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H^1 and *BMO* for certain locally doubling metric measure spaces

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Abstract. Suppose that (M,ρ,μ) is a metric measure space, which possesses two "geometric" properties, called "isoperimetric" property and approximate midpoint property, and that the measure μ is locally doubling. The isoperimetric property implies that the volume of balls grows at least exponentially with the radius. Hence the measure μ is not globally doubling. In this paper we define an atomic Hardy space $H^1(\mu)$, where atoms are supported only on "small balls", and a corresponding space $BMO(\mu)$ of functions of "bounded mean oscillation", where the control is only on the oscillation over small balls. We prove that $BMO(\mu)$ is the dual of $H^1(\mu)$ and that an inequality of John–Nirenberg type on small balls holds for functions in $BMO(\mu)$. Furthermore, we show that the $L^p(\mu)$ spaces are intermediate spaces between $H^1(\mu)$ and $BMO(\mu)$, and we develop a theory of singular integral operators acting on function spaces on M. Finally, we show that our theory is strong enough to give $H^1(\mu)$-$L^1(\mu)$ and $L^\infty(\mu)$-$BMO(\mu)$ estimates for various interesting operators on Riemannian manifolds and symmetric spaces which are unbounded on $L^1(\mu)$ and on $L^\infty(\mu)$.

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1. Introduction

Suppose that (M, ρ, μ) is a metric measure space. Assume temporarily that μ is a doubling measure; then (M, ρ, μ) is a space of homogeneous type in the sense of Coifman and Weiss. Harmonic analysis on spaces of homogeneous type has been the object of many investigations. In particular, the atomic Hardy space $H^1(\mu)$ and the space $BMO(\mu)$ of functions of bounded mean oscillation have been defined and studied in this setting. We briefly recall their definitions.

An atom a is a function in $L^1(\mu)$ supported in a ball B which satisfies appropriate "size" and cancellation conditions. Then $H^1(\mu)$ is the space of all functions in $L^1(\mu)$ that admit a decomposition of the form $\sum_j \lambda_j \, a_j$, where the a_j's are atoms and the sequence of complex numbers $\{\lambda_j\}$ is summable.

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A locally integrable function f is in $BMO(\mu)$ if

$$\sup_B \frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu < \infty,$$

where the supremum is taken over *all* balls B, and f_B denotes the average of f over B.

These spaces enjoy many of the properties of their Euclidean counterparts. In particular, the topological dual of $H^1(\mu)$ is isomorphic to $BMO(\mu)$, an inequality of John–Nirenberg type holds for functions in $BMO(\mu)$, the spaces $L^p(\mu)$ are intermediate spaces between $H^1(\mu)$ and $BMO(\mu)$ for the real and the complex interpolation methods. Furthermore, some important operators, which are bounded on $L^p(\mu)$ for all p in $(1, \infty)$, but otherwise unbounded on $L^1(\mu)$ and on $L^\infty(\mu)$, turn out to be bounded from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$. We remark that the doubling property is key in establishing these results.

There is a huge literature on this subject: we refer the reader to [15,42] and the references therein for further information.

There are interesting cases where μ is not doubling; then μ may or may not be locally doubling. An important case in which μ is not even locally doubling is that of nondoubling measures of polynomial growth treated, for instance, in [38,44, 46], where new spaces H^1 and BMO are defined, and a rich theory is developed (see also [34] for more general measures on $\mathbb{R}^n$). We also mention recent works of X.T. Duong and L. Yan [19, 20], who define a Hardy space H^1 and a space BMO of bounded mean oscillation associated to a given operator satisfying suitable estimates. This is done in metric measure spaces with the doubling property, but it is a remarkable fact that the theory works also for "bad domains" in the ambient space, to which the restriction of the measure μ may be nondoubling.

In this paper we consider the case where $\mu(M) = \infty$, and μ is a nondoubling locally doubling measure. By this we roughly mean that for every R in $\mathbb{R}^+$ balls of radius at most R satisfy a doubling condition, with doubling constant that may depend on R (see (2.1) in Section 2 for the precise definition). Important examples of this situation are complete Riemannian manifolds with Ricci curvature bounded from below, a class which includes all Riemannian symmetric spaces of the noncompact type and Damek–Ricci spaces. In recent years, analysis on complete Riemannian manifolds satisfying the local doubling condition has been the object of many investigations. For instance, see [40] and the references therein for the equivalence between a scale-invariant parabolic Harnack inequality and a scaled Poincaré inequality, and [2, 16, 39] for recents results on the boundedness of Riesz transforms on such manifolds.

Our approach to the case of locally doubling measures is inspired by a result of A.D. Ionescu [29] on rank one symmetric spaces of the noncompact type and by a recent paper of the second and third named authors concerning the analysis of the Ornstein–Uhlenbeck operator [35].

For each "scale" b in $\mathbb{R}^+$, we define spaces $H^1_b(\mu)$ and $BMO_b(\mu)$ much as in the case of spaces of homogeneous type, the only difference being that we require that the balls involved have at most radius b. So, for instance, an $H^1_b(\mu)$ atom is an atom supported in a ball of radius at most b. We remark that in the case where M is a symmetric space of the noncompact type and real rank one, the space $BMO_1(\mu)$ agrees with the space defined by Ionescu. Ionescu also proved that if p is in $(1,2)$, then $L^p(\mu)$ is an interpolation space between $L^2(\mu)$ and $BMO_1(\mu)$ for the complex method of interpolation. In the case where M is a complete noncompact Riemannian manifold with locally doubling Riemannian measure and satisfying certain additional assumptions E. Russ [39] defined a Hardy space that agrees with the space $H^1_1(\mu)$ defined above, but he did not investigate its structural properties.

We prove that under a mild "geometric" assumption, which we call property (AM) (see Section 2), $H^1_b(\mu)$ and $BMO_b(\mu)$, in fact, do not depend on the parameter b provided that b is large enough (see Section 4). Furthermore, we show that $BMO_b(\mu)$ is isomorphic to the topological dual of $H^1_b(\mu)$ (see Section 6), and that functions in $BMO_b(\mu)$ satisfy an inequality of John–Nirenberg type (see Section 5).

As far as interpolation is concerned, there is no reason to believe that in this generality $L^p(\mu)$ spaces with p in $(1,\infty)$ are interpolation spaces between $H^1_b(\mu)$ and $BMO_b(\mu)$. However, a remarkable feature of these spaces is that this is true under a simple geometric assumption on M, called property (I). Roughly speaking, M possesses property (I) if a fixed ratio of the measure of any bounded open set is concentrated near its boundary. If M possesses property (I), then a basic relative distributional inequality for the local sharp function and the local Hardy–Littlewood maximal function holds. We prove this in Section 7, by adapting to our setting some ideas of Ionescu [29]. We remark that our approach, which makes use of the dyadic cubes of M. Christ and G. David [14, 18], simplifies considerably the original proof in [29]. As a consequence of the relative distributional inequality we prove an interpolation result for analytic families of operators, analogous to that proved by C. Fefferman and E.M. Stein [23] in the classical setting.

An interesting application of the aforementioned interpolation result is to singular integral operators (Theorem 8.2). We prove that if $\mathcal{T}$ is a bounded self adjoint operator on $L^2(\mu)$ and its kernel k is a locally integrable function off the diagonal in $M \times M$ and satisfies a *local Hörmander type condition* (*i.e.* if $\nu_k < \infty$ and $\upsilon_k < \infty$, where ν_k and υ_k are defined in the statement of Theorem 8.2), then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1,\infty)$, from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$.

It is interesting to speculate about the range of applicability of the theory we develop. In particular, a natural problem is to find conditions (possibly easy to verify) under which a complete Riemannian manifold possesses all the three properties, local doubling, (I), and (AM), needed to prove the results of Sections 2-8. This problem is considered in Section 9. Suppose that M is a complete Riemannian manifold with Riemannian distance ρ and Riemannian density μ. A known fact, which is a straightforward consequence of the Bishop–Gromov comparison Theorem, is that if M has Ricci curvature bounded from below, then (M,ρ,μ) is locally doubling. Furthermore, since ρ is a length distance, (M,ρ,μ) has property (AM). We shall prove that M possesses property (I) if and only if the Cheeger isoperimetric constant $h(M)$ (see (9.1) for the definition) is strictly positive. As a consequence we shall prove that if M has Ricci curvature bounded from below, then M possesses property (I) if and only if the bottom $b(M)$ of the spectrum of M is strictly positive.

In Section 10 we apply our theory to obtain endpoint estimates for some operators on Riemannian manifolds and symmetric spaces. We show that, if the manifold has Ricci curvature bounded from below and the bottom of the spectrum is positive, then a class a spectral multipliers of the Laplacian is bounded from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$. Similar endpoint estimates hold also for a class of spherical multipliers and for "localized" Riesz transforms on noncompact Riemannian symmetric spaces. Our results complement earlier results of M. Taylor [43], J.Ph. Anker [1] and Russ [39]. Similar results on graphs with bounded geometry will appear elsewhere.

To keep the length of this paper reasonable we have considered only the case where $\mu(M) < \infty$. A detailed study of the case where $\mu(M) < \infty$ will appear in [11].

Finally, we would like to mention that, after this paper was completed, M. Taylor kindly sent us a preprint in which he develops a theory of "local" Hardy and BMO spaces on Riemannian manifolds which satisfy bounded curvature assumptions stronger than those considered in Section 9. Taylor's theory is a generalization of the theory developed by D. Goldberg in $\mathbb{R}^n$ [24]. We recall that the definition of the local atomic Hardy space $\mathfrak{h}^1(\mathbb{R}^n)$ of Goldberg is similar to that of the classical atomic Hardy space $H^1(\mathbb{R}^n)$; the only difference is that atoms supported in balls of radius larger than one do not necessarily have mean value zero. The dual of $\mathfrak{h}^1(\mathbb{R}^n)$ is the space $\mathfrak{bmo}(\mathbb{R}^n)$ of functions f such that both the mean oscillation on balls of radius at most one and the average of $|f|$ on balls of radius one are bounded. By imitating Goldberg's definition, one can define local spaces $\mathfrak{h}^1(\mu)$ and $\mathfrak{bmo}(\mu)$ also in the context of a locally doubling metric measure space (M, ρ, μ) that satisfies assumption (AM). It is easy to see that $\mathfrak{h}^1(\mu)$ is strictly larger than our $H^1(\mu)$ and $\mathfrak{bmo}(\mu)$ is strictly smaller than our $BMO(\mu)$. For instance, the characteristic function of a ball is in $H^1(\mu)$ but not in $\mathfrak{h}^1(\mu)$ and the function $x \mapsto \rho(x, x_0)$, where x_0 is a fixed point in M, is in $BMO(\mu)$ but not in $\mathfrak{bmo}(\mu)$. It is known that in $\mathbb{R}^n$ there are operators, *e.g.* the imaginary powers of the Laplacian, which are bounded from the classical space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ but fail to be bounded from $\mathfrak{h}^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The same phenomenon occurs in our context: there are metric measure spaces in which "natural" operators are bounded from $H^1(\mu)$ to $L^1(\mu)$ but not from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$. For instance, this happens for the imaginary powers of the Ornstein-Uhlenbeck operator on Gauss space [35].

2. Geometric assumptions

Suppose that (M, ρ, μ) is a metric measure space, and denote by $\mathcal{B}$ the family of all balls on M. We assume that $\mu(M) > 0$ and that every ball has finite measure. For each B in $\mathcal{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, we denote by $\kappa\, B$ the ball with centre c_B and radius $\kappa\, r_B$. For each b

in $\mathbb{R}^+$, we denote by $\mathcal{B}_b$ the family of all balls B in $\mathcal{B}$ such that $r_B \leq b$. For any subset A of M and each κ in $\mathbb{R}^+$ we denote by A_κ and A^κ the sets

$$\big\{x \in A : \rho(x, A^c) \leq \kappa\big\} \qquad \text{and} \qquad \big\{x \in A : \rho(x, A^c) > \kappa\big\}$$

respectively.

In Sections 2-8 we assume that M is *unbounded* and possesses the following properties:

(i) *local doubling property* (LD): for every b in $\mathbb{R}^+$ there exists a constant D_b such that
$$\mu\big(2B\big) \leq D_b\, \mu\big(B\big) \quad \forall\, B \in \mathcal{B}_b. \tag{2.1}$$
This property is often called *local doubling condition* in the literature, and we adhere to this terminology. Note that if (2.1) holds and M is bounded, then μ is doubling.

(ii) *isoperimetric property* (I): there exist κ_0 and C in $\mathbb{R}^+$ such that for every bounded open set A
$$\mu\big(A_\kappa\big) \geq C\, \kappa\, \mu(A) \quad \forall\, \kappa \in (0, \kappa_0]. \tag{2.2}$$
Suppose that M has property (I). For each t in $(0, \kappa_0]$ we denote by C_t the supremum over all constants C for which (2.2) holds for all κ in $(0, t]$. Then we define I_M by
$$I_M = \sup\big\{C_t : t \in (0, \kappa_0]\big\}.$$
Note that the function $t \mapsto C_t$ is decreasing on $(0, \kappa_0]$, so that
$$I_M = \lim_{t \to 0^+} C_t; \tag{2.3}$$

(iii) *property* (AM) (approximate midpoint property): there exist R_0 in $[0, \infty)$ and β in $(1/2, 1)$ such that for every pair of points x and y in M with $\rho(x, y) > R_0$ there exists a point z in M such that $\rho(x, z) < \beta\, \rho(x, y)$ and $\rho(y, z) < \beta\, \rho(x, y)$.
This is clearly equivalent to the requirement that there exists a ball B containing x and y such that $r_B < \beta\, \rho(x, y)$.

Remark 2.1. Observe that the isoperimetric property (I) implies that for every open set A of *finite measure*

$$\mu\big(A_\kappa\big) \geq C\, \kappa\, \mu(A) \quad \forall\, \kappa \in (0, \kappa_0],$$

where κ_0 and C are as in (2.2).

Indeed, suppose that A is an open set of finite measure. Fix a reference point o in M and denote by $B(o, j)$ the ball with centre o and radius j, and by $A(j)$ the set $A \cap B(o, j)$. For each κ in $(0, \kappa_0]$ denote by $A_{j,\kappa}$ the set

$$\big\{x \in A(j) : \rho\big(x, B(o, j)^c\big) \leq \kappa, \rho(x, A^c) > \kappa\big\}.$$

First we prove that
$$\lim_{j\to\infty} \mu\big(A_{j,\kappa}\big) = 0. \tag{2.4}$$
Since $\mu(A) < \infty$, for each $\epsilon > 0$ there exists J such that
$$\mu\big(A \cap B(o, J)^c\big) < \epsilon.$$
Now, if $j \geq J + \kappa$ and x is in $A_{j,\kappa}$, then x belongs also to $A \cap B(o, J)^c$, whence $\mu(A_{j,\kappa}) < \epsilon$ for all $j \geq J$, as required.

Observe that $A_{j,\kappa}$ is contained in $A(j)_\kappa$ and that $A_{j,\kappa} = A(j)_\kappa \setminus \big(B(o, j) \cap A_\kappa\big)$. Therefore
$$\mu\big(B(o, j) \cap A_\kappa\big) = \mu\big(A(j)_\kappa\big) - \mu\big(A_{j,\kappa}\big). \tag{2.5}$$
Since $\mu(A_\kappa) = \lim_{j\to\infty} \mu\big(B(o, j) \cap A_\kappa\big)$,
$$\mu(A_\kappa) = \lim_{j\to\infty} \mu\big(A(j)_\kappa\big)$$
by (2.5) and (2.4). Since $A(j)$ is a bounded open set, we may conclude that
$$\begin{aligned}\mu(A_\kappa) &\geq \lim_{j\to\infty} C\,\kappa\,\mu\big(A(j)\big)\\ &= C\,\kappa\,\mu\big(A\big),\end{aligned}$$
as required.

Remark 2.2. The local doubling property is needed for all the results in this paper, but many results in Sections 2-8 depend only on some but not all the properties (i)-(iii). In particular, Theorem 3.2 and Proposition 3.4 require only the local doubling property, Propositions 3.1 and 3.5, Lemma 7.2 and Theorem 7.3, which are key in proving the interpolation result Theorem 7.4, require property (I), but not property (AM), all the results in Sections 4, 5 and 6 require property (AM) but not property (I). In particular, property (AM) is key to prove the scale invariance of the spaces $H^1(\mu)$ and $BMO(\mu)$ defined below (Proposition 4.3). Finally, all the properties (i)-(iii) above are needed for the interpolation results in Section 7 and for the results in Section 8.

Remark 2.3. The local doubling property implies that for each $\tau \geq 2$ and for each b in $\mathbb{R}^+$ there exists a constant C such that
$$\mu\big(B'\big) \leq C\,\mu(B) \tag{2.6}$$
for each pair of balls B and B', with $B \subset B'$, B in $\mathcal{B}_b$, and $r_{B'} \leq \tau\, r_B$. We shall denote by $D_{\tau,b}$ the smallest constant for which (2.6) holds. In particular, if (2.6) holds (with the same constant) for all balls B in $\mathcal{B}$, then μ is doubling and we shall denote by $D_{\tau,\infty}$ the smallest constant for which (2.6) holds.

Remark 2.4. There are various "structural constants" which appear explicitly in the statements of some of our results. For the reader's convenience we give here a list of all the relevant constants used is Sections 2-8:

$D_{\tau,b}$	$\tau \geq 2$, $b \in \mathbb{R}^+ \cup \{\infty\}$ (see Remark 2.3)
I_M	the isoperimetric constant (see (ii) above)
R_0 and β	appear in the (AM) property (see (iii) above)
δ, C_1 and a_0	appear in the construction of dyadic cubes (see Theorem 3.2).

3. Preliminary results

Roughly speaking, if M has property (I), then a fixed ratio of the measure of any bounded open set is concentrated near its boundary. The following proposition contains a quantitative version of this statement.

Proposition 3.1. *The following hold:*

(i) *the volume growth of M is at least exponential;*
(ii) *for every bounded open set A*

$$\mu(A_t) \geq \left(1 - \mathrm{e}^{-I_M t}\right) \mu(A) \quad \forall\, t \in \mathbb{R}^+.$$

Proof. First we prove (i). Denote by o a reference point in M. For every $r > 0$ denote by V_r the measure of the ball with centre o and radius r. It is straightforward to check that $B(o,r)_\kappa \subset B(o,r) \setminus B(o, r-\kappa)$, so that for all sufficiently large r

$$\begin{aligned} V_r - V_{r-\kappa} &\geq \mu\big(B(o,r)_\kappa\big) \\ &\geq C\,\kappa\, V_r \quad \forall\, \kappa \in (0, \kappa_0] \end{aligned}$$

by property (I) (C and κ_0 are as in (2.2)). Hence

$$\begin{aligned} V_r &\geq C\,\kappa\, V_r + V_{r-\kappa} \\ &\geq \eta\, V_{r-\kappa} \quad \forall\, \kappa \in (0, \kappa_0], \end{aligned}$$

where $\eta = C\kappa + 1$. Denote by n the positive integer for which $r - n\kappa$ is in $[\kappa, 2\kappa)$. Then

$$\begin{aligned} V_r &\geq \eta^n\, V_{r-n\kappa} \\ &\geq \eta^{r/\kappa - 2}\, V_\kappa, \end{aligned}$$

as required.

Now we prove (ii). Suppose that A is a bounded open subset of M, and that t is in $(0, \kappa_0]$. Note that A^s is a bounded open subset of M. It is straightforward to check that $(A^s)_t \subset A^s \setminus A^{s+t}$. Therefore

$$\begin{aligned} \mu(A^{s+t}) - \mu(A^s) &\leq -\mu\big((A^s)_t\big) \\ &\leq -C_t\, t\, \mu(A^s) \end{aligned}$$

by property (I) (see (2.2) above). Since $s \mapsto \mu(A^s)$ is monotonic, it is differentiable almost everywhere. The inequality above and (2.3) imply that for almost every s in $\mathbb{R}^+$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mu(A^s) \leq -I_M\, \mu(A^s).$$

Notice also that $\lim_{s\to 0^+} \mu(A^s) = \mu(A)$. Therefore

$$\mu(A^s) \leq \mathrm{e}^{-I_M\, s}\, \mu(A) \quad \forall\, s \in \mathbb{R}^+,$$

and finally

$$\mu(A_s) \geq \big(1 - \mathrm{e}^{-I_M\, s}\big)\, \mu(A) \quad \forall\, s \in \mathbb{R}^+,$$

as required. □

We shall make use of the analogues in our setting of the so-called dyadic cubes Q^k_α introduced by G. David and M. Christ [14, 18] on spaces of homogeneous type. It may help to think of Q^k_α as being essentially a cube of diameter δ^k with centre z^k_α.

Theorem 3.2. *There exist a collection of open subsets* $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ *and constants* δ *in* $(0, 1)$*,* a_0*,* C_1 *in* $\mathbb{R}^+$ *such that*

(i) $\bigcup_\alpha Q^k_\alpha$ *is a set of full measure in* M *for each* k *in* $\mathbb{Z}$*;*

(ii) *if* $\ell \geq k$*, then either* $Q^\ell_\beta \subset Q^k_\alpha$ *or* $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$*;*

(iii) *for each* (k, α) *and each* $\ell < k$ *there is a unique* β *such that* $Q^k_\alpha \subset Q^\ell_\beta$*;*

(iv) $\operatorname{diam}(Q^k_\alpha) \leq C_1\, \delta^k$*;*

(v) Q^k_α *contains the ball* $B(z^k_\alpha, a_0\, \delta^k)$.

Proof. The proof of (ii)–(v) is as in [14]. In fact, the proof depends only on the metric structure of the space and not on the properties of the measure μ and is even easier in our case, because ρ is a genuine distance, rather than a quasi-distance.

The proof of (i) is again as in [14]; observe that only a local doubling property is used in the proof. □

Note that (iv) and (v) imply that for every integer k and each α in I_k

$$B(z^k_\alpha, a_0\, \delta^k) \subset Q^k_\alpha \subset B(z^k_\alpha, C_1\, \delta^k).$$

Remark 3.3. When we use dyadic cubes, we implicitly assume that for each k in $\mathbb{Z}$ the set $M \setminus \bigcup_{\alpha\in I_k} Q^k_\alpha$ has been permanently deleted from the space.

We shall denote by $\mathcal{Q}^k$ the class of all dyadic cubes of "resolution" k, *i.e.*, the family of cubes $\{Q^k_\alpha : \alpha \in I_k\}$, and by $\mathcal{Q}$ the set of all dyadic cubes. We shall need the following additional properties of dyadic cubes.

Proposition 3.4. *Suppose that b is in $\mathbb{R}^+$ and that ν is in $\mathbb{Z}$, and let C_1 and δ are as in Theorem* 3.2. *The following hold:*

(i) *suppose that Q is in $\mathcal{Q}^k$ for some $k \geq \nu$, and that B is a ball such that $c_B \in Q$. If $r_B \geq C_1\,\delta^k$, then*

$$\mu(B \cap Q) = \mu(Q); \tag{3.1}$$

if $r_B < C_1\,\delta^k$, then

$$\mu(B \cap Q) \geq D^{-1}_{C_1/(a_0\delta),a_0\delta^\nu}\,\mu(B); \tag{3.2}$$

(ii) *suppose that τ is in $[2,\infty)$. For each Q in $\mathcal{Q}$ the space $\big(Q, \rho_{|Q}, \mu_{|Q}\big)$ is of homogeneous type. Denote by $D^Q_{\tau,\infty}$ its doubling constant (see Remark 2.3 for the definition). Then*

$$\sup\left\{D^Q_{\tau,\infty} : Q \in \bigcup_{k=\nu}^{\infty} \mathcal{Q}^k\right\} \leq D_{\tau,C_1\delta^\nu}\,D_{C_1/(a_0\delta),a_0\delta^\nu};$$

(iii) *for each ball B in $\mathcal{B}_b$, let k be the integer such that $\delta^k \leq r_B < \delta^{k-1}$, and and let $\widetilde{B}$ denote the ball with centre c_B and radius $\big(1 + C_1\big)\,r_B$. Then $\widetilde{B}$ contains all dyadic cubes in $\mathcal{Q}^k$ that intersect B and*

$$\mu(\widetilde{B}) \leq D_{1+C_1,b}\,\mu(B);$$

(iv) *suppose that B is in $\mathcal{B}_b$, and that k is an integer such that $\delta^k \leq r_B < \delta^{k-1}$. Then there are at most $D_{(1+C_1)/(a_0\delta),b}$ dyadic cubes in $\mathcal{Q}^k$ that intersect B.*

Proof. First we prove (i). Our proof is a version of the proof given by Christ [14, page 613] that keeps track of the various structural constants involved.

First we prove (3.1). By Theorem 3.2 (iv) the diameter of Q is at most $C_1\,\delta^k$, so that $Q \subset B$, whence $B \cap Q = Q$, and the required formula is obvious.

To prove (3.2), denote by j the unique integer such that

$$\delta^j < \frac{r_B}{C_1} \leq \delta^{j-1}$$

and by Q^j_β the unique dyadic cube of resolution j that contains c_B. Then $j \geq k$, because $C_1\,\delta^j < r_B \leq C_1\delta^k$. The cubes Q^j_β and Q have nonempty intersection, because, they both contain c_B. Thus $Q^j_\beta \subset Q$. By Theorem 3.2 (iv) the diameter of Q^j_β is at most $C_1\,\delta^j$, which is $<r_B$ by the definition of j, so that $Q^j_\beta \subset B$. Therefore $Q^j_\beta \subset B \cap Q$, and

$$\begin{aligned}\mu(B \cap Q) &\geq \mu\big(Q^j_\beta\big)\\ &\geq \mu\big(B(z^j_\beta, a_0\,\delta^j)\big).\end{aligned}$$

Observe that $\frac{r_B}{a_0\,\delta^j} \leq \frac{C_1}{a_0\,\delta}$ and that $B(z^j_\beta, a_0\,\delta^j) \subset B$. Hence

$$\mu\big(B(z^j_\beta, a_0\,\delta^j)\big) \geq D^{-1}_{C_1/(a_0\delta),a_0\delta^\nu}\,\mu(B),$$

as required to conclude the proof of (3.2), and of (i).

Next we prove (ii). Suppose that Q is a dyadic cube in $\mathcal{Q}^k$, with $k \geq \nu$. Suppose that B and B' are balls in $\mathcal{B}$ with $B \subset B'$ such that c_B and $c_{B'}$ belong to Q and $r_{B'} \leq \tau\, r_B$. We treat the cases where $C_1\,\delta^k \leq r_B$ and $r_B < C_1\,\delta^k$ separately.

If $C_1\,\delta^k \leq r_B$, then $\mu(B' \cap Q) = \mu(Q) = \mu(B \cap Q)$.

If $r_B < C_1\,\delta^k$, then

$$\begin{aligned}\mu(B' \cap Q) &\leq \mu(B')\\ &\leq D_{\tau,C_1\delta^k}\,\mu(B)\\ &\leq D_{\tau,C_1\delta^k}\,D_{C_1/(a_0\delta),a_0\delta^\nu}\,\mu(B \cap Q),\end{aligned}$$

by the local doubling property of M and (3.2). Therefore Q is a homogeneous space with doubling constant at most $D_{\tau,C_1\delta^k}\,D_{C_1/(a_0\delta),a_0\delta^\nu}$. Since $k \geq \nu$, these doubling constants are dominated by $D_{\tau,C_1\delta^\nu}\,D_{C_1/(a_0\delta),a_0\delta^\nu}$, as required.

Now we prove (iii). Denote by Q a cube in $\mathcal{Q}^k$ that intersects B. By the triangle inequality and Theorem 3.2 (iv), Q is contained in $\widetilde{B}$. The required estimate of the measure of $\widetilde{B}$ follows from the local doubling condition (see Remark 2.3).

Finally we prove (iv). Denote by $Q_1, \ldots, Q_N$ the cubes in $\mathcal{Q}^k$ that intersect B. By (iii) each of these cubes is contained in $\widetilde{B}$. By Theorem 3.2 (v) each cube Q_j contains a ball, B'_j say, of radius $a_0\,\delta^k$, and these balls are pairwise disjoint because they are contained in disjoint dyadic cubes. By the local doubling condition $\mu(\widetilde{B}) \leq D_{(1+C_1)/(a_0\delta),b}\,\mu(B'_j)$ for all j. Therefore

$$\begin{aligned}N\,\mu(\widetilde{B}) &\leq D_{(1+C_1)/(a_0\delta),b}\sum_{j=1}^N \mu(B'_j)\\ &= D_{(1+C_1)/(a_0\delta),b}\,\mu\left(\bigcup_{j=1}^N B'_j\right)\\ &\leq D_{(1+C_1)/(a_0\delta),b}\,\mu(\widetilde{B}),\end{aligned}$$

from which the desired estimate follows. □

Our next result is a covering property enjoyed by spaces with property (I). It is key in proving relative distributional inequalities for the sharp maximal operator (see Lemma 7.2 below).

Proposition 3.5. *Suppose that ν is an integer. For every κ in $\mathbb{R}^+$, every open subset A of M of finite measure and every collection $\mathcal{C}$ of dyadic cubes of resolution at least ν such that $\bigcup_{Q\in\mathcal{C}} Q = A$, there exist mutually disjoint cubes $Q_1, \ldots, Q_k$ in $\mathcal{C}$ such that*

(i) $\sum_{j=1}^{k} \mu(Q_j) \geq \big(1 - \mathrm{e}^{-I_M\,\kappa}\big)\,\mu(A)/2$;

(ii) $\rho(Q_j, A^c) \leq \kappa$ *for every* j *in* $\{1, \ldots, k\}$.

Proof. Denote by $\widetilde{\mathcal{C}}$ the subcollection of all cubes in $\mathcal{C}$ that intersect A_κ. Clearly the cubes in $\widetilde{\mathcal{C}}$ cover A_κ and satisfy (ii).

Next we prove (i). Since two dyadic cubes are either disjoint or contained one in the other, we may consider the sequence $\{Q_j\}$ of cubes in $\widetilde{\mathcal{C}}$ which are not contained in any other cube of $\widetilde{\mathcal{C}}$. The existence of these "maximal" cubes is guaranteed by the assumption that the resolution of the cubes in $\mathcal{C}$ is bounded from below. The cubes $\{Q_j\}$ are mutually disjoint and cover A_κ. Therefore

$$\begin{aligned} \mu(A) &\geq \sum_j \mu(Q_j) \\ &\geq \mu\big(A_\kappa\big) \\ &\geq \big(1 - \mathrm{e}^{-I_M\,\kappa}\big)\,\mu(A), \end{aligned} \tag{3.3}$$

where the last inequality holds because M possesses property (I). To conclude the proof of (i) take k so large that $\sum_{j=1}^{k} \mu(Q_j) \geq (1/2) \sum_{j=1}^{\infty} \mu(Q_j)$. Then

$$\sum_{j=1}^{k} \mu(Q_j) \geq \big(1 - \mathrm{e}^{-I_M\,\kappa}\big)\,\mu(A)/2,$$

as required. □

4. The Hardy space H^1

Definition 4.1. Suppose that r is in $(1, \infty]$. A $(1, r)$-*atom* a is a function in $L^1(\mu)$ supported in a ball B in $\mathcal{B}$ with the following properties:

(i) $\|a\|_\infty \leq \mu(B)^{-1}$ if r is equal to ∞ and

$$\left(\frac{1}{\mu(B)} \int_B |a|^r \,\mathrm{d}\mu\right)^{1/r} \leq \mu(B)^{-1}$$

if r is in $(1, \infty)$;

(ii) $\displaystyle\int_B a \,\mathrm{d}\mu = 0$.

Definition 4.2. Suppose that b is in $\mathbb{R}^+$. The *Hardy space* $H_b^{1,r}(\mu)$ is the space of all functions g in $L^1(\mu)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k\, a_k, \tag{4.1}$$

where a_k is a $(1,r)$-atom *supported in a ball* B *of* $\mathcal{B}_b$, and $\sum_{k=1}^{\infty}|\lambda_k| < \infty$. The norm $\|g\|_{H_b^{1,r}(\mu)}$ of g is the infimum of $\sum_{k=1}^{\infty}|\lambda_k|$ over all decompositions (4.1) of g.

Clearly a function in $H_c^{1,r}(\mu)$ is in $H_b^{1,r}(\mu)$ for all $c < b$. We shall prove in Proposition 4.3 below that, in fact, the reverse inclusion holds whenever c is large enough. Hence for b large the space $H_b^{1,r}(\mu)$ does not depend on the parameter b, and for each pair of sufficiently large parameters b and c the norms $\|\cdot\|_{H_b^{1,r}(\mu)}$ and $\|\cdot\|_{H_c^{1,r}(\mu)}$ are equivalent.

There are cases where $H_c^{1,r}(\mu)$ and $H_b^{1,r}(\mu)$ are isomorphic spaces for each pair of parameters c and b. This happens, for instance, if M is the upper half plane, ρ the Poincaré metric and μ the associated Riemannian measure. However, if M is a homogeneous tree of degree $q \geq 1$, ρ denotes the natural distance and μ the counting measure, it is straightforward to check that $H_1^1(\mu)$ consists of the null function only, whereas $H_2^1(\mu)$ is a much richer space.

Recall that R_0 and β are the constants which appear in the definition of the (AM) property.

Proposition 4.3. *Suppose that r is in* $(1,\infty]$, *b and c are in* $\mathbb{R}^+$ *and satisfy* $R_0/(1-\beta) < c < b$. *The following hold:*

(i) *there exist a constant* C *and a nonnegative integer* N, *depending only on* M, b *and* c, *such that for each ball* B *in* $\mathcal{B}_b$ *and each* $(1,r)$*-atom* a *supported in* B *there exist at most* N $(1,r)$*-atoms* $a_1,\dots,a_N$ *with supports contained in balls* $B_1,\dots,B_N$ *in* $\mathcal{B}_c$ *and* N *constants* $\lambda_1,\dots,\lambda_N$ *such that* $|\lambda_j| \leq C$,

$$a = \sum_{j=1}^{N} \lambda_j\, a_j \qquad \textit{and} \qquad \|a\|_{H_c^{1,r}(\mu)} \leq C\,N;$$

(ii) *a function* f *is in* $H_c^{1,r}(\mu)$ *if and only if* f *is in* $H_b^{1,r}(\mu)$. *Furthermore, there exists a constant* C *such that*

$$\|f\|_{H_b^{1,r}(\mu)} \leq \|f\|_{H_c^{1,r}(\mu)} \leq C\,\|f\|_{H_b^{1,r}(\mu)} \quad \forall\, f \in H_c^{1,r}(\mu).$$

Proof. Choose β' in $(0, 1-\beta)$ such that $R_0/\beta' < c$.

First we prove (i). Suppose that B is a ball in $\mathcal{B}_b$ and that $r_B > c$, for otherwise there is nothing to prove. Denote by $\{z_1,\dots,z_{N_1}\}$ a maximal set of points in B such that $\rho(z_j,z_k) \geq \beta' r_B$ for all $j \neq k$ and each point of B is at distance at most $\beta' r_B$ from the set $\{z_1,\dots,z_{N_1}\}$. Denote by B_j the ball with centre z_j and radius $\beta' r_B$, and by B_0 the ball with centre c_B and radius $\beta' r_B$. Note that

$$\mu(B) \leq D_{1/\beta',b}\,\mu(B_0), \tag{4.2}$$

where $D_{1/\beta',b}$ is as in Remark 2.3. We consider the partition of unity $\{\psi_1, \dots, \psi_{N_1}\}$ of $\mathbf{1}_{\bigcup_j B_j}$ subordinated to the covering $\{B_1, \dots, B_{N_1}\}$ defined by $\psi_j = \mathbf{1}_{B_j} / \sum_{k=1}^{N_1} \mathbf{1}_{B_k}$. For each j in $\{1, \dots, N_1\}$ we define

$$A_j = \frac{1}{\mu(B_0)} \int_M a\,\psi_j \,\mathrm{d}\mu \qquad \text{and} \qquad \phi_j = a\,\psi_j - A_j\,\mathbf{1}_{B_0}.$$

It is straighforward to check that $a = \sum_{j=1}^{N_1} \phi_j$ and each function ϕ_j has integral 0 and is supported in $B_j \cup B_0$. Define J' and J'' by

$$J' = \{j : z_j \notin B_0\} \qquad \text{and} \qquad J'' = \{j : z_j \in B_0\}.$$

If j is in J', then z_j is not in B_0, so that $\rho(c_B, z_j) > \beta' r_B > \beta' c > R_0$. Therefore we may use property (AM) and conclude that there exists a ball B_j' containing c_B and z_j with radius $< \beta\,\rho(c_B, z_j)$. Denote by $\widetilde{B}_j$ the ball centred at $c_{B_j'}$ with radius $r_{B_j'} + \beta' r_B$. By using the triangle inequality we see that $\widetilde{B}_j$ contains $B_j \cup B_0$. Observe that $r_{\tilde{B}_j} \le (\beta + \beta')\,r_B$, which is strictly less than r_B, because we assumed that $\beta' < 1 - \beta$.

Next we check that if j is in J', then ϕ_j is a multiple of a $(1, r)$-atom: we give details in the case where $r = 2$; the cases where $r \in (1, \infty) \setminus \{2\}$ may be treated similarly, and the variations needed to treat the case where $r = \infty$ are straightforward and are omitted. By the triangle inequality

$$\begin{aligned}
\left(\frac{1}{\mu(\widetilde{B}_j)} \int_{\widetilde{B}_j} |\phi_j|^2 \,\mathrm{d}\mu\right)^{1/2} &\le \left(\frac{1}{\mu(\widetilde{B}_j)} \int_{\widetilde{B}_j} |a\,\psi_j|^2 \,\mathrm{d}\mu\right)^{1/2} \\
&\quad + |A_j| \left(\frac{1}{\mu(\widetilde{B}_j)} \int_{\widetilde{B}_j} \mathbf{1}_{B_0} \,\mathrm{d}\mu\right)^{1/2} \\
&\le \sqrt{\frac{\mu(B)}{\mu(\widetilde{B}_j)}} \left(\frac{1}{\mu(B)} \int_B |a|^2 \,\mathrm{d}\mu\right)^{1/2} \\
&\quad + \sqrt{\frac{\mu(B_0)}{\mu(\widetilde{B}_j)}}\, \frac{1}{\mu(B_0)} \int_M |a\,\psi_j| \,\mathrm{d}\mu \\
&\le \left(\sqrt{\frac{\mu(B)}{\mu(B_0)}} + \frac{\mu(B)}{\mu(B_0)}\right) \frac{1}{\mu(B)} \\
&\le \frac{2\,D_{1/\beta',b}}{\mu(B)}.
\end{aligned}$$

Observe that

$$\frac{1}{\mu(B)} \leq \frac{1}{\mu(B_0)} \leq \frac{D_{\beta/\beta'+1,b}}{\mu(\widetilde{B}_j)},$$

because B_0 is contained both in B and $\widetilde{B}_j$ and the ratio between the radii of $\widetilde{B}_j$ and B_0 is at most $\beta/\beta' + 1$. Therefore we may conclude that

$$\left(\frac{1}{\mu(\widetilde{B}_j)} \int_{\widetilde{B}_j} |\phi_j|^2 \,\mathrm{d}\mu\right)^{1/2} \leq \frac{2\, D_{1/\beta',b}\, D_{\beta/\beta'+1,b}}{\mu(\widetilde{B}_j)},$$

i.e., $\phi_j/(2\, D_{1/\beta',b}\, D_{\beta/\beta'+1,b})$ is an atom supported in the ball $\widetilde{B}_j$ of radius at most $(\beta+\beta')\, r_B$.

Now suppose that j is in J''. Then $B_j \cup B_0$ is contained in $2B_0$. Notice that $\beta' < 1-\beta < 1/2$, so that $r_{2B_0} = 2\beta' r_B < (\beta+\beta')\, r_B$. By arguing much as above, we see that $\phi_j/(2\, D_{1/\beta',b}\, D_{\beta/\beta'+1,b})$ is an atom supported in the ball $2B_0$ of radius $< (\beta+\beta')\, r_B$.

We have written a as the sum of N_1 functions ϕ_j, each of which is a multiple of an atom with constant $2\, D_{1/\beta',b}\, D_{\beta/\beta'+1,b}$. Thus, we have proved that $\|a\|_{H^{1,r}_{b(\beta+\beta')}} \leq 2\, D_{1/\beta',b}\, D_{\beta/\beta'+1,b}\, N_1$.

Now, if j is in J', and $r_{\tilde{B}_j} < c$, then $\widetilde{B}_j$ is in $\mathcal{B}_c$. Similarly, if j is in J'', and $r_{2B_0} < c$, then $2B_0$ is in $\mathcal{B}_c$. If $2B_0$ and all the balls $\widetilde{B}_j$ are in $\mathcal{B}_c$, then the proof is complete. Otherwise either $2B_0$ or some of the $\widetilde{B}_j$'s is not in $\mathcal{B}_c$, and we must iterate the construction above. It is clear that after a finite number of steps, depending on the ratio b/c, we end up with the required decomposition.

Next we prove (ii). Obviously $\|f\|_{H^{1,r}_b} \leq \|f\|_{H^{1,r}_c}$, so we only have to show that $\|f\|_{H^{1,r}_c} \leq C\, \|f\|_{H^{1,r}_b}$ for some constant C depending only on b and c and M. But this follows directly from (i). □

Definition 4.4. Suppose that r is in $(1,\infty)$. Then for every b and c in $\mathbb{R}^+$ such that $R_0/(1-\beta) < c < b$ the spaces $H^{1,r}_b(\mu)$ and $H^{1,r}_c(\mu)$ are isomorphic (in fact, they contain the same functions) by Proposition 4.3 (ii), and will simply be denoted by $H^{1,r}(\mu)$.

Later (see Section 6) we shall prove that $H^{1,r}(\mu)$ does not depend on the parameter r in $(1,\infty)$, and we shall denote $H^{1,r}(\mu)$ simply by $H^1(\mu)$.

5. The space BMO

Suppose that q is in $[1,\infty)$. For each locally integrable function f we define $N^q_b(f)$ by

$$N^q_b(f) = \sup_{B\in\mathcal{B}_b} \left(\frac{1}{\mu(B)} \int_B |f - f_B|^q \,\mathrm{d}\mu\right)^{1/q},$$

where f_B denotes the average of f over B. We denote by $BMO_b^q(\mu)$ the space of all equivalence classes of locally integrable functions f modulo constants, such that $N_b^q(f)$ is finite, endowed with the norm

$$\|f\|_{BMO_b^q(\mu)} = N_b^q(f).$$

Notice that only "small" balls enter in the definition of $BMO_b^q(\mu)$. It is a nontrivial fact, proved in Proposition 5.1 below, that $BMO_b^q(\mu)$ is independent of the parameter b, provided b is large enough, and that the norms N_b^q are all equivalent.

Proposition 5.1. *Suppose that q is in $[1,\infty)$, and b and c are positive constant such that $R_0/(1-\beta) < c < b$. Then $BMO_b^q(\mu)$ and $BMO_c^q(\mu)$ coincide and the norms N_b^q and N_c^q are equivalent.*

Proof. Obviously, if $0 < c < b$ and f is in $BMO_b^q(\mu)$, then f is in $BMO_c^q(\mu)$ and $N_c^q(f) \leq N_b^q(f)$. Thus, we only have to show that $N_b^q(f) \leq C\,N_c^q(f)$ for some constant C depending only on b and c and M. We give the proof in the case where $q = 1$; the proof in the other cases is similar.

Suppose that B is a ball in $\mathcal{B}_b$. Observe that

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |f - f_B|\,\mathrm{d}\mu &\leq \frac{2}{\mu(B)}\inf_{c\in\mathbb{C}}\int_B |f-c|\,\mathrm{d}\mu \\ &\leq \frac{2}{\mu(B)}\,\|f\|_{L^1(B)/\mathbb{C}},\end{aligned}$$

where $L^1(B)/\mathbb{C}$ is the quotient of the space $L^1(B)$ modulo the constants. Since the dual of $L^1(B)/\mathbb{C}$ is $L_0^\infty(B)$ (the space of all functions in $L^\infty(B)$ with vanishing integral, endowed with the $L^\infty(B)$ norm),

$$\|f\|_{L^1(B)/\mathbb{C}} = \sup\left\{\left|\int_B f\,\phi\,\mathrm{d}\mu\right| : \phi\in L_0^\infty(B),\ \|\phi\|_\infty \leq 1\right\}.$$

Suppose that ϕ is a function in $L_0^\infty(B)$ with $\|\phi\|_\infty \leq 1$. Then $\phi/\mu(B)$ is a $(1,\infty)$-atom and, by Proposition 4.3 (i), there exist $(1,\infty)$-atoms $a_1,\ldots,a_N$ supported in balls B_j in $\mathcal{B}_c$ whose union contains B such that $\phi/\mu(B) = \sum_{j=1}^N \lambda_j\,a_j$, with $|\lambda_j| \leq C$ and $\|a_j\|_\infty \leq 1/\mu(B_j)$, where C and N are constants which depend only on b, c and M. Thus

$$\begin{aligned}\frac{1}{\mu(B)}\left|\int_B f\,\phi\,\mathrm{d}\mu\right| &\leq C\sum_{j=0}^N \int_{B_j} |f - f_{B_j}|\,|a_j|\,\mathrm{d}\mu \\ &\leq C\sum_{j=0}^N \frac{1}{\mu(B_j)}\int_{B_j} |f - f_{B_j}|\,\mathrm{d}\mu \\ &\leq C\,N\,N_c^1(f).\end{aligned}$$

Hence $N_b^1(f) \leq 2\,C\,N\,N_c^1(f)$, as required. □

Remark 5.2. For the rest of this paper, we fix a constant $b_0 > R_0/(1-\beta)$. For each q in $[1,\infty)$ we denote by $BMO^q(\mu)$ the space $BMO^q_{b_0}(\mu)$ endowed with any of the equivalent norms N^q_b, where $b > R_0/(1-\beta)$.

Next, we want to show that $BMO^q(\mu)$ is independent of q (see the remark at the end of this section). The strategy is the same as in the classical case: it hinges on a John–Nirenberg type inequality for functions in $BMO^1(\mu)$. The original inequality was proved in [30], where classical functions of bounded mean oscillation appeared for the first time. We need the following generalization of the John–Nirenberg inequality to doubling spaces which is stated in [15] and proved in [6, Theorem 2.2] (see also [32, 34]).

Proposition 5.3. *Suppose that (X,d,μ) is a doubling metric measure space, with doubling constant D. There exist constants J_D and η_D, which depend only on D, such that for every ball B*

$$\mu\big(\big\{x \in B : \big|f(x) - f_B\big| > s\big\}\big) \leq J_D \,\mathrm{e}^{-\eta_D s/\|f\|_{BMO(X)}}\, \mu(B) \quad \forall\, s \in \mathbb{R}^+.$$

By Proposition 3.4 (ii) for each dyadic cube Q the metric measure space $(Q, \rho_{|Q}, \mu_{|Q})$ is a space of homogeneous type. Recall that a ball in Q is the intersection of Q with a ball B in $\mathcal{B}$ whose centre belongs to Q. We denote by $BMO(Q)$ the classical BMO space on Q, *i.e.* the space of all functions f in $L^1(Q,\mu_{|Q})$ such that

$$\|f\|_{BMO(Q)} = \sup_B \frac{1}{\mu(B\cap Q)} \int_{B\cap Q} \big|f - f_{B\cap Q}\big| \,\mathrm{d}\mu < \infty,$$

where the supremum is taken with respect to all balls B in $\mathcal{B}$ whose centre belongs to Q.

Theorem 5.4. *Denote by ν the unique integer such that $\delta^\nu \leq b_0 < \delta^{\nu-1}$, and by N the norm $N^1_{(1+C_1)b_0}$ on $BMO(\mu)$. The following hold:*

(i) *for each dyadic cube Q in $\bigcup_{k=\nu}^\infty \mathcal{Q}^k$ and for each f in $BMO^1(\mu)$ the restriction of f to Q is in $BMO(Q)$ and*

$$\|f\|_{BMO(Q)} \leq 2\, D_{C_1/(a_0\delta), a_0 b_0}\, N(f);$$

(ii) *there exist positive constants J and η such that for every function f in $BMO^1(\mu)$ and for every ball B in $\mathcal{B}_{b_0}$*

$$\mu\big(\{x \in B : |f(x) - f_B| > s\}\big) \leq J\, \mathrm{e}^{-\eta s/N(f)}\, \mu(B).$$

Proof. First we prove (i). Suppose that Q is in $\mathcal{Q}^k$. We have to estimate

$$\mathrm{osc}_f(B\cap Q) = \frac{1}{\mu(B\cap Q)} \int_{B\cap Q} \big|f - f_{B\cap Q}\big| \,\mathrm{d}\mu, \tag{5.1}$$

where B is a ball in (M, ρ). We shall prove that

$$\mathrm{osc}_f(B \cap Q) \le 2\, D_{C_1/(a_0\delta),a_0 b_0}\, N(f) \quad \forall\, f \in BMO^1(\mu),$$

from which (i) follows. We consider the cases where $r_B < C_1\,\delta^k$ and $r_B \ge C_1\,\delta^k$ separately.

In the case where $r_B < C_1\,\delta^k$ we compare (5.1) with the oscillation of f over B. By the triangle inequality

$$\begin{aligned}\mathrm{osc}_f(B \cap Q) &\le \frac{1}{\mu(B \cap Q)} \int_{B\cap Q} \big|f - f_B\big| \,\mathrm{d}\mu + \big|f_B - f_{B\cap Q}\big| \\ &\le \frac{2}{\mu(B \cap Q)} \int_{B\cap Q} \big|f - f_B\big| \,\mathrm{d}\mu.\end{aligned}$$

By Proposition 3.4 (i) we know that $\mu(B \cap Q) \ge D^{-1}_{C_1/(a_0\delta),a_0\delta^\nu}\, \mu(B)$; hence the right hand side in the displayed formula above may be estimated from above by

$$\frac{2\, D_{C_1/(a_0\delta),a_0\delta^\nu}}{\mu(B)} \int_B \big|f - f_B\big| \,\mathrm{d}\mu,$$

which, in turn, may be majorised by $D_{C_1/(a_0\delta),a_0 b_0}\, N(f)$, since the radius of B is at most $C_1 b_0$.

Now suppose that $r_B \ge C_1\,\delta^k$. Since $\mathrm{diam}(Q) < C_1\,\delta^k$ by Theorem 3.2 (iv), $Q \cap B = Q$. For the sake of definiteness, suppose that Q is the dyadic cube Q^k_β. Then Q^k_β contains the ball $B(z^k_\beta, a_0\,\delta^k)$. Denote by $\widetilde{B}$ the ball centred at z^k_β and radius $C_1\,\delta^k$. Now,

$$\begin{aligned}\mathrm{osc}_f(B \cap Q) &\le \frac{1}{\mu(B \cap Q)} \int_{B\cap Q} \big|f - f_{\widetilde{B}}\big| \,\mathrm{d}\mu + \big|f_{\widetilde{B}} - f_{B\cap Q}\big| \\ &\le \frac{2}{\mu(B \cap Q)} \int_{B\cap Q} \big|f - f_{\widetilde{B}}\big| \,\mathrm{d}\mu \\ &\le \frac{2}{\mu\big(B(z^k_\beta, a_0\,\delta^k)\big)} \int_{\widetilde{B}} \big|f - f_{\widetilde{B}}\big| \,\mathrm{d}\mu \\ &\le \frac{2\, D_{C_1/a_0,a_0\delta^\nu}}{\mu\big(\widetilde{B}\big)} \int_{\widetilde{B}} \big|f - f_{\widetilde{B}}\big| \,\mathrm{d}\mu,\end{aligned}$$

which is majorised by $2\, D_{C_1/a_0,a_0 b_0}\, N(f)$. The proof of (i) is complete.

Now we prove (ii). Suppose that B is in $\mathcal{B}_{b_0}$. Denote by k the unique integer such that $\delta^k \le r_B < \delta^{k-1}$ and by $Q_1, \dots, Q_N$ the dyadic cubes of resolution k

that intersect B. By Proposition 3.4 (iv) we have the estimate $N \le D_{(1+C_1)/(a_0\delta),b_0}$. Then

$$\mu\big(\{x \in B : |f(x) - f_B| > s\}\big) \le \sum_{j=1}^{N} \mu\big(\{x \in Q_j : |f(x) - f_B| > s\}\big). \tag{5.2}$$

We estimate each of the summands on the right hand side from above by

$$\mu\big(\{x \in Q_j : \big|f(x) - f_{Q_j}\big| > s/2\}\big) + \mu\big(\{x \in Q_j : \big|f_B - f_{Q_j}\big| > s/2\}\big). \tag{5.3}$$

By Proposition 5.3 and (i) the first summand in this formula is majorised by

$$J_{Q_j}\, \mathrm{e}^{-\eta_{Q_j} s/\|f\|_{BMO(Q_j)}}\, \mu(Q_j) \le J_{Q_j}\, \mathrm{e}^{-\eta_{Q_j} s/(2D_{C_1/(a_0\delta),a_0b_0} N(f))}\, \mu(Q_j).$$

Here we use the fact that since $\operatorname{diam}(Q_j)$ is finite, then Q_j is a ball in the doubling space $(Q_j, \rho_{|Q_j}, \mu_{|Q_j})$. By Proposition 3.4(ii) all the spaces $(Q_j, \rho_{|Q}, \mu_{|Q})$ are spaces of homogeneous type with doubling constant dominated by $D_{\tau,C_1b_0} D_{C_1/(a_0\delta),a_0b_0}$, which we simply denote by D'. Also, denote by η' the constant $\eta_{D'}/(2D_{C_1/(a_0\delta),a_0b_0})$. By Proposition 3.4 (iii) the ball $\widetilde{B}$ with centre c_B and radius $(1 + C_1)\, r_B$ contains $Q_1, \ldots, Q_N$ and $\mu(\widetilde{B}) \le D_{C_1+1,b_0}\, \mu(B)$. Thus, by summing over j, we see that

$$\begin{aligned} \sum_{j=1}^{N} \mu\big(\{x \in Q_j : \big|f(x) - f_{Q_j}\big| > s/2\}\big) &\le J_{D'}\, \mathrm{e}^{-\eta' s/N(f)} \sum_{j=1}^{N} \mu(Q_j) \\ &\le J_{D'}\, \mathrm{e}^{-\eta' s/N(f)}\, \mu(\widetilde{B}) \\ &\le J_{D'}\, \mathrm{e}^{-\eta' s/N(f)} D_{C_1+1,b_0}\, \mu(B). \end{aligned} \tag{5.4}$$

Now we estimate the second summand in (5.3). We claim that

$$\big|f_B - f_{Q_j}\big| \le D''\, N(f),$$

where $D'' = D_{1+C_1,b_0} + D_{(1+C_1)/(a_0b_0),a_0b_0}$. Indeed,

$$\begin{aligned} \big|f_B - f_{Q_j}\big| &\le \big|f_B - f_{\widetilde{B}}\big| + \big|f_{\widetilde{B}} - f_{Q_j}\big| \\ &\le \frac{1}{\mu(B)} \int_B \big|f - f_{\widetilde{B}}\big|\, \mathrm{d}\mu + \frac{1}{\mu(Q_j)} \int_{Q_j} \big|f - f_{\widetilde{B}}\big|\, \mathrm{d}\mu \\ &\le \frac{1}{\mu(B)} \int_B \big|f - f_{\widetilde{B}}\big|\, \mathrm{d}\mu + \frac{1}{\mu\big(B(z_j, a_0\delta^k)\big)} \int_{\widetilde{B}} \big|f - f_{\widetilde{B}}\big|\, \mathrm{d}\mu \\ &\le \frac{D''}{\mu(\widetilde{B})} \int_{\widetilde{B}} \big|f - f_{\widetilde{B}}\big|\, \mathrm{d}\mu, \end{aligned}$$

which is dominated by $D''\, N(f)$, as claimed.

Thus

$$\mu\big(\{x \in Q_j : \big|f_B - f_{Q_j}\big| > s/2\}\big) \leq \mu\big(\{x \in Q_j : D''\, N(f) > s/2\}\big).$$

Now, the right hand side is equal to $\mu(Q_j)$ when s is in $\big(0, 2\,D''\,N(f)\big)$, and to 0 when s is in $\big[2\,D''\,N(f), \infty\big)$, so that

$$\mu\big(\{x \in Q_j : \big|f_B - f_{Q_j}\big| > s/2\}\big) \leq \mathrm{e}^{2D''}\, \mathrm{e}^{-s/N(f)}\, \mu(Q_j) \quad \forall\, s \in \mathbb{R}^+.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{N} \mu\big(\{x \in Q_j : \big|f_B - f_{Q_j}\big| > s/2\}\big) &\leq \mathrm{e}^{2D''}\, \mathrm{e}^{-s/N(f)}\, \mu(\widetilde{B}) \\ &\leq \mathrm{e}^{2D''}\, D_{C_1+1,b_0}\, \mathrm{e}^{-s/N(f)}\, \mu(B).\end{aligned} \tag{5.5}$$

Now, (5.4) and (5.5) imply that

$$\begin{aligned}\mu\big(\{x \in B : |f(x) - f_B| > s\}\big) &\leq \big(J_{D'}\, \mathrm{e}^{-\eta' s/N(f)} + \mathrm{e}^{2D''}\, \mathrm{e}^{-s/N(f)}\big) D_{C_1+1,b_0}\, \mu(B) \\ &\leq J\, \mathrm{e}^{-\eta\, s/N(f)}\, \mu(B),\end{aligned}$$

where $J = \big(J_{D'} + \mathrm{e}^{2D''}\big)\, D_{C_1+1,b_0}$ and $\eta = \min(1, \eta')$, as required. □

A standard consequence of the John–Nirenberg type inequality is the following.

Corollary 5.5. *Denote by ν the unique integer such that $\delta^\nu \leq b_0 < \delta^{\nu-1}$, and by N the norm $N^1_{(1+C_1)b_0}$ on $BMO(\mu)$. The following hold:*

(i) *for every $c < \eta$*

$$\int_B \mathrm{e}^{c|f - f_B|/N(f)}\, \mathrm{d}\mu \leq \left(1 + \frac{Jc}{\eta - c}\right) \mu(B) \quad \forall\, f \in BMO(\mu) \quad \forall\, B \in \mathcal{B}_{b_0},$$

where η and J are as in Theorem 5.4 (ii)*;*

(ii) *for each q in $(1, \infty)$ there exists a constant C such that*

$$\left(\frac{1}{\mu(B)} \int_B |f - f_B|^q\, \mathrm{d}\mu\right)^{1/q} \leq C\, N(f) \quad \forall\, f \in BMO(\mu) \quad \forall\, B \in \mathcal{B}_{b_0}.$$

Proof. First we prove (i). Observe that the left hand side of (i) is equal to

$$\mu(B) + \int_1^\infty \mu\left(\{x \in B : |f - f_B| > N(f)\,(\log \beta)/c\}\right)\, \mathrm{d}\beta.$$

Changing variables and using the John–Nirenberg type inequality proved in Theorem 5.4 we see that the last integral may be estimated by

$$\mu(B)\left[1 + J \int_0^\infty \mathrm{e}^{(c-\eta)v/c}\, \mathrm{d}v\right].$$

The above integral is finite if and only if $c < \eta$ and it is equal to $c/(\eta - c)$: the required inequality follows.

Now we prove (ii). By elementary calculus, for each q in $(1, \infty)$ there exists a constant C_q such that $e^s \geq C_q\, s^q$ for every s in $\mathbb{R}^+$. Therefore (i) implies that

$$C_q \left(\frac{c}{N(f)}\right)^q \int_B |f - f_B|^q \,\mathrm{d}\mu \leq \left(1 + \frac{Jc}{\eta - c}\right) \mu(B),$$

which is equivalent to the required estimate.

The proof of the corollary is complete. □

Remark 5.6. By Corollary 5.5 (ii), if f is in $BMO^1(\mu)$, then f is in $BMO^q(\mu)$ for all q in $(1, \infty)$. Conversely, if f is in $BMO^q(\mu)$ for some q in $(1, \infty)$, then trivially it is in $BMO^1(\mu)$, hence in $BMO^r(\mu)$ for all r in $(1, \infty)$ by Corollary 5.5 (ii). Furthermore, the norms $N^1_{b_0}$ and $N^q_{b_0}$ are equivalent. In view of this observation, all spaces $BMO^q(\mu)$, q in $[1, \infty)$, coincide. We shall denote $BMO^1(\mu)$ simply by $BMO(\mu)$. We endow $BMO(\mu)$ with any of the equivalent norms N^q_b, where q is in $[1, \infty)$ and $b > R_0/(1 - \beta)$. This remark will be important in the proof of the duality between the Hardy space $H^1(\mu)$ and $BMO(\mu)$ (see Section 6 below).

6. Duality

We shall prove that the topological dual of $H^{1,r}(\mu)$ may be identified with $BMO^{r'}(\mu)$, where r' denotes the index conjugate to r. Suppose that $1 < r < s < \infty$. Then $\big(H^{1,r}(\mu)\big)^* = \big(H^{1,s}(\mu)\big)^*$, because we have proved that $BMO^{r'}(\mu) = BMO^{s'}(\mu)$ (see Remark 5.6). Observe that the identity is a continuous injection of $H^{1,s}(\mu)$ into $H^{1,r}(\mu)$, and that $H^{1,s}(\mu)$ is a dense subspace of $H^{1,r}(\mu)$. Then we may conclude that $H^{1,s}(\mu) = H^{1,r}(\mu)$.

We need some more notation and some preliminary observation. For each ball B in $\mathcal{B}_{b_0}$ let $L^2_0(B)$ denote the Hilbert space of all functions f in $L^2(\mu)$ such that the support of f is contained in B and $\int_B f \,\mathrm{d}\mu = 0$. We remark that a function f in $L^2_0(B)$ is a multiple of a $(1, 2)$-atom, and that

$$\|f\|_{H^{1,2}(\mu)} \leq \mu(B)^{1/2}\, \|f\|_{L^2(B)}. \tag{6.1}$$

Suppose that ℓ is a bounded linear functional on $H^{1,2}(\mu)$. Then, for each B in $\mathcal{B}_{b_0}$ the restriction of ℓ to $L^2_0(B)$ is a bounded linear functional on $L^2_0(B)$. Therefore, by the Riesz representation theorem there exists a unique function ℓ^B in $L^2_0(B)$ which represents the restriction of ℓ to $L^2_0(B)$. Note that for every constant η the function $\ell^B + \eta$ represents the same functional, though it is not in $L^2_0(B)$ unless η is equal to 0. Denote by $|||\ell|||_{H^{1,2}(\mu)}$ the norm of ℓ. Observe that

$$\begin{aligned}\|\ell^B\|_{L^2_0(B,\mu)} &= \sup_{\|f\|_{L^2_0(B)=1}} \left|\int_B \ell^B f \,\mathrm{d}\mu\right| \\ &\leq \sup_{\|f\|_{L^2_0(B)=1}} |||\ell|||_{H^{1,2}(\mu)} \|f\|_{H^{1,2}(\mu)} \\ &\leq \mu(B)^{1/2} \, |||\ell|||_{H^{1,2}(\mu)},\end{aligned} \tag{6.2}$$

the last inequality being a consequence of (6.1).

For every f in $BMO^{r'}(\mu)$ and every finite linear combination g of $(1,r)$-atoms the integral $\int_{\mathbb{R}^d} f\, g \,\mathrm{d}\mu$ is convergent. Denote by $H^{1,r}_{\mathrm{fin}}(\mu)$ the subspace of $H^{1,r}(\mu)$ consisting of all finite linear combinations of $(1,r)$-atoms. Then $g \mapsto \int_{\mathbb{R}^d} f\, g \,\mathrm{d}\mu$ defines a linear functional on $H^{1,r}_{\mathrm{fin}}(\mu)$. We observe that $H^{1,r}_{\mathrm{fin}}(\mu)$ is dense in $H^{1,r}(\mu)$.

Theorem 6.1. *Suppose that r is in $(1,\infty)$. The following hold*

(i) *for every f in $BMO^{r'}(\mu)$ the functional ℓ, initially defined on $H^{1,r}_{\mathrm{fin}}(\mu)$ by the rule*

$$\ell(g) = \int_{\mathbb{R}^d} f\, g \,\mathrm{d}\mu,$$

extends to a bounded functional on $H^{1,r}(\mu)$. Furthermore,

$$|||\ell|||_{H^{1,r}(\mu)} \leq \|f\|_{BMO^{r'}(\mu)};$$

(ii) *there exists a constant C such that for every continuous linear functional ℓ on $H^{1,r}(\mu)$ there exists a function f^ℓ in $BMO^{r'}(\mu)$ such that $\|f^\ell\|_{BMO^{r'}(\mu)} \leq C\, |||\ell|||_{H^{1,r}(\mu)}$ and*

$$\ell(g) = \int_{\mathbb{R}^d} f^\ell\, g \,\mathrm{d}\mu \quad \forall\, g \in H^{1,r}_{\mathrm{fin}}(\mu).$$

Proof. The proof of (i) follows the line of the proof of [15] which is based on the classical result of C. Fefferman [22, 23]. We omit the details.

Now we prove (ii) in the case where r is equal to 2. The proof for r in $(1,\infty)\setminus\{2\}$ is similar and is omitted.

Recall that for each $b > R_0/(1-\beta)$ the space $H^{1,2}(\mu)$ is isomorphic to $H^{1,2}_b(\mu)$ with norm $\|\cdot\|_{H^{1,2}_b(\mu)}$. Thus, we may interpret ℓ as a continuous linear functional on $H^{1,2}_b(\mu)$ for each $b > R_0/(1-\beta)$. Fix a reference point o in M. For each b there exists a function f_b^ℓ in $L_0^2(B(o,b))$ that represents ℓ in $B(o,b)$. Since both f_1^ℓ and the restriction of f_b^ℓ represent ℓ on $B(o,1)$, there exists a constant η_b such that

$$f_1^\ell - f_b^\ell = \eta_b$$

on $B(o,1)$. By integrating both sides of this equality on $B(o,1)$ we see that

$$\eta_b = -\frac{1}{\mu\big(B(o,1)\big)} \int_{B(o,1)} f_b^\ell \,\mathrm{d}\mu.$$

Define

$$f^\ell(x) = f_b^\ell(x) + \eta_b \quad \forall\, x \in B(o,b) \quad \forall\, b \in [1,\infty).$$

It is straightforward to check that this is a good definition. We claim that the function f^ℓ is in $BMO(\mu)$ and there exists a constant C such that

$$\|f^\ell\|_{BMO(\mu)} \leq C\, |||\ell|||_{H^{1,2}(\mu)^*} \quad \forall\, \ell \in H^{1,2}(\mu)^*.$$

Indeed, choose a ball B in $\mathcal{B}_{b_0}$. Then there exists a constant η^B such that

$$f^\ell\big|_B = \ell^B + \eta^B, \tag{6.3}$$

where ℓ^B is in $L_0^2(B)$ and represents the restriction of ℓ to $L_0^2(B)$. By integrating both sides on B, we see that $\eta^B = \big(f^\ell\big)_B$. Then, by (6.3),

$$\begin{aligned}\Big(\frac{1}{\mu(B)}\int_B \big|f^\ell - \big(f^\ell\big)_B\big|^2 \,\mathrm{d}\mu\Big)^{1/2} &= \Big(\frac{1}{\mu(B)}\int_B \big|\ell^B\big|^2 \,\mathrm{d}\mu\Big)^{1/2} \\ &\leq |||\ell|||_{H^{1,2}(\mu)},\end{aligned}$$

so that $N_{b_0}^2(f^\ell) \leq |||\ell|||_{H^{1,2}(\mu)}$, as required. □

Remark 6.2. Note that the proof of Theorem 6.1 does not apply, strictly speaking, to the case where r is equal to ∞. However, a straightforward, though tedious, adaptation to the case where μ is only locally doubling of a classical result [15], shows that $H^{1,\infty}(\mu)$ and $H^{1,2}(\mu)$ agree, with equivalence of norms. Consequently, the dual space of $H^{1,\infty}(\mu)$ is $BMO(\mu)$.

7. Estimates for the sharp function and interpolation

The main step in the proof of Fefferman–Stein's interpolation result for analytic families of operators is a certain relative distributional inequality (also referred to as "good λ inequality" in the literature) [23, Theorem 5, page 153], [42], which is a modified version of the original relative distributional inequality of D. L. Burkholder and R. F. Gundy [7, 8] for martingales.

Extensions of Fefferman–Stein's distributional inequality to spaces of homogeneous type are available in the literature (see, *e.g.*, Macías' thesis [33]). It may be worth observing that the doubling property plays a key rôle in their proof. An extension of this theory to rank one symmetric spaces of the noncompact type is due to Ionescu [29]. In this section we adapt Ionescu's ideas and arguments to our setting.

For each integer k, and each locally integrable function f, the *noncentred dyadic local Hardy–Littlewood maximal function* $\mathcal{M}_k f$ is defined by

$$\mathcal{M}_k f(x) = \sup_Q \frac{1}{\mu(Q)} \int_Q |f| \, \mathrm{d}\mu \quad \forall\, x \in M, \tag{7.1}$$

where the supremum is taken over all dyadic cubes of resolution $\geq k$ that contain x.

For each p in M we denote by $\mathcal{B}_b(p)$ the subcollection of all balls in $\mathcal{B}_b$ which contain p. For each b in $\mathbb{R}^+$ we define a *local sharp function* $f^{\sharp,b}$ of a locally integrable function f thus:

$$f^{\sharp,b}(p) = \sup_{B\in\mathcal{B}_b(p)} \frac{1}{\mu(B)} \int_B |f - f_B| \, \mathrm{d}\mu \quad \forall\, p \in M.$$

Observe that f is in $BMO(\mu)$ if and only if $\|f^{\sharp,b}\|_\infty$ is finite for some (hence for all) b in $(R_0/(1-\beta), \infty)$.

We shall need the following result, whose proof, *mutatis mutandis*, is the same as that of its Euclidean analogue.

Theorem 7.1. *Suppose that k is an integer. Then the noncentred dyadic local Hardy–Littlewood maximal operator $\mathcal{M}_k$ is bounded on $L^p(\mu)$ for every p in* $(1, \infty]$ *and of weak type* 1.

Lemma 7.2. *Define constants C_0, b', σ and D by*

$$C_0 = \max(C_1/\delta, \delta),$$

$$b' = \max(b_0, 2C_1 + C_0),$$

$$\sigma = \big(1 - \mathrm{e}^{-I_M\,\delta^3}\big)/2 \quad \textit{and}$$

$$D = D_{b'/a_0, a_0},$$

where a_0, C_1 and δ are as in Theorem 3.2, *and $D_{b'/a_0,a_0}$ is defined in Remark* 2.3. *For every η' in* $(0, 1)$, *for all positive $\varepsilon < (1-\eta')/(2D)$, and for every f in $L^1(\mu)$*

$$\mu\big(\{\mathcal{M}_2 f > \alpha,\ f^{\sharp,b'} \leq \varepsilon\,\alpha\}\big) \leq \eta\,\mu\big(\{\mathcal{M}_2 f > \eta'\,\alpha\}\big) \quad \forall\, \alpha \in \mathbb{R}^+,$$

where

$$\eta = 1 - \sigma + \frac{2\varepsilon\, D}{\sigma\,(1-\eta')}.$$

Proof. For each $\beta > 0$ we denote by $A(\beta)$ and $S(\beta)$ the sets $\{\mathcal{M}_2 f > \beta\}$ and $\{f^{\sharp,b'} > \beta\}$ respectively. The inequality to prove may then be rewritten as follows:

$$\mu\big(A(\alpha) \cap S(\varepsilon\alpha)^c\big) \le \eta\, \mu\big(A(\eta'\alpha)\big) \quad \forall\, \alpha \in \mathbb{R}^+.$$

To each x in $A(\eta'\alpha)$ we associate the maximal dyadic cube Q_x containing x of resolution at least 2 such that $|f|_{Q_x} > \eta'\alpha$. Here $|f|_{Q_x}$ denotes the average of $|f|$ on the cube Q_x. We denote by $\mathcal{C}_{\eta'\alpha}$ the collection of cubes $\{Q_x\}_{x\in A(\eta'\alpha)}$. Clearly $A(\eta'\alpha) = \bigcup_{x\in A(\eta'\alpha)} Q_x$, and $\mu\big(A(\eta'\alpha)\big) < \infty$, because $\mathcal{M}_2$ is of weak type 1. By Proposition 3.5 (with $\kappa = \delta^3$) there exist mutually disjoint cubes $Q_1, \ldots, Q_k$ in $\mathcal{C}_{\eta'\alpha}$ such that $\rho\big(Q_j, A(\eta'\alpha)^c\big) \le \delta^3$ and

$$\sum_{j=1}^{k} \mu(Q_j) \ge \sigma\, \mu\big(A(\eta'\alpha)\big). \tag{7.2}$$

We claim that if $0 < \varepsilon < (1-\eta')/(2D)$ then

$$\mu\big(Q_j \cap A(\alpha) \cap S(\varepsilon\alpha)^c\big) \le \frac{2\varepsilon\, D}{\sigma\,(1-\eta')}\, \mu\big(Q_j\big) \quad \forall\, j \in \{1, \ldots, k\}. \tag{7.3}$$

We postpone for a moment the proof of the claim and show how (7.3) implies the required conclusion. Observe that $A(\alpha) \subset A(\eta'\alpha)$ and that

$$\begin{aligned}
\mu\big(A(\alpha) \cap S(\varepsilon\alpha)^c\big) &= \mu\left(\left(A(\eta'\alpha) \setminus \bigcup_{j=1}^{k} Q_j\right) \cap A(\alpha) \cap S(\varepsilon\alpha)^c\right) \\
&\quad + \mu\left(\left(\bigcup_{j=1}^{k} Q_j\right) \cap A(\alpha) \cap S(\varepsilon\alpha)^c\right) \\
&\le (1-\sigma)\, \mu\big(A(\eta'\alpha)\big) + \frac{2\varepsilon\, D}{\sigma\,(1-\eta')} \sum_{j=1}^{k} \mu(Q_j) \\
&\le \eta\, \mu\big(A(\eta'\alpha)\big).
\end{aligned}$$

The penultimate inequality is a consequence of (7.2) and of (7.3), and the last inequality follows from the fact that the Q_j's are mutually disjoint cubes contained in $A(\eta'\alpha)$.

Thus, to conclude the proof of the lemma it remains to prove the claim (7.3). For the rest of the proof we shall denote any of the cubes $Q_1, \ldots, Q_k$ simply by Q. Denote by ν_0 the resolution of Q.

We claim that there exists a dyadic cube $\widetilde{Q}$ of the same resolution as Q such that $|f|_{\widetilde{Q}} \le \eta'\alpha$ and $\rho(Q, \widetilde{Q}) \le C_0\, \delta^{\nu_0}$.

We treat the cases where ν_0 is equal to 2 or ≥ 3 separately.

Suppose that $\nu_0 = 2$. Since $\rho\big(Q, A(\eta'\alpha)^c\big)\big) \le \delta^3$, there exists a point y in $A(\eta'\alpha)^c$ such that $\rho(Q, y) \le \delta^3$. Denote by $\widetilde{Q}$ the dyadic cube with resolution 2 which contains y. Then $\rho(Q, \widetilde{Q}) \leq \delta^3 \leq C_0\delta^2$ and $|f|_{\widetilde{Q}} \leq \eta'\alpha$, because $\widetilde{Q} \cap A(\eta'\alpha)^c \neq \emptyset$.

Now suppose that $\nu_o \geq 3$. Then the father $Q^\sharp$ of Q contains a point y in $A(\eta'\alpha)^c$, for otherwise $Q^\sharp$ would be contained in $A(\eta'\alpha)$, thereby contradicting the maximality of Q. Denote by $\widetilde{Q}$ the dyadic cube of resolution ν_0 which contains y. Then y is in $\widetilde{Q} \cap Q^\sharp$ and therefore $\widetilde{Q} \subset Q^\sharp$. Thus

$$\rho(Q, \widetilde{Q}) \leq \operatorname{diam}(Q^\sharp) \leq C_1\, \delta^{\nu_0 - 1} \leq C_0\, \delta^{\nu_0}$$

and $|f|_{\widetilde{Q}} \leq \eta'\alpha$, because $\widetilde{Q} \cap A(\eta'\alpha)^c \neq \emptyset$. This completes the proof of the claim.

To each point y in $Q \cap A(\alpha)$ we associate a maximal dyadic cube Q'_y of resolution at least 2 containing y such that $|f|_{Q'_y} > \alpha$. Denote $\mathcal{C}'$ the collection of all these cubes. By Proposition 3.5 we may select mutually disjoint cubes $Q'_1, \ldots, Q'_{k'}$ in $\mathcal{C}'$ such that

$$\sum_{j=1}^{k'} \mu(Q'_j) \geq \sigma\, \mu\big(Q \cap A(\alpha)\big). \tag{7.4}$$

Note also that $Q'_1, \ldots, Q'_{k'}$ are contained in Q. Denote by B^* a ball with centre at a point of Q and radius $b'\delta^{\nu_0}$ (recall that $b' \geq 2C_1 + C_0$). Then B^* contains both Q, whence the cubes $Q'_1, \ldots, Q'_{k'}$, and $\widetilde{Q}$. Hence

$$\mu(B^*) \leq D\, \mu(Q) \qquad \text{and} \qquad \mu(B^*) \leq D\, \mu(\widetilde{Q}) \tag{7.5}$$

by the local doubling property.

If $Q \cap A(\alpha) \cap S(\varepsilon\alpha)^c$ is nonempty, then

$$\int_{B^*} |f - f_{B^*}|\, \mathrm{d}\mu \leq \varepsilon\, \alpha\, \mu(B^*). \tag{7.6}$$

Since $\widetilde{Q} \subset B^*$ and $|f|_{\widetilde{Q}} \leq \eta'\, \alpha$,

$$\mu(\widetilde{Q})\, \big(|f_{B^*}| - \eta'\, \alpha\big) \leq \int_{B^*} |f - f_{B^*}|\, \mathrm{d}\mu$$

by the triangle inequality. Now (7.6) implies that

$$\mu(\widetilde{Q})\, \big(|f_{B^*}| - \eta'\, \alpha\big) \leq \varepsilon\, \alpha\, \mu(B^*). \tag{7.7}$$

By a similar argument

$$\big(\alpha - |f_{B^*}|\big) \sum_{j=1}^{k'} \mu(Q'_j) \leq \varepsilon\, \alpha\, \mu(B^*). \tag{7.8}$$

From (7.7) we see that $|f_{B^*}| \leq \alpha\,\big(\eta' + \varepsilon\,\frac{\mu(B^*)}{\mu(\widetilde{Q})}\big)$. By inserting this inequality in (7.8), we obtain that

$$\left(1 - \eta' - \varepsilon\,\frac{\mu(B^*)}{\mu(\widetilde{Q})}\right)\sum_{j=1}^{k'} \mu(Q'_j) \leq \varepsilon\,\mu(B^*),$$

whence

$$\sigma\,\left(1 - \eta' - \varepsilon\,\frac{\mu(B^*)}{\mu(\widetilde{Q})}\right)\,\mu(Q \cap A(\alpha) \cap S(\varepsilon\alpha)^c) \leq \varepsilon\,\mu(B^*),$$

by (7.4). Now, since $\varepsilon < (1-\eta')/(2D)$, we may use (7.5) and conclude that

$$\mu(Q \cap A(\alpha) \cap S(\varepsilon\alpha)^c) \leq \frac{2\varepsilon\,D}{\sigma\,(1-\eta')}\,\mu(Q),$$

as required. $\square$

Theorem 7.3. *For each p is in $(1,\infty)$ there exists a positive constant C such that*

$$\|f^{\sharp,b'}\|_{L^p(\mu)} \geq C\,\|f\|_{L^p(\mu)} \quad \forall\, f \in L^p(\mu),$$

where $b' = \max(b_0, 2C_1 + C_0)$ is as in the statement of Lemma 7.2.

Proof. Observe that it suffices to show that

$$\|f^{\sharp,b'}\|_{L^p(\mu)} \geq C\,\|\mathcal{M}_2 f\|_{L^p(\mu)}, \tag{7.9}$$

because $\mathcal{M}_2 f \geq |f|$ by the differentiation theorem of the integral, which is a standard consequence of Proposition 7.1.

Let η and η' be as in the statement of Lemma 7.2. By Lemma 7.2

$$\begin{aligned}
\|\mathcal{M}_2 f\|_{L^p(\mu)}{}^p &= p\int_0^\infty \alpha^{p-1}\,\mu\big(A(\alpha)\big)\,\mathrm{d}\alpha \\
&= p\int_0^\infty \alpha^{p-1}\,\mu\big(A(\alpha)\cap S(\varepsilon\alpha)^c\big)\,\mathrm{d}\alpha \\
&\quad + p\int_0^\infty \alpha^{p-1}\,\mu\big(A(\alpha)\cap S(\varepsilon\alpha)\big)\,\mathrm{d}\alpha \\
&\leq p\,\eta\int_0^\infty \alpha^{p-1}\,\mu\big(A(\eta'\alpha)\big)\,\mathrm{d}\alpha + p\int_0^\infty \alpha^{p-1}\,\mu\big(S(\varepsilon\alpha)\big)\,\mathrm{d}\alpha \\
&= p\,\eta\,\eta'^{-p}\int_0^\infty \beta^{p-1}\,\mu\big(A(\beta)\big)\,\mathrm{d}\beta + p\,\varepsilon^{-p}\int_0^\infty \beta^{p-1}\,\mu\big(S(\beta)\big)\,\mathrm{d}\beta \\
&\leq \eta\,\eta'^{-p}\,\|\mathcal{M}_2 f\|_{L^p(\mu)}{}^p + \varepsilon^{-p}\,\|f^{\sharp,b'}\|_{L^p(\mu)}{}^p.
\end{aligned}$$

Now, for a given p, we choose η' such that $\eta'^p = 1 - \sigma/4$, and then we choose ε small enough so that $\eta \leq 1 - \sigma/2$. Therefore $\eta\,\eta'^{-p} < 1$ and (7.9) follows. $\square$

As a consequence of the relative distributional inequality proved in Theorem 7.3 we prove a complex interpolation theorem. Suppose that (X^0, X^1) is an interpolation pair of Banach spaces, *i.e.*, X^0 and X^1 are Banach spaces both continuously included in a topological vector space V. For every θ in $(0, 1)$ we denote by X_θ the interpolation space $(X^0, X^1)_{[\theta]}$, obtained via Calderón's complex interpolation method [10]. The notation we adopt is consistent with that of [3, Chapter 4].

Theorem 7.4. *Suppose that* (M, d, μ) *is a metric measure space that possesses properties* (I) *and* (LD). *Suppose that* θ *is in* $(0, 1)$. *The following hold:*

(i) *if* p_θ *is* $2/(1-\theta)$, *then* $\big(L^2(M), BMO(M)\big)_{[\theta]} = L^{p_\theta}(M)$;

(ii) *if* p_θ *is* $2/(2-\theta)$, *then* $\big(H^1(M), L^2(M)\big)_{[\theta]} = L^{p_\theta}(M)$.

Proof. First we prove (i). The inclusion $L^{p_\theta}(M) \subset \big(L^2(M), BMO(M)\big)_{[\theta]}$ follows from the fact that $L^\infty(M)$ is continuously included in $BMO(M)$ and $\big(L^2(M), L^\infty(M)\big)_{[\theta]} = L^{p_\theta}(M)$. Therefore we only have to show the reverse inclusion $\big(L^2(M), BMO(M)\big)_{[\theta]} \subset L^{p_\theta}(M)$.

The idea of the proof is to linearise the sharp maximal function (see [23]). Let ϕ any measurable function which associates to every point x in M a ball $\phi(x)$ in $\mathcal{B}_1$ which contains x. Furthermore, let $\eta : M \times M \to \mathbb{C}$ be any measurable function such that $|\eta| = 1$. We consider the linear operator $S^{\phi,\eta}$ which acts on a function f in $L^2_{\mathrm{loc}}(M)$ as follows

$$S^{\phi,\eta} f(x) = \frac{1}{\mu\big(\phi(x)\big)} \int_{\phi(x)} \big[f - f_{\phi(x)}\big]\, \eta(x, \cdot)\, \mathrm{d}\mu \quad \forall\, x \in M.$$

Clearly

$$\big|S^{\phi,\eta} f\big| \le f^\sharp \qquad \text{and} \qquad \sup_{\phi,\eta} \big|S^{\phi,\eta} f\big| = f^\sharp. \tag{7.10}$$

We denote by S the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (0, 1)\}$, and denote by $\overline{S}$ its closure. We consider the class $\mathcal{F}(L^2(M), BMO(M))$ of all functions $F : \overline{S} \to L^2(M) + BMO(M)$ with the following properties:

1. $\langle F(z), \ell\rangle$ is continuous and bounded in $\overline{S}$ and analytic in S for every ℓ in $\big(L^2(M) + BMO(M)\big)^*$;
2. $F(it)$ is $L^2(M)$-continuous and $F(1+it)$ is $BMO(M)$-continuous.

We endow $\mathcal{F}(L^2(M), BMO(M))$ with the norm

$$\|F\|_{\mathcal{F}} = \sup\big\{\max(\|F(it)\|_{L^2(M)}, \|F(1+it)\|_{BMO(M)}) : t \in \mathbb{R}\big\}.$$

Now, suppose that f is in the interpolation space $\big(L^2(M), BMO(M)\big)_{[\theta]}$. Then there exists a function F in $\mathcal{F}$ such that $F(\theta) = f$. For each ϕ and η as before, consider the function $S^{\phi,\eta} F$. Denote by $\mathcal{M}_1 f$ the local noncentred Hardy–Littlewood maximal function of f, defined by

$$\mathcal{M}_1 f(x) = \sup_{B \in \mathcal{B}_1(x)} \frac{1}{\mu(B)} \int_B |f|\, \mathrm{d}\mu,$$

where $\mathcal{B}_1(x)$ denotes the collection of all balls in $\mathcal{B}_1$ that contain x. Recall that $g^\sharp \le 2\,\mathcal{M}_1 g$, and that $\mathcal{M}_1$ is bounded on $L^2(M)$. Thus,

$$\begin{aligned}\|S^{\phi,\eta} F(is)\|_{L^2(M)} &\le \|F(is)^\sharp\|_{L^2(M)} \le 2\,\|\mathcal{M}_1 F(is)\|_{L^2(M)} \\ &\le C\,\|F(is)\|_{L^2(M)}.\end{aligned} \tag{7.11}$$

Note that the constant C in the above inequality does not depend on ϕ and η.

Similarly,

$$\|S^{\phi,\eta} F(1+is)\|_{L^\infty(M)} \le \|F(1+is)^\sharp\|_{L^\infty(M)} \le \|F(1+is)\|_{BMO(M)}. \tag{7.12}$$

From these estimates we deduce that $S^{\phi,\eta}F$ belongs to the class $\mathcal{F}(L^2(M), L^\infty(M))$ and that

$$\|S^{\phi,\eta}F\|_{\mathcal{F}(L^2(M),L^\infty(M))} \le C\,\|F\|_{\mathcal{F}(L^2(M),BMO(M))}.$$

Hence

$$\|S^{\phi,\eta}F(\theta)\|_{L^{p_\theta}(M)} \le C\,\|F(\theta)\|_{(L^2(M),BMO(M))_{[\theta]}} = C\,\|f\|_{(L^2(M),BMO(M))_{[\theta]}}.$$

By taking the supremum over all ϕ and η we obtain the estimate

$$\|f^\sharp\|_{L^{p_\theta}(M)} \le C\,\|f\|_{(L^2(M),BMO(M))_{[\theta]}}. \tag{7.13}$$

By applying [11, Theorem 7.3], we may conclude that

$$\|f\|_{L^{p_\theta}(M)} \le C\,\|f\|_{(L^2(M),BMO(M))_{[\theta]}} \quad \forall\, f \in (L^2(M), BMO(M))_{[\theta]},$$

which proves the required inclusion $\big(L^2(M), BMO(M)\big)_{[\theta]} \subset L^{p_\theta}(M)$.

Denote by X_θ the interpolation space $\big(H^1(M), L^2(M)\big)_{[\theta]}$. From the duality theorem [3, Corollary 4.5.2] we deduce that if p_θ is $2/(2-\theta)$, then the topological dual of X_θ is $\big(L^2(M), BMO(M)\big)_{[\theta]}$, which is equal to $L^{p'_\theta}(M)$ by (i). Furthermore, X_θ is continuously included in $L^{p_\theta}(M)$, because $H^1(M)$ is continuously included in $L^1(M)$ and $\big(L^1(M), L^2(M)\big)_{[\theta]} = L^{p_\theta}(M)$. Since $L^2(M)$ is reflexive, the interpolation space X_θ is reflexive by a result of A. Calderón [3, Section 4.9], so that X_θ is isometrically isomorphic to $X_\theta^{**} = \big(L^{p'_\theta}(M)\big)^*$. This implies that $X_\theta = L^{p_\theta}(M)$, as required. □

A standard consequence of Theorem 7.4 is the following interpolation result for analytic families of operators.

Corollary 7.5. *Denote by S the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\}$. Suppose that $\{\mathcal{T}_z\}_{z\in\bar{S}}$ is a family of uniformly bounded operators on $L^2(\mu)$ such that $z \mapsto \int_{\mathbb{R}^d} \mathcal{T}_z f\, g \,\mathrm{d}\mu$ is holomorphic in S and continuous in $\bar{S}$ for all functions f and g in $L^2(\mu)$. Further, assume that there exists a constant A such that*

$$|||\mathcal{T}_{is}|||_{L^2(\mu)} \le A \qquad \textit{and} \qquad |||\mathcal{T}_{1+is}|||_{L^\infty(\mu);BMO(\mu)} \le A.$$

Then for every θ in $(0,1)$ the operator $\mathcal{T}_\theta$ is bounded on $L^{p_\theta}(\mu)$, where $p_\theta = 2/(1-\theta)$, and

$$|||\mathcal{T}_\theta|||_{L^{p_\theta}(\mu)} \leq A_\theta,$$

where A_θ depends only on A and on θ.

Proof. This follows directly from (i) of the previous theorem and [17, Theorem 1]. Alternatively, we may follow the line of the proof of the classical case (see, for instance, [42, Theorem 4, page 175], or [23]). □

8. Singular integrals

In this section we develop a theory of singular integral operators acting on $L^p(\mu)$ spaces.

Preliminarily, we observe the following. Recently, M. Bownik [5], following up earlier work of Y. Meyer, produced in the classical Euclidean setting an example of an operator $\mathcal{T}_B$ defined on $(1,\infty)$-atoms with

$$\sup\{\|\mathcal{T}_B a\|_{L^1(\lambda)} : a \text{ is a } (1,\infty)\text{-atom}\} < \infty,$$

that does not extend to a bounded operator from $H^1(\lambda)$ to $L^1(\lambda)$: here λ denotes the Lebesgue measure on $\mathbb{R}^n$. The problem of giving sufficient conditions for an operator uniformly bounded on atoms to extend to a bounded operator from $H^1(\lambda)$ to $L^1(\lambda)$ has been considered independently in [36, 47]. The paper [47] and most of [36] focus on the Euclidean case. However, in the last part of [36] more general settings are considered. In particular, suppose that (M,ρ,μ) is a σ-finite metric measure space with properties (LD), (I) and (AM). Then the following holds.

Proposition 8.1. *Suppose that q is in $(1,\infty)$, and that $\mathcal{T}$ is a linear operator defined on finite linear combinations of $(1,q)$-atoms, satisfying*

$$\sup\{\|\mathcal{T} a\|_{L^1(\mu)} : a \text{ is a } (1,q)\text{-atom}\} < \infty.$$

The following hold:

(i) *$\mathcal{T}$ extends to a bounded operator $\widetilde{\mathcal{T}}$ from $H^1(\mu)$ to $L^1(\mu)$ and the transpose operator $\mathcal{T}^*$ extends to a bounded operator $(\mathcal{T}^*)^{\sim}$ from $L^\infty(\mu)$ to $BMO(\mu)$;*
(ii) *if $\mathcal{T}$ is bounded on $L^2(\mu)$, then $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are consistent operators on $H^1(\mu) \cap L^2(\mu)$.*

Proof. The result [36, Theorem 4.1 and Proposition 4.2] is stated for spaces of homogeneous type. However, the proof extends *verbatim* to our setting. □

Now we assume that $\mathcal{T}$ is bounded on $L^2(\mu)$ and that there exists a locally integrable function k off the diagonal in $M \times M$ such that for every function f with support of finite measure

$$\mathcal{T} f(x) = \int_M k(x,y)\, f(y)\,\mathrm{d}\mu(y) \quad \forall\, x \notin \operatorname{supp} f.$$

We refer to k as to the *kernel of* $\mathcal{T}$. A straightforward computation shows that the kernel k^* of the (Hilbert space) adjoint $\mathcal{T}^*$ of $\mathcal{T}$ is related to the kernel k of $\mathcal{T}$ by the formula

$$k^*(y, x) = \overline{k(x, y)}. \tag{8.1}$$

In particular, if $\mathcal{T}$ is self adjoint on $L^2(\mu)$, then

$$k(y, x) = \overline{k(x, y)}. \tag{8.2}$$

The next theorem is a version in our case of a classical result which holds on spaces of homogeneous type. *Mutatis mutandis*, its proof is similar to the proof in the classical case. However, we include a sketch of the proof for the reader's convenience. See also [35] for a detailed proof of the analogous result in the Gaussian case.

Theorem 8.2. *Suppose that b is in $\mathbb{R}^+$ and $b > R_0/(1-\beta)$, where R_0 and β appear in the definition of property (AM). Suppose that $\mathcal{T}$ is a bounded operator on $L^2(\mu)$ and that its kernel k is locally integrable off the diagonal of $M \times M$. Define υ_k and ν_k by*

$$\upsilon_k = \sup_{B \in \mathcal{B}_b} \sup_{x, x' \in B} \int_{(2B)^c} \left| k(x, y) - k(x', y) \right| \, \mathrm{d}\mu(y),$$

and

$$\nu_k = \sup_{B \in \mathcal{B}_b} \sup_{y, y' \in B} \int_{(2B)^c} \left| k(x, y) - k(x, y') \right| \, \mathrm{d}\mu(x).$$

The following hold:

(i) *if ν_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1, 2]$ and from $H^1(\mu)$ to $L^1(\mu)$. Furthermore, there exists a constant C such that*

$$|||\mathcal{T}|||_{H^1(\mu);L^1(\mu)} \leq C \left(\nu_k + |||\mathcal{T}|||_{L^2(\mu)} \right);$$

(ii) *if υ_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $[2, \infty)$ and from $L^\infty(\mu)$ to $BMO(\mu)$. Furthermore, there exists a constant C such that*

$$|||\mathcal{T}|||_{L^\infty(\mu);BMO(\mu)} \leq C \left(\upsilon_k + |||\mathcal{T}|||_{L^2(\mu)} \right);$$

(iii) *if $\mathcal{T}$ is self adjoint on $L^2(\mu)$ and ν_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1, \infty)$, from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$.*

Proof. First we prove (i). In view of Proposition 8.1 it suffices to show that $\mathcal{T}$ maps $(1, 2)$-atoms in $H^1_1(\mu)$ uniformly into $L^1(\mu)$. This is done exactly as in the classical case, except that we need to consider only atoms supported in balls of $\mathcal{B}_b$. Then $\mathcal{T}$ maps $H^1(\mu)$ into $L^1(\mu)$, and, by interpolation, on $L^p(\mu)$ for all p in $(1, 2)$.

Next we prove (ii). Since the kernel $k^*(x, y)$ of the (Hilbert space) adjoint $\mathcal{T}^*$ of $\mathcal{T}$ is $\overline{k(y, x)}$, by (i) $\upsilon_{k^*} = \nu_k$ is finite and the operator $\mathcal{T}^*$ is bounded from $H^1(\mu)$ to $L^1(\mu)$. Hence $\mathcal{T}$ is bounded from $L^\infty(\mu)$ to $BMO(\mu)$. Moreover

$$|||\mathcal{T}^*|||_{L^\infty(\mu);BMO(\mu)} \leq C\,\big(\nu_k + |||\mathcal{T}|||_{L^2(\mu)}\big).$$

By interpolation $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(2, \infty)$,

Finally, we prove (iii). By (ii), $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1, 2)$ and from $H^1(\mu)$ to $L^1(\mu)$. By (8.2) also υ_k is finite. Hence, by (i), $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $[2, \infty)$ and from $L^\infty(\mu)$ to $BMO(\mu)$, thereby concluding the proof of (iii) and of the theorem. □

Remark 8.3. It is worth observing that in the case where M is a Riemannian manifold and the kernel k is "regular", then the condition $\upsilon_k < \infty$ of Theorem 8.2 (i) may be replaced by the condition $\upsilon_k' < \infty$, where

$$\upsilon_k' = \sup_{B\in\mathcal{B}_b} r_B \sup_{x\in B} \int_{(2B)^c} \big|\mathrm{grad}_x\, k(x, y)\big|\, \mathrm{d}\mu(y). \tag{8.3}$$

Similarly, the condition $\nu_k < \infty$ of Theorem 8.2 (ii) may be replaced by the condition $\nu_k' < \infty$, where

$$\nu_m' = \sup_{B\in\mathcal{B}_b} r_B \sup_{y\in B} \int_{(2B)^c} \big|\mathrm{grad}_y\, k(x, y)\big|\, \mathrm{d}\mu(x). \tag{8.4}$$

Indeed, by the mean value theorem we see that the condition

$$\sup_{B\in\mathcal{B}_1} \sup_{x,x'\in B} \rho(x, x') \int_{(2B)^c} \big|\mathrm{grad}_x\, k(x, y)\big|\, \mathrm{d}\mu(y) < \infty$$

implies the condition $\upsilon_k < \infty$ of the theorem. Since $\rho(x, x') < 2\,r_B$, (8.3) follows.

We note also that formula (8.2) imply that if $\mathcal{T}$ is self adjoint, then $\upsilon_k' < \infty$ holds if and only if $\nu_k' < \infty$ does.

9. Cheeger's isoperimetric constant and property (I)

In this section we show that the theory developed in the previous sections may be applied to an interesting class of complete noncompact Riemannian manifolds. First we recall that if (M, ρ) is a complete Riemannian manifold with Ricci curvature bounded from below, then M is a locally doubling metric space with respect to the Riemannian measure and the geodesic distance. The proof of this fact is a direct consequence of M. Gromov's variant [25] of R.L. Bishop's comparison theorem (see, for instance, [4]).

It is natural to investigate the dependence of property (I) on other geometric or analytic properties of M. Denote by $b(M)$ the bottom of the spectrum of M, defined by

$$b(M) = \inf_{f \neq 0} \frac{\int_M |\mathrm{grad}\, f|^2 \,\mathrm{d}V}{\int_M |f|^2 \,\mathrm{d}V},$$

where V denotes the Riemannian measure of M and f runs over all sufficiently smooth functions with compact support. We denote by $h(M)$ the Cheeger isoperimetric constant of M defined by

$$h(M) = \inf \frac{\sigma(\partial A)}{V(A)}, \tag{9.1}$$

where the infimum runs over all bounded open subsets A with smooth boundary ∂A and where σ denotes the $(d-1)$-dimensional measure. Cheeger proved that

$$b(M) \geq h(M)^2/4. \tag{9.2}$$

In this section we shall relate $b(M)$ and $h(M)$ to the isoperimetric constant I_M defined in Section 2.

First we need to show that in Cheeger's isoperimetric inequality $\sigma(\partial A) \geq h(M)\, V(A)$, we may replace the bounded open set A with smooth boundary by any set E of finite measure and the $(d-1)$-dimensional measure σ of the boundary by the perimeter of E. The definition of perimeter of a set in a Riemannian manifold mimics closely the definition in the Euclidean setting [21, 37].

If U is an open subset of M we shall denote by $\Lambda_c^k(U)$ the space of smooth k-forms with compact support contained in U. The divergence is the formal adjoint of the exterior derivative d, *i.e.* the operator div mapping $k+1$-forms to k-forms defined by

$$\int_M \langle \mathrm{div}\, \omega, \eta \rangle_x \,\mathrm{d}V(x) = -\int_M \langle \omega, \mathrm{d}\eta \rangle_x \,\mathrm{d}V(x) \tag{9.3}$$

for all smooth $k+1$-forms ω and all smooth k-forms η with compact support.

Given a real valued function f in $L^1(M)$, the variation of f in U is

$$\mathrm{Var}(f, U) = \sup \left\{ \int_M f\, (\mathrm{div}\, \omega) \,\mathrm{d}V : \omega \in \Lambda_c^1(U), \|\omega\|_\infty \leq 1 \right\}.$$

We say that f has bounded variation in U if $\mathrm{Var}(f, U) < \infty$. We shall denote by $BV(U)$ the space of function of bounded variation in U.

As in the Euclidean case, if f is in $BV(M)$, then the map $U \mapsto \mathrm{Var}(f, U)$ extends to a finite Borel measure on M.

A measurable set $E \subset M$ has finite perimeter if its indicator function $\mathbf{1}_E$ is in $BV(M)$. If U is a Borel set the perimeter of E in U is

$$P(E, U) = \mathrm{Var}(\mathbf{1}_E, U).$$

Since the manifold M is complete, by [27, Theorem 2.7] the Sobolev space $H^{1,1}(M)$ is the completion of the space $C_c^\infty(M)$ of smooth functions with compact support on M with respect to the norm

$$\|f\|_{H^{1,1}} = \|f\|_1 + \int_M |\text{grad } f| \, \mathrm{d}V.$$

It is an easy matter to show that $H^{1,1}(M) \subset BV(M)$ and

$$\text{Var}(f, M) = \int_M |\text{grad } f| \, \mathrm{d}V \quad \forall f \in H^{1,1}(M).$$

Note also that the space $\text{Lip}_c(M)$ of Lipschitz functions with compact support on M is contained in $H^{1,1}(M)$, by [27, Lemma 2.5].

In the Euclidean setting it is well known that BV functions may be approximated in variation by smooth functions in L^1 (see, for instance [21, Theorem 3.9]). The same result holds in the Riemannian setting [37, Proposition 1.4].

Lemma 9.1. *For every f in $BV(M)$ there exists a sequence (f_n) in $C_c^\infty(M)$ which converges to f in $L^1(M)$ and such that*

$$\text{Var}(f, M) = \lim_{n\to\infty} \text{Var}(f_n, M).$$

Now we can prove that Cheeger's isoperimetric inequality for smooth compact hypersurfaces implies an analogous inequality for the perimeters of sets of finite measure.

Proposition 9.2. *Suppose that M is a complete Riemannian manifold. If $h(M) > 0$ then for every measurable set E*

$$P(E, M) \geq h(M)\, V(E).$$

Proof. It is well known that Cheeger's isoperimetric inequality for smooth submanifolds is equivalent to the Sobolev inequality

$$\text{Var}(f, M) = \int_M |\text{grad } f| \, \mathrm{d}V \geq h(M) \int_M |f| \, \mathrm{d}V \tag{9.4}$$

for all real valued functions in $C_c^1(M)$ [12]. Suppose that E is a measurable set of finite perimeter. By Lemma 9.1 there exists a sequence (f_n) of functions in $C_c^1(M)$ such that $f_n \to \mathbf{1}_E$ in $L^1(M)$ and $\text{Var}(f_n, M) \to \text{Var}(\mathbf{1}_E, M) = P(E, M)$. Hence the desired conclusion follows from (9.4). □

The following lemma is the coarea formula for functions in $H^{1,1}(M)$. The proof uses the density of $C_c^1(M)$ in $H^{1,1}(M)$ and mimics closely the argument in the Euclidean setting [21].

Lemma 9.3. *Suppose that M is a complete Riemannian manifold. Assume that $f \in H^{1,1}(M)$. Then for every open subset U of M the sets $A^t := \{x \in U : f(x) > t\}$ have finite perimeter for a.e. t in $\mathbb{R}$ and*

$$\int_U |\mathrm{grad}\, f| \, \mathrm{d}V = \int_{\mathbb{R}} P\big(A^t, U\big) \, \mathrm{d}t.$$

Remark 9.4. We observe *en passant* that using Lemma 9.1, one can prove the following coarea formula for functions in $BV(M)$

$$\mathrm{Var}(f, M) = \int_{\mathbb{R}} P\big(A^t, M\big) \, \mathrm{d}t.$$

Now we are ready to prove the equivalence of property (I) and Cheeger's inequality. We recall that the constant I_M is defined in Section 2.

Theorem 9.5. *Suppose that M is a complete Riemannian manifold. The following hold:*

(i) *M possesses property* (I) *if and only if $h(M) > 0$. Furthermore $I_M = h(M)$;*
(ii) *if the Ricci curvature of M is bounded from below, then M possesses property* (I) *if and only if $b(M) > 0$.*

Proof. First we prove (i). Assume that M possesses property (I) and denote by A a bounded open subset of M with smooth boundary. By Proposition 3.1

$$\frac{V(A_\kappa)}{\kappa} \geq \frac{1 - \mathrm{e}^{-I_M t}}{t} \, V(A) \quad \forall\, t \in \mathbb{R}^+.$$

Hence

$$\liminf_{\kappa \to 0} \frac{V(A_\kappa)}{\kappa} \geq I_M \, V(A).$$

Since the limit in the left hand side is the lower inner Minkowski content of ∂A, which coincides with $\sigma(\partial A)$ because ∂A is smooth, we have proved that Cheeger's isoperimetric constant $h(M)$ is at least I_M.

To prove the converse, assume that $h(M) > 0$ and let A be a bounded open set in M. Since the manifold M is complete, the function f defined by $f(x) = \rho(x, A^c)$ (here A^c denotes the complementary set of A in M) is Lipschitz and $|\mathrm{grad}\, f| = 1$ almost everywhere on A. Recall that for each t in $\mathbb{R}$ we denote by A^t the set $\{x \in A : f(x) > t\}$. Notice that

$$P\big(A^s, A^t\big) = \begin{cases} P\big(A^s, M\big) & \text{if } s > t; \\ 0 & \text{if } s < t. \end{cases}$$

Thus, by the coarea formula

$$V\big(A^t\big) = \int_{A^t} |\mathrm{grad}\, f| \, \mathrm{d}V = \int_t^\infty P\big(A^s, M\big) \, \mathrm{d}s.$$

Hence, by Proposition 9.2

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(A^t\big) = -P\big(A^t, M\big) \leq -h(M)\, V\big(A^t\big) \qquad \text{for a.e. } t \in \mathbb{R}.$$

This differential inequality implies that $V\big(A^t\big) \leq e^{-h(M)\,t}\, V(A)$ for all $t > 0$, *i.e.*

$$V(A_t) = V(A) - V\big(A^t\big) \geq (1 - e^{-h(M)\,t})\, V(A) \geq h(M)\, t\, V(A) \quad \forall\, t \in (0, 1).$$

Thus M possesses property (I), and $h(M) \leq I_M$, as required to conclude the proof of (i).

To prove (ii) we recall that if M has Ricci curvature bounded below by $-K$, for some $K \geq 0$ then

$$b(M) \leq C\big(\sqrt{K}\, h(M) + h(M)^2\big), \tag{9.5}$$

where C is a constant which depends only on the dimension of M [9, 31]. This inequality, together with Cheeger's inequality (9.2), shows that the constants $h(M)$ and $b(M)$ are equivalent. The required conclusion follows directly from (i). □

Remark 9.6. We remark that property (I) is invariant under quasi-isometries. Indeed, the fact that $h(M)$ is positive is invariant under quasi-isometries [12, Remark V.2.2].

Remark 9.7. Suppose that M is a complete Riemannian manifold. If $b(M) = 0$, then M has not property (I) by Theorem 9.5 (i).

Observe also that if M has Ricci curvature bounded from below, and a spectral gap, then M has property (I). In particular noncompact Riemannian symmetric spaces and Damek-Ricci spaces have property (I).

10. Applications

In this section we illustrate some applications of the theory developed in the previous sections. Other applications will appear in [11].

The first application we consider is to spectral multipliers on certain Riemannian manifolds. Suppose that M is a complete Riemannian manifold with positive injectivity radius $\mathrm{inj}(M)$, Ricci curvature bounded from below, and positive Cheeger isoperimetric constant $h(M)$. By Cheeger's inequality the bottom $b(M)$ of the $L^2(M)$ spectrum of the Laplace–Beltrami operator $\mathcal{L}$ on M is positive. We denote by μ the Riemannian measure on M.

As shown in Section 9, under these assumptions M possesses property (I). Furthermore, there exist constants α, α', β, β', C_1 and C_2 such that

$$C_1\, (1 + r^2)^{\alpha'/2}\, \mathrm{e}^{\beta' r} \leq \mu\big(B(p, r)\big) \leq C_2\, (1 + r^2)^{\alpha/2}\, \mathrm{e}^{\beta r} \quad \forall\, r > 1.$$

We say that M has N-bounded geometry provided that the derivatives of the Riemann tensor up to the order N are uniformly bounded. Clearly if M has N-bounded geometry, then the Ricci curvature of M is bounded from below.

For each σ in $\mathbb{R}^+$ we denote by $\mathbf{S}_\sigma$ the strip $\{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < \sigma\}$.

Definition 10.1. Suppose that κ is a positive integer and that σ is in $\mathbb{R}^+$. The space $H^\infty(\mathbf{S}_\sigma; \kappa)$ is the vector space of all functions f in $H^\infty(\mathbf{S}_\sigma)$ for which there exists a positive constant C such that for each ε in $\{-1, 1\}$

$$\left| D^k f(\zeta) \right| \leq C\,(1 + |\zeta|)^{-k} \quad \forall\, k \in \{0, 1, \ldots, \kappa\} \quad \forall\, \zeta \in \mathbf{S}_\sigma. \tag{10.1}$$

If (10.1) holds, we say that f satisfies a *Mihlin–Hörmander condition of order* κ on the strip $\mathbf{S}_\sigma$. We endow $H^\infty(\mathbf{S}_\sigma; \kappa)$ with the norm

$$\|f\|_{\sigma;\kappa} = \max_{k \in \{0,1,\ldots,\kappa\}} \sup_{\zeta \in \mathbf{S}_\sigma} (1 + |\zeta|)^k \left| D^k f(\zeta) \right|.$$

The following result complements a celebrated result of Taylor [43, Theorem 1].

Theorem 10.2. *Suppose that M is an n dimensional complete Riemannian manifold with N-bounded geometry, where N is an integer $\geq n/2 + 1$. Assume that the injectivity radius* $\operatorname{inj}(M)$ *and the bottom $b(M)$ of the $L^2(M)$ spectrum of M are positive. Suppose that f is in $H^\infty(\mathbf{S}_\sigma; \kappa)$, where $\sigma \geq \beta/2$ and $\kappa >$* $\max(\alpha/2 + 1, n/2 + 1)$. *Then $f(\mathcal{L}_b^{1/2})$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$, from $L^\infty(M)$ to $BMO(\mu)$ and on $L^p(M)$ for all p in* $(1, \infty)$.

Proof. Denote by $\mathcal{L}_b$ the operator $\mathcal{L} - b\mathcal{I}$, formally defined on $L^2(\mu)$. The strategy of the proof of [43, Theorem 1] is to decompose the operator $f(\mathcal{L}_b^{1/2})$ as the sum of two operators, $f_0(\mathcal{L}_b^{1/2})$ and $f_\infty(\mathcal{L}_b^{1/2})$, where $f_\infty(\mathcal{L}_b^{1/2})$ is bounded on $L^1(\mu)$ and $L^\infty(\mu)$ and $f_0(\mathcal{L}_b^{1/2})$ is of weak type 1.

To prove the latter, Taylor [43] and Cheeger, Gromov and Taylor [13] prove that the integral kernel of $f_0(\mathcal{L}_b^{1/2})$, which is compactly supported, satisfies a Hörmander type integral condition. Taking this for granted, by Theorem 8.2, the operator $f_0(\mathcal{L}_b^{1/2})$ is bounded from $H^1(\mu)$ to $L^1(\mu)$, and from $L^\infty(\mu)$ to $BMO(\mu)$. Therefore the same is true for $f(\mathcal{L}_b^{1/2})$. The boundedness of $f(\mathcal{L}_b^{1/2})$ on $L^p(\mu)$ the follows by interpolation. □

Note that this result applies to Riemannian symmetric spaces of the noncompact type, and to Damek–Ricci spaces. In the case where M is a symmetric space of the noncompact type and real rank > 1, J.Ph. Anker [1] extended Taylor's result [43, Theorem 1] to certain multiplier operators for the spherical Fourier transform.

Suppose that G is a noncompact semisimple Lie group with finite centre, and denote by K a maximal compact subgroup thereof, and by X the associated Riemannian symmetric space of the noncompact type G/K (the G invariant metric on

X is induced by the Killing form of G). Denote by μ a G-invariant measure on X. We complement Anker's result by showing that (some) of the multiplier operators he considers satisfy natural $H^1(\mu)$-$L^1(\mu)$ and $L^\infty(\mu)$-$BMO(\mu)$ estimates.

Suppose that $G = KAN$ is an Iwasawa decomposition of G, where A is abelian and N is nilpotent. Denote by $\mathfrak{g}$ and $\mathfrak{a}$ the Lie algebras of G and A respectively, and by ρ the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$ with multiplicities. Denote by $\mathfrak{a}^*$ the dual of $\mathfrak{a}$. An important rôle in what follows is played by a certain tube $\mathbf{T}$ in the complexified dual $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}$. To define $\mathbf{T}$, denote by $\mathbf{W}$ the convex hull of the vectors $\{w \cdot \rho : w \in W\}$ in $\mathfrak{a}^*$, where W denotes the Weyl group of G. Then define $\mathbf{T} = \mathfrak{a}^* + i\mathbf{W}$.

Definition 10.3. Suppose that κ is a positive integer. The space $H^\infty(\mathbf{T}; \kappa)$ is the vector space of all Weyl invariant bounded holomorphic functions m on $\mathbf{T}$ for which there exists a positive constant C such that for every multiindex β with $|\beta| \leq \kappa$

$$\left|D^\beta m(\zeta)\right| \leq C\,(1 + |\zeta|)^{-|\beta|} \quad \forall\, \zeta \in \mathbf{T}. \tag{10.2}$$

If (10.2) holds, we say that f satisfies a *Mihlin–Hörmander condition of order* κ on the tube $\mathbf{T}$. We endow $H^\infty(\mathbf{T}; \kappa)$ with the norm

$$\|f\|_\kappa = \max_{|\beta|\leq\kappa} \sup_{\zeta\in\mathbf{T}} (1 + |\zeta|)^{|\beta|} \left|D^\beta f(\zeta)\right|.$$

Suppose that k is a K-bi-invariant distribution on G, and denote by m its spherical Fourier transform: m may be thought of as a distribution on $\mathfrak{a}^*$. If m is bounded on $\mathfrak{a}^*$, then the convolution operator $\mathcal{T}_k$, defined by

$$\mathcal{T}_k f = f * k \quad \forall\, f \in C_c^\infty(G)$$

extends to a bounded operator on $L^2(\mu)$.

Theorem 10.4. *Suppose that* $\kappa > (\dim X)/2 + 1$. *Suppose that k is a K-bi-invariant distribution such that its spherical transform m is in* $H^\infty(\mathbf{T}; \kappa)$. *Then the operator $\mathcal{T}_k$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$.*

Proof. Denote by ψ a K-bi-invariant smooth function on G with compact support which is identically 1 in a neighbourhood of the identity, and define the distributions k_0 and k_∞ by

$$k_0 = \psi\, k \qquad \text{and} \qquad k_\infty = (1 - \psi)\, k.$$

Anker [1, Theorem 1] shows that k_∞ is, in fact, a function in $L^1(G)$. Therefore, the operator $\mathcal{T}_{k_\infty}$, defined by $\mathcal{T}_{k_\infty} f = f \mapsto f * k_\infty$ extends to a bounded operator on $L^1(\mu)$ and on $L^\infty(\mu)$, and *a fortiori* to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$.

Furthermore, Anker proves that k_0 is locally integrable off the origin, and satisfies the following Hörmander type integral inequality

$$\sup_{B\in\mathcal{B}_1} \sup_{y\in B} \int_{(2B)^c} \left|k_0(y^{-1}x) - k_0(x)\right| \,\mathrm{d}\mu(x) < \infty.$$

Define the operator $\mathcal{T}_{k_0}$ by $\mathcal{T}_{k_0} f = f * k_0$. Denote by Δ the diagonal in $X \times X$, and define the locally integrable function t on $X \times X \setminus \Delta$ by

$$t(x, y) = k_0(y^{-1}x).$$

It is straightforward to check that t is the kernel (see the definition at the beginning of Section 8) of the operator $\mathcal{T}_{k_0}$, and that t satisfies conditions $\nu_t < \infty$ and $\upsilon_t < \infty$. By Theorem 8.2 (iii) the operator $\mathcal{T}_{k_0}$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $BMO(\mu)$.

Since $\mathcal{T}_k = \mathcal{T}_{k_0} + \mathcal{T}_{k_\infty}$, the required boundedness properties of $\mathcal{T}_k$ follow directly from those of $\mathcal{T}_{k_0}$ and $\mathcal{T}_{k_\infty}$. □

Our last application is to the boundedness of Riesz transforms. This is a very fashionable and interesting subject: see [2, 16] for recent results on manifolds, and the references therein for less recent results. Suppose that M is a complete Riemannian manifolds satisfying the following assumptions: the Riemannian measure μ is locally doubling, and the following scaled local Poincaré inequality holds: for every positive b there exists a constant C such that for every B in $\mathcal{B}_b$ and for every f in $C^\infty(2B)$

$$\int_B |f - f_B|^2 \, \mathrm{d}\mu \le C\, r^2 \int_{2B} |\nabla f|^2 \, \mathrm{d}\mu.$$

Suppose also that the volume growth of M is at most exponential. Note that these assumptions hold if M is a Riemannian manifold with Ricci curvature bounded from below.

We may define the "localised" Riesz transforms $\nabla(\mathcal{L} + \varepsilon)^{-1/2}$, where ε is in $\mathbb{R}^+$. Russ [39] proved that the localised Riesz transforms map local atoms uniformly into $L^1(M)$. However, in general, there is no indication that this result interpolates with the trivial $L^2(M)$ estimate to produce $L^p(M)$ boundedness for p in $(1, 2)$.

Our theory complements Russ' results. Indeed, Proposition 8.1 implies that $\nabla(\mathcal{L} + \varepsilon)^{-1/2}$ extends to a bounded operator from $H^1(M)$ into $L^1(M)$.

In the case where M possesses property (I) these result interpolate with the trivial $L^2(M)$ estimate and give that $\nabla(\mathcal{L} + \varepsilon)^{-1/2}$ extends to a bounded operator on $L^p(M)$ for all p in $(1, 2)$, a fact already known, but whose proof is far from being trivial.

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