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Harmonic mappings and distance function

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Abstract. We prove the following theorem: every quasiconformal harmonic mapping between two plane domains with $C^{1,\alpha}$ ($\alpha < 1$) and, respectively, $C^{1,1}$ compact boundary is bi-Lipschitz. This theorem extends a similar result of the author [10] for Jordan domains, where stronger boundary conditions for the image domain were needed. The proof uses distance function from the boundary of the image domain.

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1. Introduction and statement of the main result

We say that a function $u : D \to \mathbb{R}$ is ACL (absolutely continuous on lines) in the region $D \subset \mathbb{R}^2$, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has, of course, partial derivatives u_x and u_y a.e. in D. A homeomorphism $f : D \to G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-q.c), for $K \geq 1$, if f is ACL and

$$|\nabla f(z)| \leq K l(\nabla f(z)) \quad \text{a.e. on } D, \tag{1.1}$$

where

$$|\nabla f(x)| := \max_{|h|=1} |\nabla f(x)h| = |f_z| + |f_{\bar z}|$$

and

$$l(\nabla f(z)) := \min_{|h|=1} |\nabla f(z)h| = |f_z| - |f_{\bar z}|$$

(*cf.* [1, pages 23–24] and [22]). Note that, condition (1.1) can be written as

$$|f_{\bar z}| \leq k|f_z| \quad \text{a.e. on } D, \text{ where } k = \frac{K-1}{K+1} \ \textit{i.e. } K = \frac{1+k}{1-k}$$

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or in its equivalent form

$$|\nabla f(z)|^2 \leq K J_f(z),\ z \in \mathbb{U}, \tag{1.2}$$

where J_f is the Jacobian of f.

A function w is called *harmonic* in a region D if it has form $w = u + iv$ where u and v are real-valued harmonic functions on D. If D is simply connected, then there are two analytic functions g and h defined on D such that w has the representation

$$w = g + \overline{h}.$$

If w is a harmonic univalent function then, by Lewy's theorem (see [23]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r\cos x + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disc $\mathbb{U} := \{z : |z| < 1\}$ has the representation

$$w(z) = P[F](z) = \int_0^{2\pi} P(r, x - \varphi) F(e^{ix}) dx, \tag{1.3}$$

where $z = re^{i\varphi}$ and F is a bounded integrable function defined on the unit circle S^1.

In this paper we continue to study quasiconformal harmonic mappings. See [25] for the pioneering work on this topic, and [8] for related earlier results. In some recent papers, a lot of work have been done on this class of mappings ([3, 10–17, 19–21, 24, 26, 28, 29]). In these papers for the Lipschitz and the co-Lipschitz character is established quasiconformal harmonic mappings between plane domains with certain boundary conditions. In [32] the same problem is considered for hyperbolic harmonic quasiconformal selfmappings of the unit disk. Notice that, in general, quasi-symmetric self-mappings of the unit circle do not have a quasiconformal harmonic extension to the unit disk. In [25] an example is given of C^1 diffeomorphism of the unit circle onto itself whose Euclidean harmonic extension is not Lipschitz. Alessandrini and Nesi proved in [2] the following:

Proposition 1.1. *Let $F : S^1 \to \gamma \subset \mathbb{C}$ be an orientation-preserving diffeomorphism of class C^1 of S^1 onto a simple closed curve γ. Let D be the bounded domain such that $\partial D = \gamma$. Let $w = P[F] \in C^1(\overline{\mathbb{U}}; \mathbb{C})$. The mapping w is a diffeomorphism of $\mathbb{U}$ onto D if and only if*

$$J_w > 0 \text{ everywhere on } S^1. \tag{1.4}$$

From the inequalities (1.2) and (1.4), we easily deduce the following:

Corollary 1.2. *Under the assumption of Proposition* 1.1 *the harmonic mapping w is a diffeomorphism if and only if it is K-quasiconformal for some $K \geq 1$.*

In contrast to the case of the Euclidean metric, in the case of the hyperbolic metric, if $f : S^1 \mapsto S^1$ is C^1 diffeomorphism, or more generally if $f : S^{n-1} \mapsto S^{m-1}$ is a mapping with non-vanishing energy, then its hyperbolic harmonic extension is C^1 up to the boundary ([4, 5]).

To continue we need the definition of $C^{k,\alpha}$ Jordan curves ($k \in \mathbb{N}, 0 < \alpha \leq 1$). Let γ be a rectifiable curve in the complex plane. Let l be the length of γ. Let $g : [0, l] \mapsto \gamma$ be an arc-length parametrization of γ. Then $|\dot{g}(s)| = 1$ for all $s \in [0, l]$. We will say that $\gamma \in C^{k,\alpha}$, $k \in \mathbb{N}$, $0 < \alpha \leq 1$ if $g \in C^k$, and $M(k, \alpha) := \sup_{t \neq s} \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t-s|^\alpha} < \infty$. Notice this important fact: if $\gamma \in C^{1,1}$ then γ has a curvature κ_z for a.e. $z \in \gamma$ and $\operatorname{ess\,sup}\{|\kappa_z| : z \in \gamma\} \leq M(1, 1) < \infty$.

This definition can be easily extended to an arbitrary $C^{k,\alpha}$ compact 1-dimensional manifold (not necessarily connected).

The starting point of this paper is the following proposition.

Proposition 1.3. *Let $w = f(z)$ be a K-quasiconformal harmonic mapping between a Jordan domain Ω_1 with $C^{1,\alpha}$ boundary and a Jordan domain Ω with $C^{1,\alpha}$ (respectively $C^{2,\alpha}$) boundary. Consider in addition $b \in \Omega_1$ and set $a = f(b)$. Then w is Lipschitz (respectively co-Lipschitz). Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a, b) \geq 1$ such that*

$$|f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1 \tag{1.5}$$

and

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega_1, \tag{1.6}$$

respectively.

See [13] for the first part of Proposition 1.3 and [10] for its second part. In [10], it was conjectured that the second part of Proposition 1.3 remains true if we assume that Ω has $C^{1,\alpha}$ boundary only. Notice that the proof of Proposition 1.3 relies on the Kellogg-Warschawski theorem ([6, 33, 34]) from the theory of conformal mappings, which asserts that if w is a conformal mapping of the unit disk onto a domain $\Omega \in C^{k,\alpha}$, then $w^{(k)}$ has a continuous extension to the boundary ($k \in \mathbb{N}$). It also depended on Mori's theorem from the theory of quasiconformal mappings, which deals with the Hölder character of quasiconformal mappings between plane domains (see [1, 31]). In addition, Lemma 3.2 below is needed.

Using a different approach, we will extend here as stated in Theorem 1.4 the second part of Proposition 1.3 to the case of image domains with $C^{1,1}$ boundary. The proof of Theorem 1.4, given in the last section, is different form the proof of second part of Proposition 1.3, and the use of the Kellogg-Warschawski theorem for the second derivative ([34]) is avoided. The distance function is used and hence a "weaker" smoothness of the boundary of image domain is needed.

Theorem 1.4 (The main theorem). *Let $w = f(z)$ be a K-quasiconformal harmonic mapping from the unit disk $\mathbb{U}$ to a Jordan domain Ω with $C^{1,1}$ boundary. Set*

$a = f(0)$. *Then w is co-Lipschitz. More precisely, there exists a positive constant $c = c(K, \Omega, a) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega. \tag{1.7}$$

Since the composition of a quasiconformal harmonic and a conformal mapping is itself quasiconformal harmonic, using Theorem 1.4 and Kellogg's theorem for the first derivative we obtain:

Corollary 1.5. *Let $w = f(z)$ be a K-quasiconformal harmonic mapping between a plane domain Ω_1 with $C^{1,\alpha}$ compact boundary and a plane domain Ω with $C^{1,1}$ compact boundary. Consider $a_0 \in \Omega_1$ and set $b_0 = f(a_0)$. Then w is bi-Lipschitz. Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a_0, b_0) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1. \tag{1.8}$$

Proof of Corollary 1.5. Let $b = f(a) \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$, it follows that there exists a $C^{1,1}$ Jordan curve $\gamma_b \subset \overline{\Omega}$, whose interior D_b lies in Ω, and $\partial\Omega \cap \gamma_b$ is a neighborhood of b. See [13, Theorem 2.1] for an explicit construction of such a Jordan curve. Let $D_a = f^{-1}(D_b)$, and take a conformal mapping g_a of the unit disk onto D_a. Then $f_a = f \circ g_a$ is a quasiconformal harmonic mapping from the unit disk onto the $C^{1,1}$ domain D_b. From Theorem 1.4 it follows that f_a is bi-Lipschitz, and from Kellogg's theorem it follows that $f = f_a \circ g_a^{-1}$ and its inverse f^{-1} are Lipschitz in some small neighborhood of a and of $b = f(a)$ respectively. This means that ∇f is bounded in some neighborhood of a. Since $\partial\Omega_1$ is a compact, we deduce that ∇f is bounded in $\partial\Omega_1$. The same holds for ∇f^{-1} with respect to $\partial\Omega$. This implies that f is bi-Lipschitz. $\square$

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2. Auxiliary results

Let Ω be a domain in $\mathbb{R}^2$ having non-empty boundary $\partial\Omega$. The distance function from the boundaryis defined by

$$d(x) = \operatorname{dist}(x, \partial\Omega). \tag{2.1}$$

Let Ω be bounded and assume $\partial\Omega \in C^{1,1}$. These conditions on Ω imply that $\partial\Omega$ satisfies the following: at a.e. point $z \in \partial\Omega$ there exists a disk $D = D(w_z, r_z)$ depending on z such that $\overline{D} \cap (\mathbb{C} \setminus \Omega) = \{z\}$. Moreover $\mu := \operatorname{ess\,inf}\{r_z, z \in$

$\partial\Omega\} > 0$. It is easy to show that μ^{-1} bounds the curvature of $\partial\Omega$, which means that $\frac{1}{\mu} \geq \kappa_z$, for $z \in \partial\Omega$. Here κ_z denotes the curvature of $\partial\Omega$ at $z \in \partial\Omega$. Under the above conditions, we have $d \in C^{1,1}(\Gamma_\mu)$, where $\Gamma_\mu = \{z \in \overline{\Omega} : d(z) < \mu\}$ and for $z \in \Gamma_\mu$ there exists $\omega(z) \in \partial\Omega$ such that

$$\nabla d(z) = \boldsymbol{\nu}_{\omega(z)}, \tag{2.2}$$

where $\boldsymbol{\nu}_{\omega(z)}$ denotes the inner normal vector to the boundary $\partial\Omega$ at the point $\omega(z)$. See [7, Section 14.6] for the details.

Lemma 2.1. *Let $w : \Omega_1 \mapsto \Omega$ be a K-quasiconformal mapping and set $\chi = -d(w(z))$. Then*

$$|\nabla\chi| \leq |\nabla w| \leq K|\nabla\chi| \tag{2.3}$$

in $w^{-1}(\Gamma_\mu)$ for $\mu > 0$ such that $1/\mu > \kappa_0 = \operatorname{ess\ sup}\{|\kappa_z| : z \in \partial\Omega\}$.

Proof. Observe first that ∇d is a unit vector. From the identity $\nabla\chi = -\nabla d \cdot \nabla w$ it follows that

$$|\nabla\chi| \leq |\nabla d||\nabla w| = |\nabla w|.$$

For a non-singular matrix A we have

$$\begin{aligned}
\inf_{|x|=1} |Ax|^2 &= \inf_{|x|=1} \langle Ax, Ax\rangle = \inf_{|x|=1} \left\langle A^T Ax, x\right\rangle \\
&= \inf\{\lambda : \exists x \neq 0,\ A^T Ax = \lambda x\} \\
&= \inf\{\lambda : \exists x \neq 0,\ AA^T Ax = \lambda Ax\} \\
&= \inf\{\lambda : \exists y \neq 0,\ AA^T y = \lambda y\} = \inf_{|x|=1} |A^T x|^2.
\end{aligned} \tag{2.4}$$

We next denote that $(\nabla\chi)^T = -(\nabla w)^T \cdot (\nabla d)^T$, therefore for $x \in w^{-1}(\Gamma_\mu)$ we obtain

$$|\nabla\chi| \geq \inf_{|e|=1} |(\nabla w)^T e| = \inf_{|e|=1} |\nabla w\, e| = l(w) \geq K^{-1}|\nabla w|.$$

The proof of (2.3) is complete. □

Lemma 2.2. *Let $\{e_1, e_2\}$ be the canonical basis of the space $\mathbb{R}^2$. Let $w : \Omega_1 \mapsto \Omega$ be a twice differentiable mapping and let $\chi = -d(w(z))$. Then*

$$\Delta\chi(z_0) = \frac{\kappa_{w_0}}{1 - \kappa_{w_0} d(w(z_0))} |(O_{z_0} \nabla w(z_0))^T e_1|^2 - \langle(\nabla d)(w(z_0)), \Delta w\rangle\,, \tag{2.5}$$

where $z_0 \in w^{-1}(\Gamma_\mu)$, $\omega_0 \in \partial\Omega$ with $|w(z_0) - \omega_0| = \operatorname{dist}(w(z_0), \partial\Omega)$, $\mu > 0$ such that $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$ and O_{z_0} is an orthogonal transformation.

Proof. Let ν_{ω_0} be the inner unit normal vector to γ at the point $\omega_0 \in \gamma$. Let O_{z_0} be an orthogonal transformation that takes the vector e_2 to ν_{ω_0}. In complex notations one has:

$$O_{z_0} w = -i \nu_{\omega_0} w.$$

Take $\tilde{\Omega} := O_{z_0}\Omega$. Let $\tilde{d}$ be the distance function for $\tilde{\Omega}$. Then

$$d(w) = \tilde{d}(O_{z_0} w) = \text{dist}\,(O_{z_0} w, \partial\tilde{\Omega}).$$

Therefore $\chi(z) = -\tilde{d}(O_{z_0}(w(z)))$. Furthermore

$$\begin{aligned}\Delta\chi(z) = &-\sum_{i=1}^{2}(D^2\tilde{d})(O_{z_0}(w(z)))(O_{z_0}\nabla w(z)e_i, O_{z_0}\nabla w(z)e_i)\\ &-\langle\nabla d(w(z)), \Delta w(z)\rangle\,.\end{aligned}\tag{2.6}$$

To continue, we make use of the following proposition.

Proposition 2.3 ([7, Lemma 14.17]). *Let Ω be bounded and assume $\partial\Omega \in C^{1,1}$. Then, with notation as in Lemma 2.2, we have*

$$(D^2\tilde{d})(O_{z_0} w(z_0)) = \text{diag}\left(\frac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0} d}, 0\right) = \begin{pmatrix}\dfrac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0} d} & 0\\ 0 & 0\end{pmatrix},\tag{2.7}$$

where κ_{ω_0} denotes the curvature of $\partial\Omega$ at $\omega_0 \in \partial\Omega$.

Applying (2.7) we have

$$\begin{aligned}&\sum_{i=1}^{2}(D^2\tilde{d})(O_{z_0}(w(z_0)))(O_{z_0}(\nabla w(z_0))e_i, O_{z_0}(\nabla w(z_0))e_i)\\ &=\sum_{i=1}^{2}\sum_{j,k=1}^{2} D_{j,k}\tilde{d}(O_{z_0}(w(z_0)))\, D_i(O_{z_0}w)_j(z_0)\cdot D_i(O_{z_0}w)_k(z_0)\\ &=\sum_{j,k=1}^{2} D_{j,k}\tilde{d}(O_{z_0}(w(z_0)))\left\langle (O_{z_0}\nabla w(z_0))^T e_j, (O_{z_0}\nabla w(z_0))^T e_k\right\rangle\\ &=\frac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0}\tilde{d}}|(O_{z_0}\nabla w(z_0))^T e_1|^2.\end{aligned}\tag{2.8}$$

Finally we obtain

$$\Delta\chi(z_0) = \frac{\kappa_{\omega_0}}{1-\kappa_{\omega_0}\tilde{d}}|(O_{z_0}\nabla w(z_0))^T e_1|^2 - \langle(\nabla d)(w(z_0)), \Delta w\rangle\,. \qquad \square$$

3. Proof of the main theorem

The main step to establish the main theorem is the following lemma.

Lemma 3.1. *Let $w = f(z)$ be a K-quasiconformal mapping of the unit disk onto a $C^{1,1}$ Jordan domain Ω satisfying the differential inequality*

$$|\Delta w| \leq B|\nabla w|^2, \;\; B \geq 0 \tag{3.1}$$

for some $B \geq 0$. Assume in addition that $w(0) = a_0 \in \Omega$. Then there exists a constant $C(K, \Omega, B, a) > 0$ such that

$$\left|\frac{\partial w}{\partial r}(t)\right| \geq C(K, \Omega, B, a_0) \text{ for almost every } t \in S^1. \tag{3.2}$$

Proof. Let us find $A > 0$ such that the function $\varphi_w(z) = -\frac{1}{A} + \frac{1}{A}e^{-Ad(w(z))}$ is subharmonic on $\{z : d(w(z)) < \frac{1}{2\kappa_0}\}$, where

$$\kappa_0 = \operatorname{ess\,sup}\{|\kappa_w| : w \in \gamma\}.$$

Let $\chi = -d(w(z))$. Combining (2.3), (2.5) and (3.1) we get

$$|\Delta\chi| \leq 2\kappa_0|\nabla w|^2 + B|\nabla w|^2 \leq (2\kappa_0 + B)K^2|\nabla\chi|^2. \tag{3.3}$$

Take

$$g(t) = -\frac{1}{A} + \frac{1}{A}e^{At}.$$

Then $\varphi_w(z) = g(\chi(z))$. Thus

$$\Delta\varphi_w = g''(\chi)|\nabla\chi|^2 + g'(\chi)\Delta\chi. \tag{3.4}$$

Since

$$g'(\chi) = e^{-Ad(w(z))} \tag{3.5}$$

and

$$g''(\chi) = Ae^{-Ad(w(z))}, \tag{3.6}$$

it follows that

$$\Delta\varphi_w \geq (A - (2\kappa_0 + B)K^2)|\nabla\chi|^2 e^{-Ad(u(z))}. \tag{3.7}$$

In order to have $\Delta\varphi_w \geq 0$, it is enough to take

$$A = (2\kappa_0 + B)K^2. \tag{3.8}$$

Choosing

$$\varrho = \max\left\{|z| : \operatorname{dist}(w(z), \gamma) = \frac{1}{2\kappa_0}\right\},$$

we have that φ_w satisfies the conditions of the following generalization of the Hopf lemma ([9]):

Lemma 3.2 ([10]). *Let φ satisfy $\Delta\varphi \geq 0$ in $R_\varrho = \{z : \varrho \leq |z| < 1\}$, $0 < \varrho < 1$, φ be continuous on $\overline{R_\varrho}$, $\varphi < 0$ in R_ϱ, $\varphi(t) = 0$ for $t \in S^1$. Assume that the radial derivative $\frac{\partial\varphi}{\partial r}$ exists almost everywhere on S^1. Set $M(\varphi, \varrho) = \max_{|z|=\varrho} \varphi(z)$. Then the following inequality holds*

$$\frac{\partial\varphi(t)}{\partial r} > \frac{2M(\varphi, \varrho)}{\varrho^2(1 - e^{1/\varrho^2-1})} \text{ for a.e. } t \in S^1. \tag{3.9}$$

We will make use of (3.9), but under some improvement for the class of quasiconformal harmonic mappings. The idea is to make the right-hand side of (3.9) independent of the mapping w for $\varphi = \varphi_w$.

We will say that a quasiconformal mapping $f : \mathbb{U} \mapsto \Omega$ is normalized if $f(1) = w_0$, $f(e^{2\pi i/3}) = w_1$ and $f(e^{-2\pi i/3}) = w_2$, where w_0w_1, w_1w_2 and w_2w_0 are arcs of $\gamma = \partial\Omega$ having the same length $|\gamma|/3$.

In what follows we will prove that, for the class $\mathcal{H}(\Omega, K, B)$ of normalized K-quasiconformal mappings, satisfying (3.1) for some $B \geq 0$, and mapping the unit disk onto the domain Ω, the inequality (3.9) holds uniformly (see (3.10)).

Let

$$\varrho := \sup\left\{|z| : \operatorname{dist}(w(z), \gamma) = \frac{1}{2\kappa_0}, w \in \mathcal{H}(\Omega, K, B)\right\}.$$

Then there exists a sequence $\{w_n\}$, $w_n \in \mathcal{H}(\Omega, K, B)$ such that

$$\varrho_n = \max\left\{|z| : \operatorname{dist}(w_n(z), \gamma) = \frac{1}{2\kappa_0}\right\},$$

and

$$\varrho = \lim_{n\to\infty} \varrho_n.$$

Now notice that if w_n is a sequence of normalized K-quasiconformal mappings of the unit disk onto Ω then, up to taking a subsequence, w_n is a locally uniformly convergent sequence converging to some quasiconformal mapping $w \in \mathcal{H}(\Omega, K, B)$. Under the condition on the boundary of Ω, by [27, Theorem 4.4] this sequence is uniformly convergent on $\mathbb{U}$. Then there exists a sequence z_n such that $\operatorname{dist}(w_n(z_n), \gamma) = \frac{1}{2\kappa_0}$, $\lim_{n\to\infty} z_n = z_0$ and $\varrho = |z_0|$. Since w_n converges uniformly to w, it follows that $\lim_{n\to\infty} w_n(z_n) = w(z_0)$, and $\operatorname{dist}(w(z_0), \gamma) = \frac{1}{2\kappa_0}$. This implies that $\varrho < 1$. Let now

$$M(\varrho) := \sup\{M(\varphi_w, \varrho), w \in \mathcal{H}(\Omega, K, B)\}.$$

Using a similar argument we obtain that there exists a uniformly convergent sequence w_n, converging to a mapping w_0, such that

$$M(\varrho) = \lim_{n\to\infty} M(\varphi_{w_n}, \varrho) = M(\varphi_{w_0}, \varrho).$$

Thus

$$M(\varrho) < 0.$$

Placing $M(\varrho)$ instead of $M(\varrho, \varphi)$ and φ_w instead of φ in (3.9), we obtain

$$\frac{\partial \varphi_w(t)}{\partial r} > \frac{2M(\varrho)}{\varrho^2(1 - e^{1/\varrho^2 - 1})} := C(K, \Omega, B) \text{ for a.e. } t \in S^1. \tag{3.10}$$

To continue observe that

$$\frac{\partial \varphi_w(t)}{\partial r} = e^{Ad(w(z))}|\nabla d| \left|\frac{\partial w}{\partial r}(t)\right| = e^{Ad(w(z))} \left|\frac{\partial w}{\partial r}(t)\right|.$$

Combining (3.8) and (3.10) we obtain for a.e. $t \in S^1$

$$\left|\frac{\partial w}{\partial r}(t)\right| = e^{-Ad(w(z))} \frac{\partial \varphi_w(t)}{\partial r} \geq e^{-K^2} \frac{2M(\varrho)}{\varrho^2(1 - e^{1/\varrho^2 - 1})}.$$

Lemma 3.1 is now proved for a normalized mapping w. If w is not normalized then we take the composition of w and an appropprieate Möbius transformation in order to obtain the desired inequality. The proof of Lemma 3.1 is complete. □

Conclusion of the proof of Theorem 1.4. In this setting w is harmonic, therefore $B = 0$. Assume first that $w \in C^1(\overline{\mathbb{U}})$. Let $l(\nabla w)(t) = ||w_z(t)| - |w_{\bar{z}}(t)||$. Since w is K-quasiconformal, according to (3.2) we have

$$l(\nabla w)(t) \geq \frac{|\nabla w(t)|}{K} \geq \frac{\left|\frac{\partial w}{\partial r}(t)\right|}{K} \geq \frac{C(K, \Omega, 0, a_0)}{K} \tag{3.11}$$

for $t \in S^1$. Therefore, having in mind Lewy's theorem ([23]), which states that $|w_z| > |w_{\bar{z}}|$ for $z \in \mathbb{U}$, we obtain for $t \in S^1$ that $|w_z(t)| \neq 0$ and hence

$$\frac{1}{|w_z|} \frac{C(K, \Omega, 0, a_0)}{K} + \frac{|w_{\bar{z}}|}{|w_z|} \leq 1, \ t \in S^1.$$

Since $w \in C^1(\overline{\mathbb{U}})$, it follows that the functions

$$a(z) := \frac{\overline{w_{\bar{z}}}}{w_z}, \quad b(z) := \frac{1}{w_z} \frac{C(K, \Omega, 0, a_0)}{K}$$

are well-defined holomorphic functions in the unit disk having a continuous extension to the boundary. As $|a| + |b|$ is bounded on the unit circle by 1, it follows that it is bounded on the whole unit disk by 1 because

$$|a(z)| + |b(z)| \leq P[|a|_{S^1}](z) + P[|b|_{S^1}](z) = P[|a|_{S^1} + |b|_{S^1}](z), \quad z \in \mathbb{U}.$$

This in turn implies that for every $z \in \mathbb{U}$

$$l(\nabla w)(z) \geq \frac{C(K, \Omega, 0, a_0)}{K} =: C(\Omega, K, a_0). \tag{3.12}$$

This yields that

$$C(K, \Omega, a_0) \leq \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|}, \quad z_1, z_2 \in \mathbb{U}.$$

Assume now that $w \notin C^1(\overline{\mathbb{U}})$. We begin with a definition.

Definition 3.3. *Let G be a domain in $\mathbb{C}$ and let $a \in \partial G$. We will say that $G_a \subset G$ is a ∂-neighborhood of a if there exists a disk $D(a,r) := \{z : |z-a| < r\}$ such that $D(a,r) \cap G \subset G_a$.*

Let $t = e^{i\beta} \in S^1$, so that $w(t) \in \partial\Omega$. Let γ be an arc-length parametrization of $\partial\Omega$ with $\gamma(s) = w(t)$. Since $\partial\Omega \in C^{1,1}$, there exists a ∂-neighborhood Ω_t of $w(t)$ with $C^{1,1}$ Jordan boundary such that

$$\Omega_t^\tau := \Omega_t + i\gamma'(s) \cdot \tau \subset \Omega, \text{ and } \partial\Omega_t^\tau \subset \Omega \text{ for } 0 < \tau \leq \tau_t \ (\tau_t > 0)\,. \tag{3.13}$$

An example of a family Ω_t^τ such that $\partial\Omega_t^\tau \in C^{1,1}$ and with the property (3.13) has been given in [13].

Let $a_t \in \Omega_t$ be arbitrary. Then $a_t + i\gamma'(s) \cdot \tau \in \Omega_t^\tau$. Take $U_\tau = f^{-1}(\Omega_t^\tau)$. Let η_t^τ be a conformal mapping of the unit disk onto U_τ such that $\eta_t^\tau(0) = f^{-1}(a_t + i\gamma'(s) \cdot \tau)$, and $\arg \frac{d\eta_t^\tau}{dz}(0) = 0$. Then the mapping

$$f_t^\tau(z) := f(\eta_t^\tau(z)) - i\gamma'(s) \cdot \tau$$

is a harmonic K-quasiconformal mapping of the unit disk onto Ω_t satisfying the condition $f_t^\tau(0) = a_t$. Moreover

$$f_t^\tau \in C^1(\overline{\mathbb{U}}).$$

Using the case $w \in C^1(\overline{\mathbb{U}})$, it follows that

$$|\nabla f_t^\tau(z)| \geq C(K, \Omega_t, a_t).$$

On the other hand

$$\lim_{\tau \to 0+} \nabla f_t^\tau(z) = \nabla(f \circ \eta_t)(z)$$

on the compact sets of $\mathbb{U}$ as well as

$$\lim_{\tau \to 0+} \frac{d\eta_t^\tau}{dz}(z) = \frac{d\eta_t}{dz}(z),$$

where η_t is a conformal mapping of the unit disk onto $U_0 = f^{-1}(\Omega_t)$ with $\eta_t(0) = f^{-1}(a_t)$. It follows that

$$|\nabla f_t(z)| \geq C(K, \Omega_t, a_t).$$

Applying the Schwarz reflexion principle to the mapping η_t and using the formula

$$\nabla(f \circ \eta_t)(z) = \nabla f \cdot \frac{d\eta_t}{dz}(z)$$

it follows that in some ∂-neighborhood $\tilde{U}_t$ of $t \in S^1$ with smooth boundary where $(D(t, r_t) \cap \mathbb{U} \subset \tilde{U}_t$ for some $r_t > 0)$, the function f satisfies the inequality

$$|\nabla f(z)| \geq \frac{C(K, \Omega_t, a_t)}{\max\{|\eta_t'(\zeta)| : \zeta \in \overline{\tilde{U}_t}\}} =: \tilde{C}(K, \Omega_t, a_t) > 0. \tag{3.14}$$

Since S^1 is a compact set, it can be covered by a finite family $\partial\tilde{U}_{t_j} \cap S^1 \cap D(t, r_t/2)$, $j = 1, \dots, m$. It follows that the inequality

$$|\nabla f(z)| \geq \min\{\tilde{C}(K, \Omega_{t_j}, a_{t_j}) : j = 1, \dots, m\} =: \tilde{C}(K, \Omega, a_0) > 0 \tag{3.15}$$

holds in the annulus

$$\tilde{R} = \left\{ z : 1 - \frac{\sqrt{3}}{2} \min_{1 \leq j \leq m} r_{t_j} < |z| < 1 \right\} \subset \bigcup_{j=1}^{m} \tilde{U}_{t_j}.$$

This implies that the subharmonic function $S = |a(z)| + |b(z)|$ is bounded in $\mathbb{U}$. According to the maximum principle, it is bounded by 1 in the whole unit disk. This in turn implies again (3.12) and consequently

$$\frac{C(K, \Omega, a_0)}{K} |z_1 - z_2| \leq |w(z_1) - w(z_2)|, \quad z_1, z_2 \in \mathbb{U}. \qquad \square$$

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