^2 I$ for some $a > 0$ and all $x \in \Omega$ where $d_{\partial\Omega}$ is the Euclidean distance to the boundary $\partial\Omega$ of Ω. Then H_Ω is L_2-unique. Again this follows from Proposition 6.1. Define ρ_n and χ_n as in the proofs of Propositions 4.1 and 4.2, respectively. Set $\eta_n = \rho_n\,(\mathbf{1}_\Omega - \chi_n \circ d_{\partial\Omega})$. Then the η_n converge pointwise to $\mathbf{1}_\Omega$ and $\|\Gamma(\eta_n)\|_\infty \leq a\,(\log n)^{-2}$. Therefore L_2-uniqueness follows from Proposition 6.1 with $p = 2$. More generally if $|B_\delta| \leq b\,\delta$ for all bounded $B \subseteq \partial\Omega$ and all $\delta \in \langle 0, 1]$ then one can use the calculational procedure of the proof of Proposition 4.2 to deduce that if $p \in [1, 2\,]$ and $0 < C(x) \leq a\, d_{\partial\Omega}(x)^{(3p-2)/p} I$ then H_Ω is L_p-unique. In particular L_1-uniqueness follows if $0 < C(x) \leq a\, d_{\partial\Omega}(x) I$.

Although the approximation criteria for L_1-uniqueness and L_2-uniqueness in Proposition 6.1 are superficially similar they are of a totally different geometric character. The first involves the norm $\|\Gamma(\eta)\|_1$ which is related to the capacity and the second involves the norm $\|\Gamma(\eta)\|_\infty$ which is related to the Riemannian distance. In dimension one the first estimate is optimal but the second is suboptimal. This is illustrated by the following example adapted from [6] (see also [9, 28]).

Example 6.7. Assume $d = 1$ and $\Omega = \langle -1, 1\rangle$. Further let H be the operator with domain $C_c^\infty(-1, 1)$ and action $H\varphi = -(c\,\varphi')'$ where $c(x) = (1 - x^2)^\delta$. Then $c \in W^{1,\infty}(-1, 1)$ if and only if $\delta \geq 1$. Set $W(x) = \int_0^x c^{-1}$. Thus $H^*W = 0$. It follows that H is L_p-unique for $p \in [1, \infty\rangle$ if and only if $W \notin L_q(-1, 1)$ where q is conjugate to p (see [6, Proposition 3.5]). Hence H is L_1-unique for all $\delta \geq 1$ and L_p-unique for $p > 1$ if and only if $\delta > (2p - 1)/p$. In particular it is L_2-unique if and only if $\delta > 3/2$ and L_p-unique for all $p \in [1, \infty\rangle$ if and only if

$\delta \geq 2$. Alternatively, H is Markov unique for all $\delta \geq 1$ by [9], Theorem 3.5. Thus Markov uniqueness and L_1-uniqueness are simultaneously valid in agreement with Theorem 1.3.

The L_1-uniqueness can be verified by the criterion of Proposition 6.1. Define η_n by $\eta_n(x) = 1 - W(x)/W(1-n^{-1})$ if $x \in [0, n^{-1}\rangle$, $\eta_n(x) = 0$ if $x \geq 1 - n^{-1}$ and $\eta_n(-x) = \eta_n(x)$ for all $x \geq 0$. Since $\delta \geq 1$ it follows that η_n converges monotonically upward to $\mathbf{1}_{\langle -1,1\rangle}$ as $n \to \infty$. But $\Gamma(\eta_n) = c\,|\eta_n'|^2$. Thus $h(\eta_n) = \|\Gamma(\eta_n)\|_1 = 2\,W(1-n^{-1})^{-1} \to 0$ as $n \to \infty$. Therefore L_1-uniqueness of H follows for all $\delta \geq 1$. But $\|\Gamma(\eta_n)\|_\infty \sim n^{(2-\delta)}$ and this is bounded if and only if $\delta \geq 2$. Therefore the L_2-uniqueness only follows for $\delta \geq 2$ and not for the full range $\delta > 3/2$.

Note that the Riemannian distance corresponding to the metric c^{-1} is given by $d(x\,;y) = |\int_y^x c^{-1/2}|$. Thus the distance from the origin to the boundary, $d(0\,;1) = d(0\,;-1)$, is finite for all $\delta \in [1,2\rangle$. Therefore if $\delta \in \langle 3/2, 2\rangle$ then the distance to the boundary is finite but H is nonetheless essentially self-adjoint.

6.2. Sets of capacity zero

Let A be a closed subset of $\overline{\Omega}$ with $|A| = 0$. In this subsection we assume that the coefficients c_{ij} are real, symmetric, $c_{ij} \in W^{1,\infty}(\Omega)$ and $C(x) > 0$ for all $x \in \Omega\backslash A$. Then we define the operators H_Ω and $H_{\Omega\backslash A}$ with the coefficients c_{ij} on $C_c^\infty(\Omega)$ and $C_c^\infty(\Omega\backslash A)$, respectively. All the foregoing considerations apply to $H_{\Omega\backslash A}$ because the matrix of coefficients C is non-degenerate on $\Omega\backslash A$ but they do not necessarily apply to H_Ω since C can be degenerate on A. Nevertheless $H_\Omega \supseteq H_{\Omega\backslash A}$. Hence uniqueness criteria for $H_{\Omega\backslash A}$ give sufficient conditions for uniqueness of H_Ω. For example if $H_{\Omega\backslash A}$ is Markov unique then H_Ω is Markov unique. But Markov uniqueness of $H_{\Omega\backslash A}$ is equivalent to the boundary $\partial(\Omega\backslash A)$ having zero capacity and this is equivalent to $\partial\Omega$ and A both having zero capacity. Thus the boundary condition $\mathrm{cap}_\Omega(\partial\Omega) = 0$ is sufficient for H_Ω to be Markov unique if in addition the degeneracy set A has zero capacity. This typically occurs for one of two reasons. Either $d(A) \leq d-2$ and $\mathrm{cap}_\Omega(A) = 0$ independently of the behaviour of the coefficients in the neighbourhood of A or $d(A)$ is arbitrary and the coefficients have a correspondingly strong degeneracy on A (see Proposition 4.2). We illustrate these possibilities with two simple examples.

Example 6.8. Let $\Omega = \mathbb{R}^d$ and consider the operator $H = -\sum_{i,j=1}^d \partial_i c_{ij} \partial_j$ acting on $C_c^\infty(\mathbb{R}^d)$ with $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ and $(c_{ij}) > 0$ on the complement A^{c} of a closed set A with $|A| = 0$ and $\sup_{\delta \in \langle 0,1]} \delta^{-2}|B_\delta| < \infty$ for all bounded $B \subseteq A$. Then H is L_1-unique and Markov unique. This follows because $\partial A^{\mathrm{c}} = A$ and $\mathrm{cap}_\Omega(A) = 0$ by the estimates of Proposition 4.2. Therefore H is L_1-unique and Markov unique on $C_c^\infty(\mathbb{R}^d\backslash A)$ by Theorems 1.2 and 1.3.

Example 6.9. Again let $\Omega = \mathbb{R}^d$ and $H = -\sum_{i,j=1}^d \partial_i c_{ij} \partial_j$ the operator acting on $C_c^\infty(\mathbb{R}^d)$ with coefficients $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ and $(c_{ij}) > 0$ on the complement $A^{\mathrm{c}} = \mathbb{R}^d\backslash A$ of a closed set A with $|A| = 0$. Further assume that $A^{\mathrm{c}} = \Omega_1 \cup \Omega_2$

with $\Omega_1 \cap \Omega_2 = \emptyset$. Now assume $\operatorname{cap}_\Omega(A) = 0$. Since $A = \partial(\Omega_1 \cup \Omega_2) = \overline{\Omega}_1 \cap \overline{\Omega}_2$ it follows that H is L_1-unique and Markov unique on $C_c^\infty(\Omega_1 \cup \Omega_2)$. Therefore H is L_1-unique and Markov unique on $C_c^\infty(\mathbb{R}^d)$. Moreover, the unique Markov extension, the Friedrichs extension H^F, must coincide with the Friedrichs extension of H on $C_c^\infty(\Omega_1 \cup \Omega_2) = C_c^\infty(\Omega_1) + C_c^\infty(\Omega_2)$. But it follows readily from the definition of the Friedrichs extension that this latter operator is of the form $H^F_{\Omega_1} \oplus H^F_{\Omega_2}$ on $L_2(\Omega_1) \oplus L_2(\varphi_2)$. Therefore the semigroup S_t^F generated by H^F leaves the subspaces $L_2(\Omega_1)$ and $L_2(\Omega_2)$ invariant.

6.3. Irreducibility and ergodicity

In Example 6.8 the set $\mathbb{R}^d \backslash A$ on which the coefficients of the operator H are non-degenerate has two disjoint components Ω_1 and Ω_2. Consequently the corresponding Markov semigroup has two invariant subspaces $L_2(\Omega_1)$ and $L_2(\Omega_2)$. We conclude by giving a general result that relates connectedness of the set of non-degeneracy and ergodicity of the corresponding Friedrichs semigroup.

The absence of non-trivial invariant subspaces is variously defined as ergodicity or irreducibility of a semigroup. The property can be characterized by strict positivity. In particular the positive semigroup S_t on $L_2(\Omega)$ is defined to be irreducible if for every $t > 0$ and every positive, nonzero, $\varphi \in L_2(\Omega)$ one has $S_t\varphi > 0$ almost everywhere (see, for example [24, Definition 2.8]). This is clearly equivalent to the requirement that $(\varphi, S_t\psi) > 0$ for all positive, nonzero, $\varphi, \psi \in L_2(\Omega)$ and for all $t > 0$.

Now consider the submarkovian semigroups generated by extensions of H_Ω always under the assumptions of Theorem 1.1. The following proposition extends [24, Theorem 4.5] to this situation.

Proposition 6.10. *Let S_t^F denote the semigroup generated by the Friedrichs extension H_Ω^F of H_Ω. The following conditions are equivalent:*

I. *S_t^F is irreducible,*
II. *Ω is connected.*

Moreover if these conditions are satisfied then each semigroup generated by a submarkovian extension of H_Ω is irreducible.

Proof. First note that if K_Ω is a submarkovian extension of H_Ω then the semigroup e^{-tK_Ω} dominates S_t^F by Theorem 1.1.V. It follows immediately that irreducibility of S_t^F implies irreducibility of e^{-tK_Ω}.

I $\Rightarrow$ II This follows by the foregoing discussion. If Ω is not connected then S_t^F is not irreducible.

II $\Rightarrow$ I Let Ω_n be an increasing family of connected open subsets of Ω with $\Omega_n \Subset \Omega_{n+1}$ and $\Omega = \bigcup_{n\geq 1} \Omega_n$. Then $H_n = H_\Omega|_{C_c^\infty(\Omega_n)}$ is a strongly elliptic operator on $L_2(\Omega_n)$. Let H_n^F denote the Friedrichs extension of H_n and $S_t^{(n)}$ the corresponding semigroup. The extension H_n^F corresponds to Dirichlet boundary conditions on H_n. It follows that $S_t^{(n)}$ is irreducible on $L_2(\Omega_n)$ by [24], Theorem 4.5. But $S_t^F\varphi \geq S_t^{(n+1)}\mathbf{1}_{\Omega_{n+1}}\varphi \geq S_t^{(n)}\mathbf{1}_{\Omega_n}\varphi$ for all positive $\varphi \in L_2(\Omega)$ by [10, Corollary 2.3]. Therefore S_t^F is irreducible on $L_2(\Omega)$. □

A. Strong ellipticity

In this appendix we recall some basic properties of the operator $L = -\sum_{i,j=1}^{d} \partial_i\, c_{ij}\, \partial_j$, with real symmetric Lipschitz continuous coefficients c_{ij} and domain $D(L) = C_c^\infty(\mathbb{R}^d)$, acting on $L_2(\mathbb{R}^d)$ under the hypothesis of strong ellipticity.

Proposition A.1. *Assume $c_{ij} = c_{ji} \in W^{1,\infty}(\mathbb{R}^d)$ and that $C = (c_{ij}) \geq \mu I > 0$ uniformly over $\mathbb{R}^d$. Then one has the following.*

I. *L is essentially self-adjoint,*
II. *$D(\overline{L}) = W^{2,2}(\mathbb{R}^d)$,*
III. *$(I + \overline{L})^{-1}$ is a bounded operator from $W^{-\delta,2}(\mathbb{R}^d)$ to $W^{2-\delta,2}(\mathbb{R}^d)$ for all $\delta \in [0, 1]$.*

The conclusions of the proposition are well known. If the operator is strongly elliptic and $c_{ij} \in W^{2,\infty}(\mathbb{R}^d)$ then the result follows in its entirety from [25] or [11] but we have not found a suitable reference for the complete statement with $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$. We briefly sketch the proof.

Sketch of proof of Proposition A.1. First, since the operator L is symmetric on $L_2(\mathbb{R}^d)$ it is closable and its closure $\overline{L}$ is self-adjoint if and only if the range condition $R(\kappa I + \overline{L}) = L_2(\mathbb{R}^d)$ is satisfied for large positive κ. Since the coefficients $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ the latter condition can be established by the following variant of Levi's parametrix argument (see, for example, [17]).

Fix $y \in \mathbb{R}^d$ and introduce the constant coefficient operator $L_y = -\sum_{i,j=1}^{d} c_{ij}(y)\, \partial_i\, \partial_j$ with domain $W^{2,2}(\mathbb{R}^d)$. Then L_y is a positive self-adjoint operator which generates a translationally invariant submarkovian semigroup with an integral kernel

$$K_t^{(y)}(x) = (\det C(y))^{-1}(4\pi t)^{-d/2} e^{-(x, C(y)^{-1}x)/4t} \ . \tag{A.1}$$

The kernel $R_\kappa^{(y)}$ of the resolvent $(\kappa I + L_y)^{-1}$ is then given by

$$R_\kappa^{(y)}(x) = \int_0^\infty dt\, e^{-\kappa t} K_t^{(y)}(x) \ .$$

Next define R_κ as a bounded operator on the spaces $L_p(\mathbb{R}^d)$ by

$$(R_\kappa\varphi)(x) = \int_{\mathbb{R}^d} dy\, R_\kappa^{(y)}(x-y)\,\varphi(y)\;.$$

It follows that if $\kappa \geq 1$ then $R_\kappa L_2(\mathbb{R}^d) \subseteq D(\overline{L})$ and

$$(\kappa I + \overline{L})R_\kappa = I + Q_\kappa$$

where the Q_κ are the bounded operators

$$\begin{aligned}(Q_\kappa\varphi)(x) &= -\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} dy\,\Big(\partial_i(c_{ij}(x)-c_{ij}(y))\partial_j R_\kappa^{(y)}\Big)(x-y)\,\varphi(y)\\ &= -\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} dy\,\Big((\partial_i c_{ij})(x)(\partial_j R_\kappa^{(y)})(x-y)\\ &\qquad + (c_{ij}(x)-c_{ij}(y))(\partial_i\partial_j R_\kappa^{(y)})(x-y)\Big)\,\varphi(y).\end{aligned}$$

Since the coefficients $c_{ij} \in W^{1,\infty}(\mathbb{R}^d)$ one has bounds $|c_{ij}(x)-c_{ij}(y)| \leq a\,(|x-y|\wedge 1)$ for some $a > 0$. Therefore it follows that Q_κ satisfy bounds $\|Q_\kappa\|_{2\to 2} \leq b\,\kappa^{-1/2}$ for all $\kappa \geq 1$. Thus $\|Q_\kappa\|_{2\to 2} < 1$, the operator $I + Q_\kappa$ has a bounded inverse and

$$(\kappa I + \overline{L})R_\kappa(I+Q_\kappa)^{-1} = I$$

for all large κ. Then the range of $(\kappa I + \overline{L})$ is $L_2(\mathbb{R}^d)$ and $\overline{L}$ is self-adjoint.

Secondly, to deduce that $D(\overline{L}) = W^{2,2}(\mathbb{R}^d)$, with equivalent norms, we note that

$$\|L\varphi\|_2^2 = \sum_{i,j,k,l=1}^{d}(\partial_j\, c_{ij}\,\partial_i\varphi, \partial_k\, c_{kl}\,\partial_l\varphi) = \sum_{i,j,k,l=1}^{d}(\partial_k\partial_i\varphi, c_{ij}\,c_{kl}\,\partial_j\partial_l\varphi) + \mathrm{LOT}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ where LOT denotes a sum of lower order terms. But if A denotes the $d^2\times d^2$-matrix with coefficients $a_{(ik),(jl)} = c_{ij}c_{kl}$ then $A = C\otimes C$. Thus if $\lambda\, I \geq C \geq \mu\, I > 0$ then $\lambda^2\, I \geq A \geq \mu^2\, I$ uniformly on $\mathbb{R}^d$ and

$$\lambda^2\,\|\Delta\varphi\|_2^2 \geq \sum_{i,j,k,l=1}^{d}(\partial_k\partial_i\varphi, c_{ij}\,c_{kl}\,\partial_j\partial_l\varphi) \geq \mu^2\,\|\Delta\varphi\|_2^2 \tag{A.2}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ where Δ denotes the usual Laplacian. The lower order terms can, however, be bounded by standard estimates. For each $\varepsilon \in \langle 0, 1]$ there is a $c_\varepsilon > 0$ such that

$$|\mathrm{LOT}| \leq \varepsilon\,\|\Delta\varphi\|_2^2 + c_\varepsilon\,\|\varphi\|_2^2 \tag{A.3}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. The identity $D(\overline{L}) = W^{2,2}(\mathbb{R}^d)$ follows straightforwardly by combination of the estimates (A.2) and (A.3) since the $W^{2,2}$-norm and the graph norm on $D(\Delta)$ are equivalent and $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$.

Thirdly, if $\delta \geq 0$ then $W^{\delta,2}(\mathbb{R}^d) = D((I + \Delta)^{\delta/2})$ with the graph norm and $W^{-\delta,2}(\mathbb{R}^d)$ is defined as the dual space. These spaces are a scale of spaces for real interpolation. Since, by the foregoing, $(I+\overline{L})^{-1}L_2(\mathbb{R}^d) = D(\overline{L}) = W^{2,2}(\mathbb{R}^d)$, *i.e.* the resolvent is a bounded operator from $L_2(\mathbb{R}^d)$ to $W^{2,2}(\mathbb{R}^d)$, it suffices to prove that $(I + \overline{L})^{-1}$ is bounded from $W^{-1,2}(\mathbb{R}^d)$ to $W^{1,2}(\mathbb{R}^d)$. Therefore it suffices to prove that $(I+\Delta)^{1/2}(I+\overline{L})^{-1}(I+\Delta)^{1/2}$ extends to a bounded operator on $L_2(\mathbb{R}^d)$. This follows, however, because

$$\begin{aligned}
&\|(I + \Delta)^{1/2}(I + \overline{L})^{-1}(I + \Delta)^{1/2}\varphi\|_2^2 \\
&\qquad \leq (1 \vee \mu^{-1})\,(\varphi, (I + \Delta)^{1/2}(I + \overline{L})^{-1}(I + \Delta)^{1/2}\varphi) \\
&\qquad \leq (1 \vee \mu^{-1})\,\|\varphi\|_2\,\|(I + \Delta)^{1/2}(I + \overline{L})^{-1}(I + \Delta)^{1/2}\varphi\|_2
\end{aligned}$$

by strong ellipticity. □

References

[1] T. ARIYOSHI and M. HINO, *Small-time asymptotic estimates in local Dirichlet spaces*, Electron. J. Probab. **10** (2005), 1236–1259.

[2] S. ALBEVERIO, S. KUSUOKA and M. RÖCKNER, *On partial integration in infinite-dimensional space and applications to Dirichlet forms*, Proc. London Math. Soc. **42** (1990), 122–136.

[3] R. AZENCOTT, *Behavior of diffusion semi-groups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240.

[4] N. BOULEAU and F. HIRSCH, “Dirichlet Forms and Analysis on Wiener Space”, de Gruyter Studies in Mathematics, Vol. 14, Walter de Gruyter & Co., Berlin, 1991.

[5] M. BIROLI and U. MOSCO, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. **169** (1995), 125–181.

[6] M. CAMPITI, G. METAFUNE and D. PALLARA, *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum **57** (1998), 1–36.

[7] E.-B. DAVIES, *L^1 properties of second order elliptic operators*, Bull. London Math. Soc. **17** (1985), 417–436.

[8] E.-B. DAVIES, *Heat kernel bounds, conservation of probability and the Feller property*, J. Anal. Math. **58** (1992), 99–119. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.

[9] A. EBERLE, “Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators”, Lect. Notes in Math. 1718. Springer-Verlag, Berlin, 1999.

[10] A. F. M. TER ELST and D. W. ROBINSON, *Conservation and invariance properties of submarkovian semigroups*, J. Ramanujan. Math. Soc. **24** (2009), 1–13.

[11] A. F. M. TER ELST and D. W. ROBINSON, *Uniform subellipticity*, J. Operator Theory **62** (2009), 125–149.

[12] A. F. M. TER ELST, D. W. ROBINSON, A. SIKORA and Y. ZHU, *Dirichlet forms and degenerate elliptic operators*, In: “Partial Differential Equations and Functional Analysis”, Koelink, E., Neerven, J. van, Pagter, B. de and Sweers, G. (eds.), Operator Theory: Advances and Applications, Vol. 168, Birkhäuser, 2006, 73–95. Philippe Clement Festschrift.

[13] A. F. M. TER ELST, D. W. ROBINSON, A. SIKORA and Y. ZHU, *Second-order operators with degenerate coefficients*, Proc. London Math. Soc. **95** (2007), 299–328.
[14] K. J. FALCONER, "Fractal geometry", Second edition, Mathematical Foundations and Applications, John Wiley & Sons Inc., Hoboken, NJ, 2003.
[15] M. FUKUSHIMA, Y. OSHIMA and M. TAKEDA, "Dirichlet forms and symmetric Markov processes", de Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, 1994.
[16] K. O. FRIEDRICHS, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, I*, Math. Ann. **109** (1934), 465–487.
[17] A. FRIEDMAN, "Partial Differential Equations of Parabolic Type", Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
[18] M. P. GAFFNEY, *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. **12** (1959), 1–11.
[19] D. GATZOURAS, *Lacunarity of self-similar and stochastically self-similar sets*, Trans. Amer. Math. Soc. **352** (2000), 1953–1983.
[20] D. GILBARG and N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", Second edition, Grundlehren der mathematischen Wissenschaften 224. Springer-Verlag, Berlin, 1983.
[21] T. KATO, "Perturbation Theory for Linear Operators", Second edition, Grundlehren der mathematischen Wissenschaften 132, Springer-Verlag, Berlin, 1980.
[22] M. G. KREIN, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications, I*, Mat. Sbornik N.S. **20(62)** (1947), 431–495.
[23] Z. M. MA and M. RÖCKNER, "Introduction to the Theory of (Non Symmetric) Dirichlet Forms", Universitext. Springer-Verlag, Berlin, 1992.
[24] E. M. OUHABAZ, "Analysis of Heat Equations on Domains", London Mathematical Society Monographs Series, Vol. 31, Princeton University Press, Princeton, NJ, 2005.
[25] D. W. ROBINSON, *Commutator theory on Hilbert space*, Canad. J. Math. **37** (1987), 1235–1280.
[26] D. W. ROBINSON and A. SIKORA, *Degenerate elliptic operators: capacity, flux and separation*, J. Ramanujan Math. Soc. **22** (2007), 385–408.
[27] D. W. ROBINSON and A. SIKORA, *Analysis of degenerate elliptic operators of Grušin type*, Math. Z. **260** (2008), 475–508.
[28] D. W. ROBINSON and A. SIKORA, *Degenerate elliptic operators in one-dimension*, J. Evol. Equ. **10** (2010), 731–759.
[29] K. T. STURM, *The geometric aspect of Dirichlet forms*, In: "New Directions in Dirichlet Forms", AMS/IP Stud. Adv. Math., Vol. 8, Amer. Math. Soc., Providence, RI, 1998, 233–277.
[30] M. TAKEDA, *Two classes of extensions for generalized Schrödinger operators*, Potential Anal. **5** (1996), 1–13.

Centre for Mathematics
and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, ACT 0200, Australia
derek.robinson@anu.edu.au

Department of Mathematics
Macquarie University
Sydney, NSW 2109, Australia
sikora@ics.mq.edu.au